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HYPERGEOMETRIC FUNCTIONS AND EICHLER INTEGRALS

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ABSTRACT. Duke and Imamoḡlu express the Eichler integrals associated to modular forms of weight 3 in terms of generalized hypergeometric functions. We extend this result to most general modular forms of weight 3.

1. Introduction

In the classical theory of modular forms one associates to a cusp form $f(\tau)$ of integral weight $k \geq 2$ its Eichler integral

$$\int_{\tau}^{i\infty} (\tau - \sigma)^{k-2} f(\sigma) d\sigma$$

where the integral is to be taken over the vertical line $\sigma = \tau + i\mathbb{R}^+$ in $\mathcal{H}$.

And a hypergeometric series is a power series in which the ratios of successive coefficients a_n is a rational function of n. This series, if convergent, will define a hypergeometric function. Hypergeometric functions are solutions to the hypergeometric differential equation.

The hypergeometric differential equation is

$$z(1-z)\frac{d^2w}{dz^2} + [c - (a+b+1)z]\frac{dw}{dz} - abw = 0. \tag{1}$$

Then what is the solution of this equation? To answer this question we introduce the ${}_2F_1$, the classical standard hypergeometric series.

The classical standard hypergeometric series is given by

$${}_2F_1(a_1, a_2; b_1|x) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the rising factorial or Pochhammer symbol. Then ${}_2F_1(a, b; c|w)$ is a solution of (1).

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In this paper we will use generalized hypergeometric series defined for $|x| < 1$ by

$$F(x) = F(a_1, \cdots, a_m; b_1, \cdots, b_{m-1}|x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_m)_n}{(b_1)_n \cdots (b_{m-1})_n} \frac{x^n}{n!}, \qquad (2)$$

where no $(b_k)_n = 0$. It is well-known that for any fixed choice of $b \in \{1, b_1, \cdots, b_{m-1}\}$, the function $x^{b-1}F(x)$ satisfies an m-th order hypergeometric equation.

To state the result of Duke and Imamoḡlu we need the Eisenstein series

$$E_4(\tau) = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n \text{ and } E_6(\tau) = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n,$$

and the normalized discriminant function

$$\Delta(\tau) = \frac{1}{1728}(E_4^3(\tau) - E_6^2(\tau)) = q \prod_{n \geq 1} (1 - q^n)^{24},$$

where $\sigma_s(n) = \sum_{d|n} d^s$. Let $j(\tau)$ be the classical modular invariant given by

$$j(\tau) = E_4^3(\tau)/\Delta(\tau)$$

and let $x = 1 - 1728/j$ and $t = 1 - x$. It is a classical fact that a pair of linearly independent solutions to the hypergeometric equation

$$t(1-t)Y'' + (1 - \frac{3}{2}t)Y' - \frac{5}{144}Y = 0$$

is given by

$$F_1(t) = F(\frac{1}{12}, \frac{5}{12}; 1|t) \text{ and } F_2(t) = \tau(t)F_1(t),$$

where $\tau(t)$ is the inverse of $t(\tau)$. Then we have the following result [1]:

Theorem 1.1. *We have*

$$\int_{\tau}^{i\infty} (\tau - \sigma) \frac{\Delta(\sigma)}{E_6(\sigma)^{\frac{3}{2}}} d\sigma = -\frac{4(1-t)^{\frac{1}{4}}}{(12)^3\pi^2} \frac{F(1-t)}{F_1(t)} - \frac{\sqrt{6}i}{(12)^3\pi^2}\tau - \frac{\sqrt{6}log(5 + 2\sqrt{6})}{(12)^3\pi^2}$$

where $F(x) = F(\frac{1}{3}, \frac{2}{3}, 1; \frac{3}{4}, \frac{5}{4}|x)$.

Here the modular form in the integral is of weight 3. We extend this result to most general modular forms of weight 3.

