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VIDEO TRAFFIC MODELING BASED ON GEO^Y/G/∞ INPUT PROCESSES

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ABSTRACT. With growing applications of wireless video streaming, an efficient video traffic model featuring modern high-compression techniques is more desirable than ever, because the wireless channel bandwidths are ever limited and time-varying. We propose a modeling and analysis method for video traffic by a class of stochastic processes, which we call 'Geo^Y/G/∞ input processes'. We model video traffic by Geo^Y/G/∞ input process with *gamma-distributed* batch sizes Y and *Weibull-like* autocorrelation function. Using four real-encoded, full-length video traces including action movies, a drama, and an animation, we evaluate our modeling performance against existing model, transformed-M/G/∞ input process, which is one of most recently proposed video modeling methods in the literature. Our proposed Geo^Y/G/∞ model is observed to consistently provide conservative performance predictions, in terms of packet loss ratio, within acceptable error at various traffic loads of interest in practical multimedia streaming systems, while the existing transformed-M/G/∞ fails. For real-time implementation of our model, we analyze G/D/1/K queueing systems with Geo^Y/G/∞ input process to upper estimate the packet loss probabilities.

1. INTRODUCTION

With the advance of video compression technologies, next generation ubiquitous networks are expected to accommodate heavily compressed streaming video traffic on wireless and/or wireline channels. An efficient video traffic model featuring modern high compression techniques is more desirable than ever, because the wireless channel bandwidths are ever limited and time-varying. Quality of service management requires sophisticated resource control schemes based on up-to-date video traffic models which are verified with real video traces with the state-of-the-art of video compression techniques such as MPEG4 or H.264 [1].

It has been shown that packetized video traffic streams have both long-range dependence and short-range dependence properties [2, 3, 4, 5]. Therefore, a video modeling method based

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on either of long-range dependence or short-range dependence is not expected to gurantee the modeling performance for full length video streams over practical communication systems. A modeling method, 'transformed-M/G/∞ input process' has recently been proposed [3] with autocorrelation function $\rho(k) = e^{-\beta\sqrt{k}}$ as a compromise between long-range dependence models with $\rho(k) = e^{-\beta \log k}$ and Markovian or short-range dependence models with $\rho(k) = e^{-\beta k}$. As shown in the performance evaluation section of [3], the transformed-M/G/∞ input process model outperforms traditional existing models. From a practical application point of view, however, its drawback is that it is not tractable in queueing analysis due to the *transformation* of the marginal distribution. Therefore, it is not readily applicable as real time algorithms such as real-time call admission control, real-time packet scheduling, and so on. Also, in our extensive simulations, the transformed-M/G/∞ input process model does not estimate exact queueing performance in all situations.

In this paper, we propose a class of stochastic process which we call 'Geo^Y/G/∞ input process' characterizing both long-range and short-range dependences and yet tractable in queueing analyses for real-time applications. In order to model the number of video packets in a fixed-size time slot, we propose Geo^Y/G/∞ input process with *gamma-distributed* batch sizes Y and *Weibull-like* autocorrelation function. The model parameters are determined by matching the marginal statistics and autocorrelation of video frame size to the corresponding statistics of the model. A desirable video traffic model is expected to be mathematically tractable so that it can be real-time implementable as a practical packet scheduler. In this context, we analyze a discrete-time single server, infinite-buffer queue with Geo^Y/G/∞ input process by large deviation theory. We obtain an upper bound of loss ratio with a form simple enough to be real-time implementable. The derived upper bound of loss ratio helps to guarantee quality of service because it gives conservative packet loss levels in real systems such as routers, scheduling servers, etc.

The rest of this paper is organized as follows. In Section II we introduce our proposed Geo^Y/G/∞ input processes. In Section III we present the procedure for parameter matching between real video traces and Geo^Y/G/∞ input processes in terms of autocorrelation function and marginal distribution. Discrete-time single server queue with Geo^Y/G/∞ input processes is analyzed in Section IV. Finally, a conclusion is drawn in Section V.

2. GEO^Y/G/∞ INPUT PROCESSES

In this section, we introduce our proposed stochastic process, which we call 'Geo^Y/G/∞ input process.' Consider a discrete-time queueing system where b_{t+1} denotes the number of arrivals at the start of time slot $[t, t+1)$, $t = \cdots, -1, 0, 1, \cdots$. Let us now call the arrival process $\{b_t\}$ a 'Geo^Y/G/∞ input process' when it is given as the busy server process of a discrete-time Geo^Y/G/∞ system, as an extension of M/G/∞ input process [3, 7, 8].

