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MULTIPLICITY RESULTS AND THE M-PAIRS OF TORUS-SPHERE VARIATIONAL LINKS OF THE STRONGLY INDEFINITE FUNCTIONAL

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ABSTRACT. Let $I \in C^{1,1}$ be a strongly indefinite functional defined on a Hilbert space H. We investigate the number of the critical points of I when I satisfies two pairs of Torus-Sphere variational linking inequalities and when I satisfies m $(m \geq 2)$ pairs of Torus-Sphere variational linking inequalities. We show that I has at least four critical points when I satisfies two pairs of Torus-Sphere variational linking inequality with $(P.S.)^*_c$ condition. Moreover we show that I has at least $2m$ critical points when I satisfies m $(m \geq 2)$ pairs of Torus-Sphere variational linking inequalities with $(P.S.)^*_c$ condition. We prove these results by Theorem 2.2 (Theorem 1.1 in [1]) and the critical point theory on the manifold with boundary.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Let $I \in C^{1,1}$ be a strongly indefinite functional defined on a Hilbert Space H. In this paper, we investigate the number of the critical points of I when I satisfies m $(m \geq 2)$ pairs of Torus-Sphere variational linking inequalities and $(P.S.)^*_c$ condition, $m \in N$. We show that I has at least two critical points each when I satisfies each one pair of Torus-Sphere variational linking inequality and $(P.S.)^*_c$ condition. We prove these results by use of Theorem 2.2 and the critical point theory on the manifold with boundary. In the case that I is not strongly indefinite functional Marino, A., Micheletti, A.M., Pistoia, Schechter, M., Tintarev. K., and Rabinowitz, P., proved in Theorem (3.4) of [4], [7] and [8] a theorem of existence of two solutions when I satisfies one pair of Sphere-Torus variational linking inequality by the mountain pass theorem and degree theory. Marino, A., Micheletti, A. M. and Pistoia, A. proved in Theorem (8.4) of [5] a theorem of existence of three solutions when I satisfies two pairs of Sphere-Torus variational linking inequalities and $(P.S.)_c$ condition by the mountain pass theorem and degree theory. In this paper we obtain the following results for the strongly indefinite functional case:

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Theorem 1.1. *(Two pairs of Torus-Sphere variational links) Let H be a Hilbert space with a norm $\|\cdot\|$, which is topological direct sum of the four subspaces X_0, X_1, X_2 and X_3. Let $I \in C^{1,1}(H,R)$ be a strongly indefinite functional. Assume that*

(1) $\dim X_i < \infty,\ i = 1, 2$;

(2) *There exist a small number $\rho > 0$, $r^{(1)} > 0$ and $R^{(1)}$ such that*

$$r^{(1)} < R^{(1)} \text{ and } \sup_{\Sigma_{R^{(1)}}(S_1(\rho),X_0)} I < \inf_{S_{r^{(1)}}(X_1\oplus X_2\oplus X_3)} I,$$

where $S_1(\rho) = \{u \in X_1 |\ \|u\| = \rho\}$;

(3) *There exist a small number $\rho > 0$, $r^{(2)} > 0$ and $R^{(2)} > 0$ such that*

$$r^{(2)} < R^{(2)} \text{ and } \sup_{\Sigma_{R^{(2)}}(S_2(\rho),X_0\oplus X_1)} I < \inf_{S_{r^{(2)}}(X_2\oplus X_3)} I,$$

where

$$S_{r^{(2)}}(X_2 \oplus X_3) = \{u \in X_2 \oplus X_3 |\ \|u\| = r^{(2)}\},$$

$$\begin{aligned}\Sigma_{R^{(2)}}(S_2(\rho), X_0 \oplus X_1) \;=\; & \{u = u_0 + u_1 + u_2 |\ u_2 \in S_2(\rho), u_0 \in X_0, u_1 \in X_1, \|u_2\| = \rho,\\ & 1 \le \|u_0 + u_1 + u_2\| = R^{(2)}\}\\ & \cup\{u = u_0 + u_1 + u_2 |\ u_2 \in S_2(\rho), u_0 \in X_0, u_1 \in X_1,\\ & \|u_2\| = \rho, 1 \le \|u_0 + u_1\| \le R^{(2)}\};\end{aligned}$$

