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MIXED FINITE VOLUME METHOD ON NON-STAGGERED GRIDS FOR THE SIGNORINI PROBLEM

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ABSTRACT. In this work we propose a mixed finite volume method for the Signorini problem which are based on the idea of Keller's finite volume box method. The triangulation may consist of both triangles and quadrilaterals. We choose the first-order nonconforming space for the scalar approximation and the lowest-order Raviart–Thomas vector space for the vector approximation. It will be shown that our mixed finite volume method is equivalent to the standard nonconforming finite element method for the scalar variable with a slightly modified right-hand side, which are crucially used in a priori error analysis.

1. INTRODUCTION

In this paper we consider the following Signorini Problem (which is a simplification of the unilateral contact problem)

$$-\nabla\cdot(A\nabla u) = f \qquad \text{in } \Omega, \tag{1}$$

$$u = 0 \qquad \text{on } \Gamma_D, \tag{2}$$

$$A\nabla u\cdot\boldsymbol{n} = g \qquad \text{on } \Gamma_N, \tag{3}$$

$$u \ge 0, \quad A\nabla u\cdot\boldsymbol{n} \ge 0, \quad u(A\nabla u\cdot\boldsymbol{n}) = 0 \qquad \text{on } \Gamma_C. \tag{4}$$

Here Ω is a bounded polygonal domain in $\mathbb{R}^2$ with the Lipschitz boundary $\partial\Omega$, and $\boldsymbol{n}$ denotes the unit outward normal to $\partial\Omega$. The coefficient A is assumed to be symmetric and uniformly positive definite, i.e., there exist positive constants $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 \boldsymbol{z}^T\boldsymbol{z} \le \boldsymbol{z}^T A(\boldsymbol{x})\boldsymbol{z} \le \alpha_2 \boldsymbol{z}^T\boldsymbol{z} \qquad \forall \boldsymbol{z}\in\mathbb{R}^2, \boldsymbol{x}\in\overline{\Omega}.$$

The boundary $\partial\Omega$ is split into three disjoint parts, namely, the closed Dirichlet part Γ_D with positive measure, the Neumann part Γ_N and the possible contact zone Γ_C. In order to avoid some technical difficulty arising from $H^1_{00}(\Gamma_C)$, we assume that Γ_D and Γ_C do not touch.

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Let $\mathcal{K}$ be the closed convex subset of $H^1_D(\Omega) := \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\}$ defined by

$$\mathcal{K} = \{v \in H^1_D(\Omega) : v|_{\Gamma_C} \geq 0\}.$$

Then the primal variational formulation of problem (1)–(4) is given as follows: find $u \in \mathcal{K}$ such that for all $v \in \mathcal{K}$,

$$a(u, v - u) \geq L(v - u), \tag{5}$$

where

$$a(u,v) = \int_\Omega A\nabla u \cdot \nabla v \, d\boldsymbol{x}, \qquad L(v) = \int_\Omega f v \, d\boldsymbol{x} + \int_{\Gamma_N} g v \, ds.$$

The finite element approximations based on this variational inequality have been studied in many works; for example, we refer to [5, 9, 10] for some early results and to [1, 2, 3, 11] for improved error estimates. When one is more interested in approximation of the vector variable $\boldsymbol{\sigma} = A\nabla u$, it is desirable to rewrite (1)–(4) in the mixed form

$$\boldsymbol{\sigma} = A\nabla u, \qquad \nabla \cdot \boldsymbol{\sigma} + f = 0 \qquad \text{in } \Omega, \tag{6}$$

$$u = 0 \qquad \text{on } \Gamma_D, \tag{7}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = g \qquad \text{on } \Gamma_N, \tag{8}$$

$$u \geq 0, \quad \boldsymbol{\sigma} \cdot \boldsymbol{n} \geq 0, \quad u(\boldsymbol{\sigma} \cdot \boldsymbol{n}) = 0 \qquad \text{on } \Gamma_C. \tag{9}$$

Dual mixed finite element methods, which were very successful for the variational equation (cf. [4]), have been also proposed for this system in [14, 15].

The purpose of this paper is to apply the mixed finite volume method on non-staggered grids proposed in [6] for the Poisson problem to the Signorini problem. In this method the vector approximation is sought in the lowest-order Raviart–Thomas space (like the dual mixed finite element method), while the scalar approximation is sought in the edge-based nonconforming space from [8, 13]. It was discovered in [6] that this mixed finite volume method possesses many attractive features when compared with the dual mixed finite element method, some of which are listed here:

- One can easily decouple the scalar and vector variables in the discrete mixed system, which results in a nonconforming finite element method for the scalar variable only.
- The nonconforming finite element system for the scalar only involves much less number of unknowns, and thus is easier to solve than the dual mixed finite element method.
- Once the scalar approximation is computed, the vector approximation can be recovered from it in a local and direct manner.
- The recovered vector approximation has continuous normal components over the whole domain and satisfies the local mass conservation, as in the dual mixed finite element method.