2. **General Formula**

Let $F_1(t) = F(\frac{1}{12}, \frac{5}{12}, 1|t)$ and $p(x) \in \mathbb{C}[x]$ such that $p(0) \neq 0$. Then we get the following general formula:

Theorem 2.1. *We get these relations:*

(1) $\int_\tau^{i\infty}(\tau-\sigma)\frac{\Delta(\sigma)}{E_6(\sigma)^{\frac{3}{2}}}(1-t(\sigma))^{\frac{3}{4}+r}p(1-t(\sigma))d\sigma = \sum_{l=1}^{m-1}\frac{a_l(1-t)^{r+l}}{F_1(t)}$
$$+\frac{a_m(1-t)^{r+m}}{F_1(t)}F(1-t)+a_m\frac{\Gamma(r+m+1)}{\Gamma(r+m+\frac{5}{12})}2\pi i\tau+O(1)$$
for some $a_l, a \in \mathbb{C}$ *where* $m = deg(p)-1, r+m \notin \{-1,-2,\cdots\}\cup\{-\frac{1}{2},-\frac{3}{2},\cdots\}$ *and* $F(t) = F(r+m+\frac{1}{12}, r+m+\frac{5}{12}, 1; r+m+\frac{1}{2}, r+m+1|t)$.

(2) $\int_\tau^{i\infty}(\tau-\sigma)\frac{\Delta(\sigma)}{E_6(\sigma)^{\frac{3}{2}}}(1-t(\sigma))^{\frac{3}{4}}t(\sigma)^r p(t(\sigma))d\sigma = \sum_{l=1}^{m-1}\frac{b_l t^{r+l}}{F_1(t)}+\frac{b_m t^{r+m}}{F_1(t)}G(t)$
for some $b_l \in \mathbb{C}$ *where* $m = deg(p)-1, r+m \notin \{-1,-2,\cdots\}$ *and* $G(t) = F(r+m+\frac{1}{12}, r+m+\frac{5}{12}, 1; r+m+1, r+m+1|t)$.

Remark 2.2. (1) *If* r *and* $p(x)$ *are given then* a, a_l *and* b_l *can be computed.*
(2) *If we put* $r = -\frac{3}{4}$ *and* $p(x) = 1$ *in (1) of Theorem2.1, then we get the result of the Duke and Imamoḡlu.*

3. **Proof Of The Theorem**

Proofs of (1) and (2) of Theorem2.1 are similar. So we will prove only (1). Write $u = t(\tau)$ and let

$$H(u) := 4\pi^2 F_1(u)\int_{i\infty}^{\tau(u)}(\sigma-\tau(u))\frac{1728\Delta(\sigma)}{E_6(\sigma)^{\frac{3}{2}}}(1-t(\sigma))^{\frac{3}{4}+r}p(1-t(\sigma))d\sigma. \tag{3}$$

By changing variables $\sigma \mapsto t = t(\sigma)$ we get

$$H(u) = 2\pi i\int_0^u (F_1(t)F_2(u)-F_1(u)F_2(t))(1-t)^{r-\frac{1}{2}}p(1-t)dt.$$

Now apply the differential operator

$$L_u = u(1-u)\frac{d^2}{du^2}+(1-\frac{3}{2}u)\frac{d}{du}-\frac{5}{144}$$

to this integral to get

$$L_u(H(u)) = (1-u)^r p(1-u).$$

Letting $x = 1-u$ this equation can be written

$$x(1-x)Y''+(\frac{1}{2}-\frac{3}{2}x)Y'-\frac{5}{144}Y = x^r p(x). \tag{4}$$

By using the method of Frobenius we see that

$$\sum_{l=1}^{m-1} a_l x^l + a_m x^{r+m} F(x)$$

is a solution where m is the degree of $p(x)$ and $F(x) = F(r+m+\frac{1}{12}, r+m+\frac{5}{12}, 1; r+m+\frac{1}{2}, r+m+1|x)$. Thus it follows that for some constants a and b we have

$$H(t) = aF_1(t) + bF_2(t) + \sum_{l=1}^{m-1} a_l x^l + a_m x^{r+m} F(x).$$

From (3) we get for some constants $b_l, c, d \in \mathbb{R}$

$$\int_{i\infty}^{\tau} (\tau - \sigma) \frac{\Delta(\sigma)}{E_6(\sigma)^{\frac{3}{2}}} (1 - t(\sigma))^{\frac{3}{4}+r} p(1 - t(\sigma)) d\sigma$$

$$= \sum_{l=1}^{m-1} \frac{b_l}{F_1(t)} (1-t)^l + \frac{b_m}{F_1(t)} (1-t)^{r+m} F(1-t) + c\tau + d.$$

In order to compute the constant c, let $\tau = iy$ and take $y \to \infty$. From the asymptotic formula

$$F(a, b, c; d, e|1-t) = -\frac{\Gamma(d)\Gamma(e)}{\Gamma(a)\Gamma(b)\Gamma(c)} (2\pi i \tau) + O(1), \text{ as } y \to \infty$$

we compute the constant c.

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