(4) $R^{(2)} < R^{(1)} \Rightarrow \Delta_R^{(2)}(S_2(\rho), X_0 \oplus X_1) \subset \Sigma_{R^{(1)}}(S_1(\rho), X_0)$;

(5) $\beta^{(1)} = \sup_{\Delta_{R^{(1)}}(S_1(\rho),X_0)} I < +\infty$, *where*

$$\begin{aligned}\Delta_{R^{(1)}}(S_1(\rho), X_0) \;=\; & \{u = u_0 + u_1 |\ u_1 \in S_1(\rho), u_0 \in X_0,\\ & \|u_1\| = \rho, 1 \le \|u_0 + u_1\| \le R^{(1)}\};\end{aligned}$$

(6) $(P.S.)^*_c$ *condition holds for any $c \in [\alpha^{(1)}, \beta^{(1)}]$, where*

$$\alpha^{(1)} = \inf_{S_{r^{(2)}}(X_2\oplus X_3)} I;$$

(7) *There exists one critical point e in $X_0 \oplus X_3$ with $I(e) < \alpha^{(1)}$.*
Then there exist at least four distinct critical points except e, u^1_j, $j = 1, 2$, in X_1, u^2_j, $j = 1, 2$ in X_2, of I with

$$\begin{aligned}\alpha^{(1)} &= \inf_{S_{r^{(2)}}(X_2\oplus X_3)} I \le I(u^2_j) \le \sup_{\Delta_{R^{(2)}}(S_2(\rho),X_0\oplus X_1)} I\\ &\le \sup_{\Sigma_{R^{(1)}}(S_1(\rho),X_0)} I < \inf_{S_{r^{(1)}}(X_1\oplus X_2\oplus X_3)} I\\ &\le I(u^1_j) \le \sup_{\Delta_{R^{(1)}}(S_1(\rho),X_0)} I = \beta^{(1)} < +\infty.\end{aligned}$$

Theorem 1.2. *(m pairs of Torus-Sphere variational links) Let H be a Hilbert space with a norm $\|\cdot\|$, which is a topological direct sum of the $m+2$ subspaces $X_0, X_1, \cdots, X_m$ and X_{m+1}. Let $I \in C^{1,1}(H,R)$ be a strongly indefinite functional. Assume that*

(1) $\dim(X_i) < \infty$, $i = 1, \cdots, m$;

(2) *There exist a small number $\rho > 0$, $r^{(k)} > 0$ and $R^{(k)} > 0$ such that*

$$r^{(k)} < R^{(k)} \text{ and } \sup_{\Sigma_{R^{(k)}}(S_k(\rho), X_0\oplus\cdots\oplus X_{k-1})} I < \inf_{S_{r^{(k)}}(X_k\oplus\cdots\oplus X_{m+1})} I,$$

$k = 1, \cdots m$;

(3) $R^{(k)} < R^{(k-1)} \Rightarrow$

$$\Delta_{R^{(k)}}(S_k(\rho), X_0 \oplus \cdots \oplus X_{k-1}) \subset \Sigma_{R^{(k-1)}}(S_{k-1}(\rho), X_0 \oplus \cdots X_{k-2}),$$

$k = 1, \cdots, m$;

(4) $\beta^{(m)} = \sup_{\Delta_{R^{(1)}}(S_1(\rho), X_0)} I < +\infty$;

(5) $(P.S.)_c^*$ *condition holds for any* $c \in [\alpha^{(m)}, \beta^{(m)}]$, *where*

$$\alpha^{(m)} = \inf_{S_{r^{(m)}}(X_m \oplus X_{m+1})} I;$$

(6) *There exists one critical points e in $X_0 \oplus X_{m+1}$ with $I(e) < \alpha^{(m)}$.*