It will be shown that these advantages carry over to the Signorini problem without modification. In particular, we only need to deal with the nonconforming finite element method to establish error estimates for the mixed finite volume method of the Signorini problem. Although a satisfactory analysis is presented in [11] for the $P1$ nonconforming element, it seems to apply to triangular meshes only. We give an alternative approach which treats both triangular and quadrilateral cases.

The remainder of the paper is organized as follows. In the next section we introduce some notation and relevant finite element spaces. In Section 3, the mixed finite volume method for the Signorini problem is constructed and analyzed. The close relationship to the nonconforming finite element method is established together with the existence and uniqueness of a solution. Finally, in Section 4, we derive some error estimates for both the scalar and vector variables under the usual regularity assumptions.

2. Preliminaries

To discretize the problem (6)–(9), we construct a regular triangulation $\mathcal{T}_h$ over the domain Ω consisting of triangles and/or quadrilaterals such that every nonempty intersection of two elements is either a vertex or a complete edge. The mesh size is defined by

$$h = \max_{T\in\mathcal{T}_h} h_T, \qquad h_T = \operatorname{diam}(T).$$

We shall assume throughout the paper that every quadrilateral element of $\mathcal{T}_h$ is a h^2-perturbation of a parallelogram: the distance between the midpoints of the two diagonals of T is $O(h_T^2)$. This assumption is satisfied, e.g., if $\mathcal{T}_h$ is obtained through successive refinement of an initial mesh by connecting the midpoints of opposite edges.

Let $\mathcal{E}_h$ and $\mathcal{E}_\Omega$ be the collection of all edges and all interior edges of $\mathcal{T}_h$, respectively. The sets of boundary edges are denoted by $\mathcal{E}_D$, $\mathcal{E}_N$ and $\mathcal{E}_C$, according to whether the edge belongs to Γ_D, Γ_N or Γ_C. It is assumed that the triangulation $\mathcal{T}_h$ respects the boundary conditions, i.e., there is no change of boundary conditions within an edge of $\mathcal{T}_h$. We also use the notation $\mathcal{E}_T$ to denote the set of all edges of an element T.

Now we define the finite element spaces for the scalar and vector variables. For the scalar approximation we choose the first-order nonconforming finite element which is defined locally on a generic element T by

$$\mathbb{NC}_1(T) = \begin{cases} \mathbb{P}_1(T) & \text{if } T \text{ is a triangle,} \\ \mathbb{RQ}_1(T) & \text{if } T \text{ is a quadrilateral,} \end{cases}$$

where $\mathbb{P}_k(T)$ is the space of all polynomials on T of degree at most k, and $\mathbb{RQ}_1(T)$ is the Rannacher–Turek space

$$\mathbb{RQ}_1(T) := \{\hat{v}\circ F_T^{-1} : \hat{v} = a + b\hat{x} + c\hat{y} + d(\hat{x}^2 - \hat{y}^2),\ a,b,c,d \in \mathbb{R}\},$$

with F_T being the invertible bilinear transformation from $\hat{T} = [0,1]^2$ onto T. It is known that the integral averages over the edges of T can be used as degrees of freedom for the space $\mathbb{NC}_1(T)$. For each $E \in \mathcal{E}_T$, we denote by $\phi_E^{(T)} \in \mathbb{NC}_1(T)$ the local basis function satisfying

$$\int_{E'} \phi_E^{(T)}\, ds = \delta_{E,E'}|E| \qquad \forall E' \in \mathcal{E}_T,$$

where $|D|$ is the measure of a set D.

The global nonconforming finite element space on a quadrangular mesh $\mathcal{T}_h$ is defined by (see [8] for triangular meshes and [13] for quadrilateral meshes)

$$\mathcal{NC}_h = \left\{ v_h \in L^2(\Omega) : v_h|_T \in \mathbb{NC}_1(T) \ \ \forall T \in \mathcal{T}_h, \text{ and } \int_E [\![v_h]\!] \, ds = 0 \ \ \forall E \in \mathcal{E}_\Omega \right\},$$

$$\mathcal{NC}_{h,D} = \left\{ v_h \in \mathcal{NC}_h : \int_E v_h \, ds = 0 \ \ \forall E \in \mathcal{E}_D \right\},$$

where $[\![v]\!]|_E$ is the jump of v across the edge E. The global basis function ϕ_E associated with the edge $E = \partial T_1 \cap \partial T_2$ can be obtained by patching the local functions $\phi_E^{(T_i)}$ together.