For a Geo^Y/G/∞ system, the arrival is according to a batch Bernoulli process where the batch size $Y_t \in \{0, 1, \cdots\}$ in time slot t forms an i.i.d. random process $\{Y_t;\ t = \cdots, -1, 0, 1, \cdots\}$. During t-th time slot, $Y_t = y_t$ new customers arrive into the Geo^Y/G/∞ system; then, customer

i ($i = 1, \ldots, y_t$) is presented to its own server who begins service from the next slot with service duration $\sigma_{t,i} \in \{1, 2, \cdots\}$ time slots. We take $\{\sigma_{t,i}\}$ to be i.i.d. rv's.

We have the number of busy servers (or, equivalently, number of customers) b_t at time slot t, which is given by

$$b_t = \sum_{s=-\infty}^{t} \sum_{i=1}^{Y_s} \mathbf{1}[\sigma_{s,i} > t - s], \tag{1}$$

where $\mathbf{1}[\cdot]$ is the indicator function.

The correlated process $\{b_t;\ t = \cdots, -1, 0, 1, \cdots\}$ is easily shown to be stationary and ergodic. To characterize the marginal statistics, the probability generating function (PGF) of b_t is represented as

$$\sum_{i=0}^{\infty} z^i \mathrm{P}[b_t = i] = \prod_{k=0}^{\infty} X\left(a_k z + (1 - a_k)\right) \tag{2}$$

where $a_k \equiv \mathrm{P}[\sigma > k]$ and $X(z)$ is the PGF of batch size, such that

$$X(z) \equiv \sum_{i=0}^{\infty} z^i \mathrm{P}[Y = i].$$

From (2), the mean and variance of b_t are given by, respectively,

$$\mathrm{E}[b] = \mathrm{E}[Y]\mathrm{E}[\sigma] \tag{3}$$

and

$$\mathrm{Var}[b] = \mathrm{E}[b] + (\mathrm{Var}[Y] - \mathrm{E}[Y]) \sum_{i=0}^{\infty} \mathrm{P}[\sigma > i]^2. \tag{4}$$

The covariance structure of $\{b_t; t = \cdots, -1, 0, 1, \cdots\}$ is given by

$$\begin{aligned} \Gamma(k) &\equiv \mathrm{cov}\,[b_t, b_{t+k}] \\ &= \mathrm{E}[Y] \sum_{i=k}^{\infty} \mathrm{P}[\sigma > i] + (\mathrm{Var}[Y] - \mathrm{E}[Y]) \sum_{i=0}^{\infty} \mathrm{P}[\sigma > i]\mathrm{P}[\sigma > i + k], \end{aligned} \tag{5}$$

where $k = 0, 1, \cdots$, and the autocorrelation function (ACF) is defined as

$$\rho(k) \equiv \frac{\Gamma(k)}{\Gamma(0)} = \frac{\Gamma(k)}{\mathrm{Var}[b]}. \tag{6}$$

We also define the forward recurrence time, $\hat{\sigma}$, associated with the service time σ such that

$$\mathrm{P}[\hat{\sigma} = r] \equiv \frac{\mathrm{P}[\sigma \geq r]}{\mathrm{E}[\sigma]},\ r = 1, 2, \cdots,$$

and define

$$v_t \equiv -\ln \mathrm{P}[\hat{\sigma} > t]. \tag{7}$$

TABLE 1. Summary of the MPEG4-encoded VBR traces used in the study (QCIF, Qp=5)

Movie	Length (time)	Number of frames	Mean frame size [packets]	Variance of frame size [packets^2]
Terminator 2	2:16:02	195,909	134.84	3829.4
The Fifth Element	2:05:49	181,195	157.10	4529.1
The English Patient	2:41:18	199,998	91.0	2238.9
Shrek	1:29:58	129,566	128.65	4356.5