Then there exist at least $2m$ distinct critical points except e, u_j^k, $j = 1, 2$, in X_k, $1 \le k \le m$, of I with

$$\begin{aligned}
\alpha^{(m)} &= \inf_{S_{r^{(m)}}(X_m\oplus X_{m+1})} I \le I(u_j^m) \le \sup_{\Delta_{R^{(m)}}(S_m(\rho), X_0\oplus\cdots\oplus X_{m-1})} I \\
&\le \sup_{\Sigma_{R^{(m-1)}}(S_{m-1}(\rho), X_0\oplus\cdots\oplus X_{m-2})} I < \inf_{S_{r^{(m-1)}}(X_{m-1}\oplus X_m\oplus X_{m+1})} I \\
&\le I(u_j^{m-1}) \le \sup_{\Delta_{R^{(m-1)}}(S_{m-1}(\rho), X_0\oplus\cdots\oplus X_{m-2})} I \le \cdots \le \sup_{\Sigma_{R^{(k)}}(S_k(\rho), X_0\oplus\cdots\oplus X_{k-1})} I \\
&< \inf_{S_{r^{(k)}}(X_k\oplus\cdots\oplus X_{m+1})} I \le I(u_j^k) \le \sup_{\Delta_{R^{(k)}}(S_k(\rho), X_0\oplus\cdots\oplus X_{k-1})} I \\
&\le \sup_{\Sigma_{R^{(k-1)}}(S_{k-1}(\rho), X_0\oplus\cdots\oplus X_{k-2})} I \le I(u_j^{k-1}) \le \sup_{\Delta_{R^{(k-1)}}(S_{k-1}(\rho), X_0\oplus\cdots\oplus X_{m-2})} I \\
&\le \cdots < \inf_{S_{r^{(1)}}(X_1\oplus\cdots\oplus X_{m+1})} I \le I(u_j^1) \le \sup_{\Delta_{R^{(1)}}(S_1(\rho), X_0)} I = \beta^{(m)}.
\end{aligned}$$

For the proofs of the main results we use Theorem 2.2 and the critical point theory on the manifold with boundary. Since the functional I is strongly indefinite functional, it is convenient to use the notion of the limit relative category instead of the relative category and the $(P.S.)_c^*$ condition which is a suitable version of the Palais-Smale condition. We restrict the functional I to the manifold C_k with boundary, where C_k is introduced in section 4. We study the geometry and topology of the sub-levels of I and $\tilde{I}_k$ and investigate the limit relative category of the sub-level sets of $\tilde{I}_k$ and $(P.S.)^*_c$ condition in C_k. By Theorem 2.2 and the the critical point theory on the manifold with boundary, we obtain at least two distinct critical points of $\tilde{I}_k$, in each linked subspace X_k, $k = 1, \cdots, m$. So we obtain at least two distinct critical points of I, in each linked subspace X_k, $k = 1, \cdots, m$.

2. CRITICAL POINT THEORY ON THE MANIFOLD WITH BOUNDARY

Now, we consider the critical point theory on the manifold with boundary. Let H be a Hilbert space and M be the closure of an open subset of H such that M can be endowed with the structure of C^2 manifold with boundary. Let $f : W \to R$ be a $C^{1,1}$ functional, where W is an open set containing M. For applying the usual topological methods of critical points theory we need a suitable notion of critical point for f on M. Since the functional $I(u)$ is strongly indefinite, the notion of the $(P.S.)^*_c$ condition and the limit relative category (see [2]) is a useful tool for the proof of the main theorems.

Definition 2.1. If $u \in M$, the lower gradient of f on M at u is defined by

$$grad^-_M f(u) = \begin{cases} \nabla f(u) & \text{if } u \in int(M), \\ \nabla f(u) + [< \nabla f(u), \nu(u) >]^- \nu(u) & \text{if } u \in \partial M, \end{cases} \tag{2.1}$$

where we denote by $\nu(u)$ the unit normal vector to ∂M at the point u, pointing outwards. We say that u is a lower critical for f on M, if $grad^-_M f(u) = 0$.

Let $(H_n)_n$ be a sequence of closed finite dimensional subspace of H with $\dim H_n < +\infty$, $H_n \subset H_{n+1}$, $\cup_{n \in N} H_n$ is dense in H.

Let $M_n = M \cap H_n$, for any n, be the closure of an open subset of H_n and has the structure of a C^2 manifold with boundary in H_n. We assume that for any n there exists a retraction $r_n : M \to M_n$. For given $B \subset H$, we will write $B_n = B \cap H_n$.