For the vector approximation we adopt the lowest order Raviart–Thomas space (cf. [4])

$$\mathbb{RT}_0(T) = \begin{cases} (\mathbb{P}_0(T))^2 \oplus (x,y)\mathbb{P}_0(T) & \text{if } T \text{ is a triangle,} \\ (\mathbb{P}_0(T))^2 \oplus (x\mathbb{P}_0(T), y\mathbb{P}_0(T)) & \text{if } T \text{ is a rectangle.} \end{cases}$$

On a general quadrilateral T, this space is built via the Piola transformation

$$\mathbb{RT}_0(T) := \{ (\det DF_T)^{-1} DF_T \hat{\boldsymbol{\tau}} : \hat{\boldsymbol{\tau}} \in \mathbb{RT}_0(\hat{T}) \},$$

where DF_T is the Jacobian matrix of $F_T : \hat{T} \to T$. The global space is then defined to be

$$\mathcal{RT}_h = \{ \boldsymbol{\tau} \in H(\mathrm{div}, \Omega) : \boldsymbol{\tau}|_T \in \mathbb{RT}_0(T) \ \ \forall T \in \mathcal{T}_h \}.$$

It is well known that $\boldsymbol{\tau} \in \mathcal{RT}_h$ has *continuous and constant* normal components on the edges of T which can be used as degrees of freedom.

3. Mixed finite volume method

In this section we define a mixed finite volume method for the Signorini problem (6)–(9) and establish its close relationship to the nonconforming finite element method for (1)–(4).

The discretization is simply accomplished by integrating the strong form of the Signorini problem (6)–(9) (multiplied by some test functions for the first equation of (6)) locally over each element $T \in \mathcal{T}_h$ and each edge $E \in \mathcal{E}_D \cup \mathcal{E}_N \cup \mathcal{E}_C$. We seek the approximate variables in the pair $\mathcal{RT}_h \times \mathcal{NC}_h$ defined in the previous section, leading to the following discrete system: find $(\boldsymbol{\sigma}_h, u_h) \in \mathcal{RT}_h \times \mathcal{NC}_h$ such that for all $v_h \in \mathbb{NC}_1(T)$,

$$\int_T (\boldsymbol{\sigma}_h - A\nabla u_h) \cdot \nabla v_h \, d\boldsymbol{x} = 0, \quad \int_T (\nabla \cdot \boldsymbol{\sigma}_h + f) \, d\boldsymbol{x} = 0 \qquad \forall T \in \mathcal{T}_h, \tag{10}$$

$$\int_E u_h \, ds = 0 \qquad \forall E \in \mathcal{E}_D, \tag{11}$$

$$\boldsymbol{\sigma}_h \cdot \boldsymbol{n}|_E - \frac{1}{|E|} \int_E g \, ds = 0 \qquad \forall E \in \mathcal{E}_N, \tag{12}$$

$$\int_E u_h \, ds \geq 0, \quad \boldsymbol{\sigma}_h \cdot \boldsymbol{n}|_E \geq 0, \quad \int_E u_h (\boldsymbol{\sigma}_h \cdot \boldsymbol{n}) \, ds = 0 \qquad \forall E \in \mathcal{E}_C. \tag{13}$$

It is worthwhile to mention that, unlike standard finite element methods, all boundary conditions (7)–(9) are handled in strong ways.

Now we show that the discrete problem (10)–(13) can be reduced to the following standard nonconforming finite element method: find $u_h \in \mathcal{K}_h$ such that for all $v_h \in \mathcal{K}_h$,

$$a_h(u_h, v_h - u_h) \geq \bar{L}(v_h - u_h), \tag{14}$$

where $\mathcal{K}_h$ is the closed convex subset of $\mathcal{NC}_{h,D}$ given by

$$\mathcal{K}_h = \left\{ v_h \in \mathcal{NC}_{h,D} : \int_E v_h \, d\boldsymbol{x} \geq 0 \quad \forall E \in \mathcal{E}_C \right\},$$

and

$$a_h(u, v) = \sum_{T \in \mathcal{T}_h} \int_T A\nabla u \cdot \nabla v \, d\boldsymbol{x}, \qquad \bar{L}(v) = \int_\Omega \bar{f} v \, d\boldsymbol{x} + \int_{\Gamma_N} \bar{g} v \, ds.$$

Here $\bar{f}$ and $\bar{g}$ represent the piecewise integral averages taken over $\mathcal{T}_h$ and $\mathcal{E}_N$, respectively, defined by

$$\bar{f}|_T = \frac{1}{\det DF_T} \int_T f \, dx, \qquad \bar{g}|_E = \frac{1}{|E|} \int_E g \, ds.$$

Note that $\bar{f}|_T$ is not a constant unless T is a parallelogram and that the second equation of (10) can be written as $\nabla \cdot \boldsymbol{\sigma}_h + \bar{f} = 0$.

Theorem 3.1. *Let* $(\boldsymbol{\sigma}_h, u_h) \in \mathcal{RT}_h \times \mathcal{NC}_h$ *be a solution of the mixed finite volume method* (10)–(13). *Then* u_h *is a solution of* (14), *and we have for all* $T \in \mathcal{T}_h$ *and* $E \in \mathcal{E}_T$

$$\boldsymbol{\sigma}_h \cdot \boldsymbol{n}_T|_E = \frac{1}{|E|} \left(\int_T A\nabla u_h \cdot \nabla \phi_E^{(T)} \, d\boldsymbol{x} - \int_T \bar{f} \phi_E^{(T)} \, d\boldsymbol{x} \right), \tag{15}$$

where $\boldsymbol{n}_T$ *is the unit normal outward to* T.