3. VIDEO TRAFFIC MODELING BY GEO^Y/G/∞ INPUT PROCESS WITH GAMMA-DISTRIBUTED BATCHES Y

In this section, we propose a video traffic modeling scheme with Geo^Y/G/∞ input process with gamma-distributed batches, Y. In our study, we examined real video traces including action movies, a drama, and an animation as in Table 1. Each video is full-length in quarter common intermediate format (QCIF) format at 24 frames/sec, that is most widely-used format in wireless video transmission. We use public domain Microsoft MPEG4 Visual Reference Software version 2 FDMA1-2.3-001213, to encode/decode video data. In our experiments, we encode each frame as an I frame so that we obtain as many samples as possible to get exact statistics from a real video trace. The quantization step (Qp) is set to be 5, which is correspond to about 200 Kbps video traffic that is suitable for current 3-rd generation (3G) wireless channels. In the table, the mean and standard deviation of the frame size is represented in [packets] where the packet size is 24 Bytes as is the 192-bit payload size in General Packet Radio Service (GPRS) [6].

Our modeling with Geo^Y/G/∞ input process is by matching the moments and autocorrelation function of $\{b_t\}$ with those of the sequence of frame size. We note that our proposed model can be applied to modern Group of Picture (GOP) based video compression with I, P and B frames by taking the modeling unit as a GOP size instead of a frame size.

From Table 1, we match the mean and variance of frame size with E$[b]$ and Var$[b]$, respectively. We then match the autocorrelation function as Weibull-like characteristics $\rho(k) = e^{-\beta k^{\alpha}}(0 < \alpha < 1,\ \beta > 0)$ in terms of least mean squared error. The fitting parameters are shown in Table 2 for our proposed method together with $\rho(k) = e^{-\beta_M \sqrt{k}}$ from transformed-M/G/∞ input process, or 't-M/G/∞,' [3].

We show the curve fitting results for autocorrelation function (ACF) with *The Fifth Element* in Fig. 1. As expected, the curve $\rho(k) = e^{-0.078k^{0.42}}$ from Geo^Y/G/∞ better fits to the real trace than $\rho(k) = e^{-0.043\sqrt{k}}$ from t-M/G/∞, in terms of mean squared error.

Given measured E$[b]$, Var$[b]$, α, and β, we now determine the distribution of the service time, σ, for Geo^Y/G/∞ input process. Let us define the distribution of σ

$$C(k) \equiv \mathrm{P}[\sigma > k]. \tag{8}$$

TABLE 2. Parameter estimation for autocorrelation function (ACF)

Movie	Geo^Y/G/∞		t-M/G/∞ [2]
	α	β	β_M
Terminator 2	0.48	0.065	0.055
The Fifth Element	0.42	0.078	0.043
The English Patient	0.55	0.040	0.056
Shrek	0.61	0.013	0.030

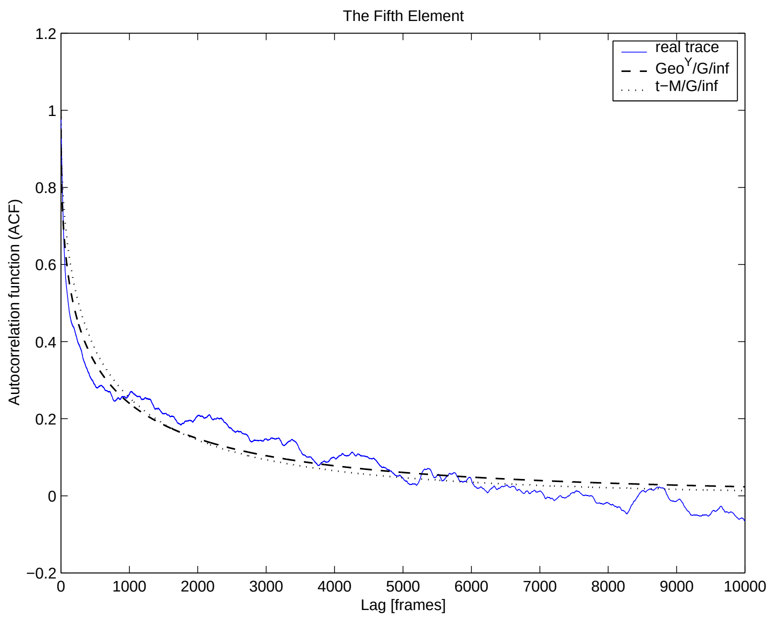

FIGURE 1. Autocorrelation function fitting for movie *The Fifth Element*

Manipulate (5) and (6) to obtain

$$(\mathrm{E}[Y] + \mathrm{Var}[Y]) \sum_{k=n}^{\infty} C(k)$$
$$= \mathrm{Var}[b] e^{-\beta n^{\alpha}} + \sum_{k=0}^{\infty} C(n+k) \left\{ \mathrm{Var}[Y] \left(1 - C(k)\right) + \mathrm{E}[Y] C(k) \right\}. \tag{9}$$