Definition 2.2. Let $c \in R$. We say that f satisfies the $(P.S.)^*_c$ condition with respect to $(M_n)_n$, on the manifold with boundary M, if for any sequence $(k_n)_n$ in N and any sequence $(u_n)_n$ in M such that $k_n \to \infty$, $u_n \in M_{k_n}$, $\forall n$, $f(u_n) \to c$, $grad^-_{M_{k_n}} f(u_n) \to 0$, there exists a subsequence of $(u_n)_n$ which converges to a point $u \in M$ such that $grad^-_M f(u) = 0$.

Let Y be a closed subspace of M.

Definition 2.3. Let B be a closed subset of M with $Y \subset B$. We define the relative category $cat_{M,Y}(B)$ of B in (M,Y), as the least integer h such that there exist $h + 1$ closed subsets U_0, $U_1, \ldots, U_h$ with the following properties:

$B \subset U_0 \cup U_1 \cup \ldots \cup U_h$;

$U_1, \ldots, U_h$ are contractible in M;

$Y \subset U_0$ and there exists a continuous map $F : U_0 \times [0, 1] \to M$ such that

$$\begin{aligned} F(x, 0) &= x && \forall x \in U_0, \\ F(x, t) &\in Y && \forall x \in Y,\ \forall t \in [0, 1], \\ F(x, 1) &\in Y && \forall x \in U_0. \end{aligned}$$

If such an h does not exist, we say that $cat_{M,Y}(B) = +\infty$.

Definition 2.4. Let (X,Y) be a topological pair and $(X_n)_n$ be a sequence of subsets of X. For any subset B of X we define the limit relative category of B in (X,Y), with respect to $(X_n)_n$, by

$$cat^*_{(X,Y)}(B) = \lim\sup_{n\to\infty} cat_{(X_n,Y_n)}(B_n).$$

Let Y be a fixed subset of M. We set

$$\mathcal{B}_{\mathrm{i}} = \{\mathrm{B} \subset \mathrm{M} |\ \mathrm{cat}^*_{(\mathrm{M},\mathrm{Y})}(\mathrm{B}) \geq \mathrm{i}\},$$

$$c_i = \inf_{B\in\mathcal{B}_{\mathrm{i}}} \sup_{x\in B} f(x).$$

We have the following multiplicity theorem, which was proved in [6].

Theorem 2.1. *Let $i \in N$ and assume that*
(1) $c_i < +\infty$,
(2) $\sup_{x\in Y} f(x) < c_i$,
(3) *the $(P.S.)^*_{c_i}$ condition with respect to $(M_n)_n$ holds.*
Then there exists a lower critical point x such that $f(x) = c_i$. If

$$c_i = c_{i+1} = \ldots = c_{i+k-1} = c,$$

then

$$cat_M(\{x \in M | f(x) = c,\ grad^-_M f(x) = 0\}) \geq k.$$

Jung and Choi [1] prove the following theorem which will be used to prove the main results:

Theorem 2.2. *(One pair of Torus-Sphere variational link) Let H be a Hilbert space with a norm $\|\cdot\|$, which is topological direct sum of the three subspaces X_0, X_1 and X_2. Let $I \in C^{1,1}(H,R)$ be a strongly indefinite functional. Assume that*

(1) $\dim X_1 < +\infty$;
(2) *There exist a small number $\rho > 0$, $r > 0$ and $R > 0$ such that $r < R$ and*

$$\sup_{\Sigma_R(S_1(\rho),X_0)} I < \inf_{S_r(X_1\oplus X_2)} I,$$

where

$$S_1(\rho) = \{u \in X_1 |\ \|u\| = \rho\},$$
$$S_r(X_1 \oplus X_2) = \{u \in X_1 \oplus X_2 |\ \|u\| = r\},$$
$$B_r(X_1 \oplus X_2) = \{u \in X_1 \oplus X_2 |\ \|u\| \leq r\},$$

$$\begin{aligned}\Sigma_R(S_1(\rho),X_0) &= \{u = u_1 + u_2 |\ u_1 \in S_1(\rho), u_2 \in X_0, \|u_1\| = \rho,\\ &\quad 1 \leq \|u_1 + u_2\| = R\} \cup \{u = u_1 + u_2 |\ u_1 \in S_1(\rho),\\ &\quad \|u_1\| = \rho, 1 \leq \|u_2\| \leq R\},\\ \Delta_R(S_1(\rho),X_0) &= \{u = u_1 + u_2 |\ u_1 \in S_1(\rho), u_2 \in X_0, \|u_1\| = \rho, 1 \leq \|u_1 + u_2\| \leq R\};\end{aligned}$$