Proof. By the first equation of (10), it follows that for $v_h \in \mathcal{K}_h$,

$$a_h(u_h, v_h - u_h) = \sum_{T \in \mathcal{T}_h} \int_T \boldsymbol{\sigma}_h \cdot \nabla(v_h - u_h) \, d\boldsymbol{x}.$$

Now, integrating by parts and using the continuity properties of $\mathcal{RT}_h$ and $\mathcal{NC}_h$, we obtain

$$\begin{aligned} a_h(u_h, v_h - u_h) &= \sum_{T \in \mathcal{T}_h} \left\{ \int_{\partial T} \boldsymbol{\sigma}_h \cdot \boldsymbol{n}_T (v_h - u_h) \, ds - \int_T \nabla \cdot \boldsymbol{\sigma}_h (v_h - u_h) \, d\boldsymbol{x} \right\} \\ &= \sum_{E \in \mathcal{E}_C} \int_E \boldsymbol{\sigma}_h \cdot \boldsymbol{n} (v_h - u_h) \, ds + \bar{L}(v_h - u_h), \end{aligned}$$

where we used the boundary conditions (11)–(12) in the last equality. On the other hand, it follows directly from the discrete contact condition (13) that

$$\sum_{E \in \mathcal{E}_C} \int_E \boldsymbol{\sigma}_h \cdot \boldsymbol{n} (v_h - u_h) \, ds = \sum_{E \in \mathcal{E}_C} \int_E \boldsymbol{\sigma}_h \cdot \boldsymbol{n} \, v_h \, ds \geq 0,$$

which shows that u_h is a solution of (14).

The second result (15) is easily derived from (10) as follows:

$$|E|\,\boldsymbol{\sigma}_h\cdot\boldsymbol{n}|_E = \int_{\partial T}\boldsymbol{\sigma}_h\cdot\boldsymbol{n}\,\phi_E^{(T)}\,ds = \int_T(\boldsymbol{\sigma}_h\cdot\nabla\phi_E^{(T)} + \nabla\cdot\boldsymbol{\sigma}_h\,\phi_E^{(T)})\,d\boldsymbol{x}$$
$$= \int_T A\nabla u_h\cdot\nabla\phi_E^{(T)}\,d\boldsymbol{x} - \int_T \bar{f}\phi_E^{(T)}\,d\boldsymbol{x}.$$

This completes the proof. □

To prove the converse result of Theorem 3.1, we need the following well-known lemma (see pp. 58 of [10]).

Lemma 3.2. *$u_h \in \mathcal{K}_h$ is a solution of* (14) *if and only if*

$$a_h(u_h, v_h) \geq \bar{L}(v_h) \quad \forall v_h \in \mathcal{K}_h \quad \textit{and} \quad a_h(u_h, u_h) = \bar{L}(u_h).$$

Now we are ready to state the following theorem.

Theorem 3.3. *Let $u_h \in \mathcal{K}_h$ be a solution of the variational inequality* (14) *and let $\boldsymbol{\sigma}_h|_T \in \mathbb{RT}_0(T)$ be defined by* (15) *for all $T \in \mathcal{T}_h$. Then $(\boldsymbol{\sigma}_h, u_h)$ is a solution of the mixed finite volume method* (10)–(13).

Proof. We need to verify (10), (12) and the latter two conditions of (13). By summing (15) over $E \in \mathcal{E}_T$ and using the fact that $\sum_{E\in\mathcal{E}_T}\phi_E^{(T)} \equiv 1$, it follows that

$$\int_{\partial T}\boldsymbol{\sigma}_h\cdot\boldsymbol{n}_T\,d\boldsymbol{x} = \sum_{E\in\mathcal{E}_T}\left(\int_T A\nabla u_h\cdot\nabla\phi_E^{(T)}\,d\boldsymbol{x} - \int_T\bar{f}\phi_E^{(T)}\,d\boldsymbol{x}\right) = -\int_T f\,d\boldsymbol{x},$$

which yields the second equation of (10). By (15) we also obtain for all $E \in \mathcal{E}_T$

$$\int_T\boldsymbol{\sigma}_h\cdot\nabla\phi_E^{(T)}\,d\boldsymbol{x} = \int_{\partial T}\boldsymbol{\sigma}_h\cdot\boldsymbol{n}_T\,\phi_E^{(T)}\,ds - \int_T\nabla\cdot\boldsymbol{\sigma}_h\phi_E^{(T)}\,d\boldsymbol{x}$$
$$= |E|\,\boldsymbol{\sigma}_h\cdot\boldsymbol{n}_T|_E + \int_T\bar{f}\phi_E^{(T)}\,d\boldsymbol{x}$$
$$= \int_T A\nabla u_h\cdot\nabla\phi_E^{(T)}\,d\boldsymbol{x},$$

implying that for all $v_h \in \mathbb{NC}_1(T)$,

$$\int_K(\boldsymbol{\sigma}_h - A\nabla u_h)\cdot\nabla v_h\,d\boldsymbol{x} = 0.$$

This proves (10).