We calculate the distribution of σ, $C(n)$, by successive iteration according to the following steps:

- Step 1: (Initialization) For $n = 0, 1, \cdots$, calculate

$$C_{old}(n) \Leftarrow \frac{e^{-\beta n^{\alpha}} - e^{-\beta (n+1)^{\alpha}}}{1 - e^{-\beta}} \tag{10}$$

Here the left arrow '$\Leftarrow$' means assignment of the value on right hand side the term on left hand side.

- Step 2: Calculate

$$\begin{aligned} \mathrm{E}[Y] &\Leftarrow \frac{\mathrm{E}[b]}{\sum_{k=0}^{\infty} C_{old}(k)} \\ \mathrm{Var}[Y] &\Leftarrow \frac{\mathrm{Var}[b] - \mathrm{E}[b]}{\sum_{k=0}^{\infty} C_{old}(k)^2} + \mathrm{E}[Y] \end{aligned}$$

- Step 3: For $n = 0, 1, \cdots$, calculate

$$C_{new}(n) \Leftarrow \frac{\mathrm{Var}[b]\left(e^{-\beta n^{\alpha}} - e^{-\beta (n+1)^{\alpha}}\right) + A(n) - A(n+1)}{\mathrm{E}[Y] + \mathrm{Var}[Y]} \tag{11}$$

where

$$A(n) \equiv \sum_{k=0}^{\infty} C_{old}(n+k) \left\{\mathrm{Var}[Y]\left(1 - C_{old}(k)\right) + \mathrm{E}[Y] C_{old}(k)\right\}.$$

- Step 4: Assure that $C_{new}(0) = 1$.
- Step 5: If $C_{old}(n)$ and $C_{new}(n)$ are close to each other within a criterion, then $C(n) \Leftarrow C_{old}(n)$ and stop. Otherwise, update $C_{old}(n) \Leftarrow C_{new}(n)$ for $n = 0, 1, \cdots$ and go to Step 2.

All that is left now is to find the distribution of the batch size Y. From extensive studies with various distribution functions, we observe best modeling performance with gamma-distributed Y for our proposed Geo^Y/G/∞ input process model. Note that we have already obtained E$[Y]$ and Var$[Y]$ when calculating the distribution of σ in the above successive iteration. We propose that

$$\mathrm{P}[Y \leq y] = \frac{\gamma(\eta, (y+1)/\theta)}{\Gamma(\eta)}, (y = 0, 1, \cdots) \tag{12}$$

where the scale parameter θ and shape parameter η are given by

$$\theta = \frac{\mathrm{Var}[Y]}{\mathrm{E}[Y]}, \tag{13}$$

$$\eta = \frac{\mathrm{E}[Y]}{\theta}. \tag{14}$$

Here $\gamma(a, z)$ is a lower incomplete gamma function and $\Gamma(a)$ is a complete gamma function.

In summary, we determine model parameters Geo^Y/G/∞ input process with gamma-distributed batch according to the following steps:

(i) Match $\rho(k)$ with real trace in terms of minimum mean squared error to obtain α and β.