(3) $\beta = \sup_{\Delta_R(S_1(\rho),X_0)} I < +\infty$;

(4) $(P.S.)^*_c$ *condition holds for any* $c \in [\alpha, \beta]$ *where*

$$\alpha = \inf_{S_r(X_1 \oplus X_2)} I;$$

(5) *There exists one critical point* e *in* $X_0 \oplus X_2$ *with* $I(e) < \alpha$.
Then there exist at least two distinct critical points except e, u_i, $i = 1, 2$, *in* X_1, *of* I *with*

$$\inf_{S_r(X_1 \oplus X_2)} I \leq I(u_i) \leq \sup_{\Delta_R(S_1(\rho), X_0)} I.$$

3. PROOF OF THEOREM 1.1

We will apply Theorem 2.2 to the case when H is the topological direct sum of $X_0 \oplus X_1$, X_2 and X_3 and to the case when H is the topological direct sum of X_0, X_1 and $X_2 \oplus X_3$. By the conditions (1), (2), (3), (4), we have that

$$\begin{aligned} \alpha^{(1)} &= \inf_{S_{r^{(2)}}(X_2 \oplus \cdots \oplus X_{m+1})} I \leq \sup_{\Delta_{R^{(2)}}(S_2(\rho), X_0 \oplus X_1)} I \leq \sup_{\Sigma_{R^{(1)}}(S_1(\rho), X_0)} I \\ &< \inf_{S_{r^{(1)}}(X_1 \oplus \cdots \oplus X_{m+1})} I \leq \sup_{\Delta_{R^{(1)}(S_1(\rho), X_0)}} I. \end{aligned} \tag{3.1}$$

The condition (6) implies that I satisfies $(P.S.)^*_c$ condition for any c with

$$\inf_{S_{r^{(1)}}(X_1 \oplus \cdots \oplus X_{m+1})} I \leq c \leq \sup_{\Delta_{R^{(1)}}(S_1(\rho), X_0)} I. \tag{3.2}$$

and I also satisfies $(P.S.)^*_\gamma$ condition for any γ with

$$\inf_{S_{r^{(2)}}(X_2 \oplus \cdots \oplus X_{m+1})} I \leq \gamma \leq \sup_{\Delta_{R^{(2)}}(S_2(\rho), X_0 \oplus X_1)} I \tag{3.3}$$

By the condition (5),

$$\sup_{\Delta_{R^{(1)}}(S_1(\rho), X_0)} I = \beta < +\infty. \tag{3.4}$$

Now, we apply Theorem 2.2 to the case when H is the topological direct sum of X_0, X_1 and $X_2 \oplus X_3$. In this case we set the smooth manifold

$$C^{(1)} = \{u \in H | \, \|P_{X_1} u\| \geq 1\},$$

$\psi^{(1)} : H \backslash (X_0 \oplus (X_2 \oplus X_3)) \to H$ by

$$\psi^{(1)}(u) = u - \frac{P_{X_1} u}{\|P_{X_1} u\|} = P_{X_0 \oplus (X_2 \oplus X_3)} u + \left(1 - \frac{1}{\|P_{X_1} u\|}\right) P_{X_1} u$$

and $\tilde{I}_1 = I \cdot \psi^{(1)} \in C^{1,1}_{loc}(C^{(1)}, H)$. Then by Theorem 2.2 with the conditions (1), (2), (4), (5), (7) and (3.2), I has at least two critical points u^1_j, $j = 1, 2$, in X_1, except e, with

$$\inf_{S_{r^{(1)}}(X_1 \oplus \cdots \oplus X_{m+1})} I \leq I(u^1_j) \leq \sup_{\Delta_{R^{(1)}}(S_1(\rho), X_0)} I. \tag{3.5}$$