Now let us derive the Neumann condition (12). Since both ϕ_E and $-\phi_E$ belong to $\mathcal{K}_h$ for $E \in \mathcal{E}_N$, it follows by taking $v_h = \phi_E$ (with $E \subset \partial T$) in Lemma 3.2 that

$$\int_T A\nabla u_h\cdot\nabla\phi_E^{(T)}\,d\boldsymbol{x} = \int_T\bar{f}\phi_E^{(T)}\,d\boldsymbol{x} + \int_E\bar{g}\phi_E^{(T)}\,ds,$$

which gives by (15)

$$\boldsymbol{\sigma}_h \cdot \boldsymbol{n}|_E = \frac{1}{|E|}\int_E \bar{g}\phi_E^{(T)}\,ds = \bar{g}|_E = \frac{1}{|E|}\int_E g\,ds.$$

On the other hand, taking $v_h = \phi_E$ for $E \in \mathcal{E}_C \cap \mathcal{E}_T$ in Lemma 3.2 leads to

$$\boldsymbol{\sigma}_h \cdot \boldsymbol{n}|_E = \frac{1}{|E|}\left(\int_T A\nabla u_h \cdot \nabla\phi_E^{(T)}\,d\boldsymbol{x} - \int_T \bar{f}\phi_E^{(T)}\,d\boldsymbol{x}\right) \geq 0.$$

Finally, substituting in Lemma 3.2

$$u_h = \sum_{E\in\mathcal{E}_h\setminus\mathcal{E}_D} \alpha_E\phi_E, \qquad \alpha_E = \frac{1}{|E|}\int_E u_h\,ds,$$

we obtain

$$\sum_{T\in\mathcal{T}_h}\sum_{E\in\mathcal{E}_C\cap\mathcal{E}_T} \alpha_E\left(\int_T A\nabla u_h \cdot \nabla\phi_E^{(T)}\,d\boldsymbol{x} - \int_T \bar{f}\phi_E^{(T)}\,d\boldsymbol{x}\right) = 0.$$

This immediately results in

$$\sum_{E\in\mathcal{E}_C} \alpha_E\,|E|\boldsymbol{\sigma}_h \cdot \boldsymbol{n}|_E = 0.$$

Since all terms in this summation are nonnegative, they should vanish, i.e.,

$$\alpha_E\,|E|\boldsymbol{\sigma}_h \cdot \boldsymbol{n}|_E = 0 \qquad \text{or} \qquad \int_E u_h(\boldsymbol{\sigma}_h \cdot \boldsymbol{n})\,ds = 0$$

for all $E \in \mathcal{E}_C$. This proves the contact conditions (13). □

As a corollary of the two theorems established above, we can deduce the existence and uniqueness of a solution of our mixed finite volume method.

Corollary 3.4. *The mixed finite volume method* (10)–(13) *has a unique solution* $(\boldsymbol{\sigma}_h, u_h)$.

Proof. It is known that the nonconforming finite element method (14) has a unique solution u_h (cf. [11]). Hence uniqueness of a solution $(\boldsymbol{\sigma}_h, u_h)$ follows from Theorem 3.1, and its existence from Theorem 3.3. □

Remark 3.5. *On a triangular element T, it is easy to show that $\boldsymbol{\sigma}_h$ can be also computed by the Marini formula (cf.* [7, 12]*)*

$$\boldsymbol{\sigma}_h|_T = \bar{A}\nabla u_h - \frac{\bar{f}}{2}(\boldsymbol{x} - \boldsymbol{x}_T),$$

where $\boldsymbol{x}_T = (x_T, y_T)$ is the mass center of T. On a rectangular element T with size $h_x \times h_y$, one can obtain

$$\boldsymbol{\sigma}_h|_T = A\nabla u_h - \frac{\bar{f}}{h_x + h_y}(hy(x - x_T), hx(y - y_T))$$

in the case of a piecewise constant scalar-valued A. We refer to [6] *for more details.*

4. A Priori Error Estimates

In this section we establish some error estimates for the solution of the mixed finite volume method (10)–(13). Since u_h is the solution of the nonconforming finite element method (14), we may follow the analysis of [11] which, however, seems to apply to triangular meshes only. We give an alternative proof here which treats quadrilateral meshes as well.

To begin with, let us define the broken H^1 norm

$$|v|_{1,h} = \left(\sum_{T \in \mathcal{T}_h} |v|_{1,T}^2 \right)^{1/2}$$

and the interpolation operator $I_h : H_D^1(\Omega) \to \mathcal{NC}_{h,D}$ by

$$\int_E I_h v \, ds = \int_E v \, ds \qquad \forall E \in \mathcal{E}_h.$$

It is well known that

$$|v - I_h v|_{1,h} \le C h^s \|v\|_{s,\Omega} \qquad (3/2 < s \le 2).$$

All subsequent error estimates will be based on the following lemma.