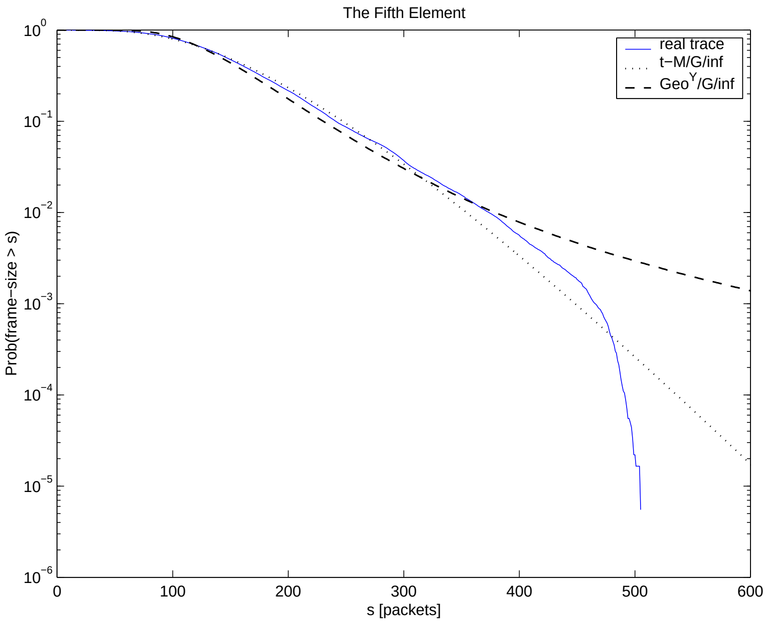

FIGURE 2. Complementary frame-size distribution for *The Fifth Element*

(ii) Measure the mean and variance of frame size to be matched with E$[b]$ and Var$[b]$, respectively.
(iii) Calculate the distribution of σ, $C(k)$, together with E$[Y]$ and Var$[Y]$.
(iv) Finally, calculate the parameters θ and η for gamma distribution for Y.

The marginal distribution of the frame size is illustrated in Fig. 2 where the complementary frame-size distribution is shown for *The Fifth Element*. With QCIF-encoded video traces, *The Fifth Element*, the tail of the distribution drops rapidly yielding the maximum frame size, $f_{\max} = 506$ [packets], rather than Pareto tail [3]. This is expected because QCIF-encoded video frames are highly compressed, which are commonly used in transmission on limited-bandwidth wireless channel. In Fig. 2, dashed line is from our proposed Geo^Y/G/∞ input process. It is observed that Geo^Y/G/∞ input process gives a more slowly decaying tail than the real trace. We only use gamma transformation for t-M/G/∞ input process due to rapidly-dropping tail. It is noted that for both models, the fitting is based on matching the mean and variance between models and real traces.

Finally, we evaluate modeling performance in terms of packet loss ratio (PLR). A summary of the simulation results to two significant digits is given in Table 3 for each of the four video traces at various offered traffic loads: heavy load ($U = 0.7$), moderate load ($U = 0.5$), and light load ($U = 0.3$). With *The Fifth Element*, we use $U = 0.4$ for light load, because no packet losses are observed with the real trace for $U = 0.3$. For the modeling results, we generate synthesis traces and apply them into a G/D/1/K queueing system to estimate packet loss probabilities. We conduct the discrete-event simulation at the frame level.

The buffer size is varied from 10 to 3000. It is observed that increasing buffer barely provides improvement in performance, which is remarkable especially for higher load. In contrast, reducing the load (i.e., increasing bandwidth) improves the packet loss ratio significantly.

In the heavy load regime, both t-M/G/∞ and Geo^Y/G/∞ models provide acceptable predictions of PLR, with Geo^Y/G/∞ being more accurate. In the moderate regime, Geo^Y/G/∞ model yields more accurate estimations than t-M/G/∞ model which underestimates PLR by about 50buffer size of 1000 [packets]. In the light load regime, Geo^Y/G/∞ model overestimates PLR, while t-M/G/∞ model underestimates it and is overly sensitive to the buffer size as shown with *Terminator 2*. Also, it is desirable for a model to give overestimated, i.e., conservative PLR rather than underestimated PLR, because implementing real systems based on underestimated PLR may not guarantee quality of service. Overall, Geo^Y/G/∞ model is observed to consistently provide conservative performance predictions within acceptable error at various traffic loads. It is difficult to analyze queueing systems with t-M/G/∞ input processes. Geo^Y/G/∞ model is mathematically tractable as shown in the next section

4. Analysis of Queueing Systems with Geo^Y/G/∞ Input Processes

It is desirable to show that our proposed Geo^Y/G/∞ Input Processes is applicable in real-time implementation is practical video streaming stream on the communication systems. We now consider a discrete-time single-server queue with infinite buffer and constant release rate of c packets/slot under first-come first-served discipline, of which the arrival is Geo^Y/G/∞ input process. The goal of this section is to obtain an upper bound of the packet loss ratio in a form simple enough to be real-time implementable.