Next we apply Theorem 2.2 once more to the case when H is the topological direct sum of $X_0 \oplus X_1$, X_2 and X_3. In this case we set the smooth manifold

$$C^{(2)} = \{u \in H | \ \|P_{X_2} u\| \geq 1\},$$

$\psi^{(2)} : H \backslash ((X_0 \oplus X_1) \oplus X_3) \to H$ by

$$\psi^{(2)}(u) = u - \frac{P_{X_2} u}{\|P_{X_2} u\|} = P_{(X_0 \oplus X_1) \oplus X_3} u + \left(1 - \frac{1}{\|P_{X_2} u\|}\right) P_{X_2} u$$

and $\tilde{I}_2 = I \cdot \psi^{(2)} \in C^{1,1}_{loc}(C^{(2)}, H)$. Then by Theorem 2.2 with the conditions (1), (3), (7), (3.3) and (3.4), I has at least two critical points , u_j^2, $j = 1, 2$, in X_2, except e, with

$$\inf_{S_{r^{(2)}}(X_2 \oplus \cdots \oplus X_{m+1})} I \leq I(u_j^2) \leq \sup_{\Delta_{R^{(2)}}(S_2(\rho), X_0 \oplus X_1)} I. \tag{3.6}$$

Using the condition (4), we can combine (3.5) with (3.6). Then we have

$$\begin{aligned} \alpha^{(1)} &= \inf_{S_{r^{(2)}}(X_2 \oplus \cdots \oplus X_{m+1})} I \leq I(u_j^2) \leq \sup_{\Delta_{R^{(2)}}(S_2(\rho), X_0 \oplus X_1)} I \leq \sup_{\Sigma_{R^{(1)}}(S_1(\rho), X_0)} I \\ &< \inf_{S_{r^{(1)}}(X_1 \oplus \cdots \oplus X_{m+1})} I \leq I(u_j^1) \leq \sup_{\Delta_{R^{(1)}}(S_1(\rho), X_0)} I = \beta^{(1)}. \end{aligned}$$

Thus I has at least four nontrivial distinct critical points except e. So we prove the theorem.

4. PROOF OF THEOREM 1.2

We will apply Theorem 2.2 m times to the case when H is the topological direct sum of $X_0 \oplus X_1 \oplus \cdots \oplus X_{k-1}$, X_k, $X_{k+1} \oplus \cdots \oplus X_{m+1}$, for each $1 \leq k \leq m$. The conditions (1), (2) and (3) implies that

$$\begin{aligned} \alpha^{(m)} &= \inf_{S_{r^{(m)}}(X_m \oplus X_{m+1})} I \leq \sup_{\Delta_{R^{(m)}}(S_m(\rho), X_0 \oplus \cdots \oplus X_{m-1})} I \\ &\leq \sup_{\Sigma_{R^{(m-1)}}(S_{m-1}(\rho), X_0 \oplus \cdots \oplus X_{m-2})} I < \cdots \\ &< \inf_{S_{r^{(k)}}(X_k \oplus \cdots \oplus X_{m+1})} I \\ &\leq \sup_{\Delta_{R^{(k)}}(S_k(\rho), X_0 \oplus \cdots \oplus X_{k-1})} I \leq \sup_{\Sigma_{R^{(k-1)}}(S_{k-1}(\rho), X_0 \oplus \cdots \oplus X_{k-2})} I \\ &< \inf_{S_{r^{(k-1)}}(X_{k-1} \oplus \cdots \oplus X_{m+1})} I < \cdots \\ &\leq \sup_{\Sigma_{R^{(1)}}(S_1(\rho), X_0)} I < \inf_{S_{r^{(1)}}(X_1 \oplus \cdots \oplus X_{m+1})} I \\ &\leq \sup_{\Delta_{R^{(1)}}(S_1(\rho), X_0)} I = \beta^{(m)}. \end{aligned} \tag{4.1}$$

The condition (5) implies that I satisfies $(P.S.)^*_{c^{(k)}}$ condition for any $c^{(k)}$ with

$$\inf_{S_{r^{(k)}}(X_k\oplus\cdots\oplus X_{m+1})} I \le c^{(k)} \le \sup_{\Delta_{R^{(k)}}(S_k(\rho),X_0\oplus\cdots\oplus X_{k-1})} I, \qquad k=1,\cdots,m. \tag{4.2}$$