Lemma 4.1. *Let $u \in \mathcal{K}$ and $u_h \in \mathcal{K}_h$ be the solutions of* (5) *and* (14)*, respectively. If $u \in H^s(\Omega)$ for $\frac{3}{2} < s \le 2$, then there exists a constant $C > 0$ independent of the mesh size such that*

$$|u - u_h|_{1,h} \le C \left\{ h^{s-1} \|u\|_{s,\Omega} + \operatorname{osc}(f, g) + \left| \int_{\Gamma_C} \left(\overline{\frac{\partial u}{\partial n}} - \frac{\partial u}{\partial n} \right) (u - \bar{u}) \, ds \right|^{\frac{1}{2}} \right\},$$

where $\operatorname{osc}(f, g)$ *is the data oscillation term defined by*

$$\operatorname{osc}(f, g) = \left(\sum_{T \in \mathcal{T}_h} h_T^2 \|f - \bar{f}\|_{0,T}^2 + \sum_{E \in \mathcal{E}_N} h_E \|g - \bar{g}\|_{0,E}^2 \right)^{\frac{1}{2}}$$

Proof. We recall the following abstract error estimate due to Falk [9] (see also [11])

$$|u - u_h|_{1,h} \le C \inf_{v_h \in \mathcal{K}_h} \left\{ |u - v_h|_{1,h}^2 + a_h(u, v_h - u_h) - \bar{L}(v_h - u_h) \right\}^{\frac{1}{2}}.$$

Using the integration by parts and the weak continuity of $\mathcal{NC}_h$, one can obtain

$$\begin{aligned} a_h(u, v_h - u_h) - \bar{L}(v_h - u_h) &= \int_\Omega (f - \bar{f})(v_h - u_h) \, d\boldsymbol{x} + \int_{\Gamma_N} (g - \bar{g})(v_h - u_h) \, ds \\ &\quad + \sum_{T \in \mathcal{T}_h} \int_{\partial T \setminus \Gamma_N} \left(\frac{\partial u}{\partial n} - \overline{\frac{\partial u}{\partial n}} \right) (v_h - u_h) \, ds \\ &\quad + \int_{\Gamma_C} \overline{\frac{\partial u}{\partial n}} (v_h - u_h) \, ds \\ &:= J_1 + J_2 + J_3. \end{aligned}$$

The first term J_1 are easily bounded, since we have for any constants c_T and c_E

$$J_1 = \sum_{T\in\mathcal{T}_h} \int_T (f-\bar{f})(v_h - u_h - c_T)\, d\boldsymbol{x} + \sum_{E\in\mathcal{E}_N} \int_E (g-\bar{g})(v_h - u_h - c_E)\, ds,$$
$$\leq C\Big(\sum_{T\in\mathcal{T}_h} h_T^2 \|f-\bar{f}\|_{0,T}^2 + \sum_{E\in\mathcal{E}_N} h_E \|g-\bar{g}\|_{0,E}^2 \Big)^{1/2} |v_h - u_h|_{1,h}.$$

The second term J_2, which arises from the a priori error analysis of the variational equation, is bounded in the same way, yielding

$$J_2 \leq C h^{s-1} \|u\|_{s,\Omega} |v_h - u_h|_{1,h}.$$

The crucial part is to estimate the remaining part J_3. Taking $v_h = I_h u \in \mathcal{K}_h$, we obtain

$$J_3 = \int_{\Gamma_C} \overline{\frac{\partial u}{\partial n}} (I_h u - u_h)\, ds \leq \int_{\Gamma_C} \overline{\frac{\partial u}{\partial n}}\, I_h u\, ds = \int_{\Gamma_C} \overline{\frac{\partial u}{\partial n}}\, u\, ds.$$

Moreover, the saturation condition results in

$$\int_{\Gamma_C} \overline{\frac{\partial u}{\partial n}}\, u\, ds = \int_{\Gamma_C} \Big(\overline{\frac{\partial u}{\partial n}} - \frac{\partial u}{\partial n} \Big) u\, ds = \int_{\Gamma_C} \Big(\overline{\frac{\partial u}{\partial n}} - \frac{\partial u}{\partial n} \Big)(u-\bar{u})\, ds.$$

Now combining all these results together gives the desired result. □

Note that $\mathrm{osc}(f,g) = o(h)$ if $f|_T \in L^2(T)\ \ \forall T \in \mathcal{T}_h$ and $g|_E \in H^{\frac{1}{2}+\epsilon}(E)\ \ \forall E \in \mathcal{E}_N$. This will be assumed throughout the remainder of this paper. Thus it suffices to estimate the term

$$\int_{\Gamma_C} \Big(\overline{\frac{\partial u}{\partial n}} - \frac{\partial u}{\partial n} \Big)(u-\bar{u})\, ds = \sum_{E\in S_h} \int_E \Big(\overline{\frac{\partial u}{\partial n}} - \frac{\partial u}{\partial n} \Big)(u-\bar{u})\, ds,$$

where $S_h \subset \mathcal{E}_C$ is the set of edges on which the constraint changes from binding to non-binding. This is done in the following two theorems which extend the results of [11] to quadrilateral cases.

Theorem 4.2. *Suppose that the number of points on Γ_C at which the constraint changes from binding to non-binding is finite. Then we have for $u \in H^s(\Omega)$ with $\frac{3}{2} < s < 2$*

$$|u-u_h|_{1,h} \leq C h^{s-1} \|u\|_{s,\Omega}.$$

and for $u \in H^2(\Omega)$

$$|u-u_h|_{1,h} \leq C h |\log h|^{\frac{1}{4}} \|u\|_{2,\Omega}.$$

Proof. For $\frac{3}{2} < s < 2$, we have the following imbedding result

$$\|v\|_{0,p,\Gamma_C} \leq C \|v\|_{s-\frac{3}{2},\Gamma_C}, \qquad p = 1/(2-s) > 2.$$

This yields together with the Hölder inequality

$$\Big\| \frac{du}{d\tau} \Big\|_{0,E} \leq h^{\frac{1}{2}-\frac{1}{p}} \Big\| \frac{du}{d\tau} \Big\|_{0,p,E} \leq h^{\frac{1}{2}-\frac{1}{p}} \Big\| \frac{du}{d\tau} \Big\|_{0,p,\Gamma_C} \leq C h^{\frac{1}{2}-\frac{1}{p}} \|u\|_{s-\frac{1}{2},\Gamma_C},$$

where $\frac{du}{d\tau}$ is the tangential derivative of u along Γ_C. Hence it follows that

$$\int_E \left(\overline{\frac{\partial u}{\partial n}} - \frac{\partial u}{\partial n}\right)(u - \bar{u})\, ds \le Ch^{s-\frac{3}{2}} \left\|\frac{\partial u}{\partial n}\right\|_{s-\frac{3}{2},E} h \left\|\frac{du}{d\tau}\right\|_{0,E}$$
$$\le Ch^{s-\frac{1}{p}} \|u\|_{s-\frac{1}{2},E} \|u\|_{s-\frac{1}{2},\Gamma_C}.$$

Summing over $E \in S_h$ and using the fact that S_h is a uniformly finite set, we conclude that

$$\int_{\Gamma_C} \left(\overline{\frac{\partial u}{\partial n}} - \frac{\partial u}{\partial n}\right)(u - \bar{u})\, ds \le Ch^{s-\frac{1}{p}} \|u\|^2_{s-\frac{1}{2},\Gamma_C} \le Ch^{2(s-1)} \|u\|^2_{s,\Omega}.$$

In the case of $s = 2$, one can use the imbedding result

$$\|v\|_{0,p,\Gamma_C} \le Cp^{\frac{1}{2}} \|v\|_{\frac{1}{2},\Gamma_C}, \qquad 2 \le p < \infty$$

to deduce that

$$\int_E \left(\overline{\frac{\partial u}{\partial n}} - \frac{\partial u}{\partial n}\right)(u - \bar{u})\, ds \le Cp^{\frac{1}{2}} h^{2-\frac{1}{p}} \|u\|_{\frac{3}{2},E} \|u\|_{\frac{3}{2},\Gamma_C}$$
$$\le Ch^2 |\log(h)|^{\frac{1}{2}} \|u\|_{\frac{3}{2},E} \|u\|_{\frac{3}{2},\Gamma_C}$$

by taking $p = |\log h|$. The rest of the proof follows similarly as in the case of $\frac{3}{2} < s < 2$. □

Remark 4.3. *As note in* [1, 11], *the factor* $|\log h|^{\frac{1}{4}}$ *can be removed if* $u|_{\Gamma_C} \in W^{1,\infty}(\Gamma_C)$, *as we have*

$$\left\|\frac{du}{d\tau}\right\|_{0,E} \le h^{\frac{1}{2}} \left\|\frac{du}{d\tau}\right\|_{0,\infty,E},$$

which leads to

$$|u - u_h|_{1,h} \le Ch(\|u\|_{2,\Omega} + \|u\|_{1,\infty,\Gamma_C}).$$

With a stronger regularity condition on u, the optimal $O(h)$ convergence rate can be obtained even without the assumption that the number of points on Γ_C at which the constraint changes from binding to non-binding is finite. We refer to [11] for discussion on the $P1$ nonconforming finite element method.

Theorem 4.4. *If* $u \in W^{2,p}(\Omega)$ *for some* $p > 2$, *then we have*

$$|u - u_h|_{1,h} \le Ch\|u\|_{2,p,\Omega}.$$

Proof. We start with the following inequality from the proof of the previous theorem

$$\int_E \left(\overline{\frac{\partial u}{\partial n}} - \frac{\partial u}{\partial n}\right)(u - \bar{u})\, ds \le Ch^{\frac{1}{2}} \left\|\frac{\partial u}{\partial n}\right\|_{\frac{1}{2},E} h \left\|\frac{du}{d\tau}\right\|_{0,E}.$$