By the condition (4),

$$\sup_{\Delta_{R^{(1)}}(S_1(\rho),X_0)} I = \beta^{(m)} < +\infty, \tag{4.3}$$

We apply Theorem 2.2 to the case when H is the topological direct sum of $X_0 \oplus X_1 \oplus \cdots \oplus X_{k-1}$, X_k, $X_{k+1} \oplus \cdots \oplus X_{m+1}$, $k=1,\cdots,m$. In this case we set

$$C^{(k)} = \{u \in H |\ \|P_{X_k}u\| \ge 1\}, \qquad k=1,\cdots,m.$$

$\psi^{(k)} : H \backslash \{(X_0 \oplus X_1 \oplus \cdots \oplus X_{k-1}) \oplus (X_{k+1} \oplus \cdots \oplus X_{m+1})\} \longrightarrow H$ by

$$\psi^{(k)}(u) = u - \frac{P_{X_k}u}{\|P_{X_k}u\|} = P_{(X_0\oplus\cdots\oplus X_{k-1})}\oplus(X_{k+1}\oplus\cdots\oplus X_{m+1})u + \left(1 - \frac{1}{\|P_{X_k}u\|}\right) P_{X_k}u,$$

$k=1,\cdots,m$, and

$$\tilde{I}_k = I \cdot \psi^{(k)} \in C^{1,1}_{loc}(C^{(k)}, H), \qquad k=1,\cdots,m.$$

Then by Theorem 2.2 with the conditions (1), (2), (3), (5), (6), (4.2) and (4.3), I has at least two critical points u^k_j, $j=1,2$, in X_k, except e $k=1,\cdots,m$ with

$$\begin{aligned}\inf_{S_{r^{(k)}}(X_k\oplus\cdots\oplus X_{m+1})} I \;&\le\; I(u^k_j) \le \sup_{\Delta_{R^{(k)}}(S_k(\rho),X_0\oplus\cdots\oplus X_{k-1})} I \\ &\le\; \sup_{\Sigma_{R^{(k-1)}}(S_{k-1}(\rho),X_0\oplus\cdots\oplus X_{k-2})} I < \inf_{S_{r^{(k-1)}}(X_{k-1}\oplus\cdots\oplus X_{m+1})} I.\end{aligned} \tag{4.4}$$

Using the condition (3), we can combine (4.4) for all $k=1,\cdots,m$. So we have

$$\begin{aligned}\alpha^{(m)} \;&=\; \inf_{S_{r^{(m)}}(X_m\oplus X_{m+1})} I \le I(u^m_j) \le \sup_{\Delta_{R^{(m)}}(S_m(\rho),X_0\oplus\cdots\oplus X_{m-1})} I \\ &\le\; \sup_{\Sigma_{R^{(m-1)}}(S_{m-1}(\rho),X_0\oplus\cdots\oplus X_{m-2})} I < \cdots \\ &<\; \inf_{S_{r^{(k)}}(X_k\oplus\cdots\oplus X_{m+1})} I \le I(u^k_j) \\ &\le\; \sup_{\Delta_{R^{(k)}}(S_k(\rho),X_0\oplus\cdots\oplus X_{k-1})} I \le \sup_{\Sigma_{R^{(k-1)}}(S_{k-1}(\rho),X_0\oplus\cdots\oplus X_{k-2})} I \\ &<\; \inf_{S_{r^{(k-1)}}(X_{k-1}\oplus\cdots\oplus X_{m+1})} I \le I(u^{k-1}_j) \le \cdots \\ &\le\; \sup_{\Sigma_{R^{(1)}}(S_1(\rho),X_0)} I < \inf_{S_{r^{(1)}}(X_1\oplus\cdots\oplus X_{m+1})} I \\ &\le\; I(u^1_j) \le \sup_{\Delta_{R^{(1)}}(S_1(\rho),X_0)} I = \beta^{(m)}.\end{aligned}$$

Thus I has at least $2m$ distinct critical points except e. Thus we prove the theorem.

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