Following the argument in the proof of Lemma 4.11 of [2], we can prove that for $E \in S_h$,

$$\left\|\frac{du}{d\tau}\right\|_{0,E} \le Ch^{1-\frac{1}{p}} \left\|\frac{du}{d\tau}\right\|_{1-\frac{1}{p},E} \le Ch^{1-\frac{1}{p}} \|u\|_{2-\frac{1}{p},E},$$

which gives

$$\int_E \left(\overline{\frac{\partial u}{\partial n}} - \frac{\partial u}{\partial n}\right)(u - \bar{u})\, ds \le Ch^{2+\frac{1}{2}-\frac{1}{p}} \left\| \frac{\partial u}{\partial n} \right\|_{\frac{1}{2},E} \|u\|_{2-\frac{1}{p},E} \le Ch^2 \|u\|^2_{2-\frac{1}{p},E}.$$

Now summing over $E \in S_h$ and invoking the inequality

$$\sum_{E\in\, S_h} \|u\|^2_{2-\frac{1}{p},E} \le \|u\|^2_{2-\frac{1}{p},\Gamma_C} \le C\|u\|^2_{2-\frac{1}{p},p,\Gamma_C} \le C\|u\|^2_{2,p,\Omega},$$

we get the desired result. □

Now we are in a position to derive an error estimate for $\boldsymbol{\sigma} - \boldsymbol{\sigma}_h$. For this sake, we define the Raviart–Thomas projection $\Pi_h : H(\mathrm{div}, \Omega) \to \mathcal{RT}_h$ by (cf. [4])

$$\int_E \Pi_h \boldsymbol{\sigma} \cdot \boldsymbol{n}_T \, ds = \int_E \boldsymbol{\sigma} \cdot \boldsymbol{n}_T \, ds \qquad \forall E \in \mathcal{E}_T,\ T \in \mathcal{T}_h.$$

The following lemma implies that the error estimate for $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}$ can be deduced from that of $|u - u_h|_{1,h}$.

Lemma 4.5. *Let* $(\boldsymbol{\sigma}_h, u_h) \in \mathcal{RT}_h \times \mathcal{NC}_h$ *be the solution of the mixed finite volume method* (10)–(13). *Then we have*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \le C(\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{0,\Omega} + |u - u_h|_{1,h}).$$

Proof. By the definition of Π_h, it is easy to see that

$$\nabla \cdot (\Pi_h \boldsymbol{\sigma}) + \bar{f} = 0,$$

from which it follows that

$$\begin{aligned} |E|\, \Pi_T\boldsymbol{\sigma} \cdot \boldsymbol{n}_T|_E &= \int_{\partial T} \Pi_T \boldsymbol{\sigma} \cdot \boldsymbol{n}_T\, \phi_E^{(T)}\, ds \\ &= \int_T \Pi_T\boldsymbol{\sigma} \cdot \nabla\phi_E^{(T)}\, d\boldsymbol{x} - \int_T \bar{f}\phi_E^{(T)}\, d\boldsymbol{x}. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} |E|\, (\Pi_T\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \boldsymbol{n}_T|_E &= \int_T (\Pi_T\boldsymbol{\sigma} - \boldsymbol{\sigma}) \cdot \nabla\phi_E^{(T)}\, d\boldsymbol{x} + \int_T A\nabla(u - u_h) \cdot \nabla\phi_E^{(T)}\, d\boldsymbol{x} \\ &\le (\|\boldsymbol{\sigma} - \Pi_T\boldsymbol{\sigma}\|_{0,T} + \alpha_2 \|\nabla(u - u_h)\|_{0,T}) \|\nabla\phi_E^{(T)}\|_{0,T}. \end{aligned}$$

The scaling argument shows that $\|\nabla\phi_E^{(T)}\|_{0,T} \le C$ and

$$\|\Pi_T\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{0,T} \le C \sum_{E\in\mathcal{E}_T} |E|^2 \big|(\Pi_T\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \boldsymbol{n}_T|_E\big|^2.$$

So the proof is completed by using the bound obtained above and summing over $T \in \mathcal{T}_h$. □

Combining this lemma with the approximation result

$$\|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{0,\Omega} \le C h^{s-1} \|\boldsymbol{\sigma}\|_{s-1,\Omega} \qquad (3/2 < s \le 2)$$

and Theorems 4.2 and 4.4, we get the following theorem.

Theorem 4.6. *If $u \in W^{2,p}(\Omega)$ for some $p > 2$, then we have*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \le Ch(\|u\|_{2,p,\Omega} + \|\boldsymbol{\sigma}\|_{1,\Omega}).$$

When the number of points on Γ_C at which the constraint changes from binding to non-binding is finite, we have for $u \in H^s(\Omega)$ with $\frac{3}{2} < s < 2$

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \le C h^{s-1}(\|u\|_{s,\Omega} + \|\boldsymbol{\sigma}\|_{s-1,\Omega}),$$

and for $u \in H^2(\Omega)$

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \le Ch|\log h|^{\frac{1}{4}}(\|u\|_{2,\Omega} + \|\boldsymbol{\sigma}\|_{1,\Omega}).$$

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