Let q_t denote the number of cells remaining in the buffer by the end of slot $[t-1,t)$, and let b_{t+1} denote the number of new cells which arrive at the start of time slot $[t,t+1)$. Then the buffer content sequence $\{q_t, t=\cdots,-1,0,1,\cdots\}$ evolves according to the Lindley recursion

$$q_{t+1} = [q_t + b_{t+1} - c]^+,\ t=\cdots,-1,0,1,\cdots \tag{15}$$

for some initial condition q.

$\{q_t; \cdots,-1,0,1,\cdots\}$ is uniquely determined by equation (15) if $\mathrm{E}[Y]\mathrm{E}[\sigma] < c$, and it is stationary. The queueing system will reach statistical equilibrium if $\mathrm{E}[Y]\mathrm{E}[\sigma] < c$. The stationary and ergodic input process $\{b_t;\ t=0,1,\cdots\}$ is reversible sequence and the steady state buffer content is given by

$$q_0 =_{st} \sup\{\tilde{S}_t - ct,\ t=0,1,\cdots\} \tag{16}$$

with

$$\tilde{S}_0 = 0;\quad \tilde{S}_t = b_{-1} + b_{-2} + \cdots + b_{-t},\ t=1,2,\cdots. \tag{17}$$

We note that, for each $t=0,1,\cdots$,

$$b_t = b_t^{(0)} + b_t^{(a)}, \tag{18}$$

TABLE 3. Modeling Performance : Mean packet loss ratio at three different loads, U

Buffer	$U = 0.3$			$U = 0.5$			$U = 0.7$		
size [pkts]	real	t-M/G/∞	Geo^Y/G/∞	real	t-M/G/∞	Geo^Y/G/∞	real	t-M/G/∞	Geo^Y/G/∞
10	3.6E-4	3.1E-5	3.1E-3	1.5E-2	8.3E-3	1.8E-2	5.6E-2	5.2E-2	5.7E-2
50	3.5E-4	2.0E-5	2.8E-3	1.5E-2	7.4E-3	1.8E-2	5.5E-2	4.9E-2	5.7E-2
100	3.5E-4	1.4E-5	2.6E-3	1.5E-2	6.9E-3	1.7E-2	5.4E-2	4.7E-2	5.6E-2
500	2.9E-4	2.8E-6	1.7E-3	1.4E-2	5.4E-3	1.4E-2	5.0E-2	4.2E-2	5.1E-2
1000	2.1E-4	3.5E-7	1.3E-3	1.3E-2	4.5E-3	1.3E-2	4.7E-2	3.8E-2	4.8E-2
2000	1.2E-4	0	8.2E-4	1.2E-2	3.5E-3	1.0E-2	4.2E-2	3.4E-2	4.4E-2
3000	8.5E-5	0	6.0E-4	1.2E-2	2.9E-3	8.8E-3	3.9E-2	3.1E-2	4.0E-2

(a) Terminator 2

Buffer	$U = 0.4(*)$			$U = 0.5$			$U = 0.7$		
size [pkts]	real	t-M/G/∞	Geo^Y/G/∞	real	t-M/G/∞	Geo^Y/G/∞	real	t-M/G/∞	Geo^Y/G/∞
10	1.6E-3	7.4E-4	6.7E-3	9.1E-3	5.2E-3	1.4E-2	4.9E-2	4.3E-2	4.8E-2
50	1.5E-3	6.5E-4	6.4E-3	9.0E-3	4.7E-3	1.4E-2	4.8E-2	4.1E-2	4.8E-2
100	1.5E-3	6.0E-4	6.0E-3	8.9E-3	4.4E-3	1.3E-2	4.8E-2	4.0E-2	4.7E-2
500	1.3E-3	4.7E-4	4.6E-3	8.2E-3	3.5E-3	1.1E-2	4.4E-2	3.6E-2	4.4E-2
1000	1.2E-3	4.0E-4	3.7E-3	7.6E-3	3.0E-3	9.9E-3	4.1E-2	3.3E-2	4.1E-2
1500	1.1E-3	3.5E-4	3.2E-3	7.1E-3	2.7E-3	8.9E-3	3.9E-2	3.1E-2	3.9E-2
2000	1.1E-3	3.2E-4	2.8E-3	6.7E-3	2.4E-3	8.1E-3	3.7E-2	3.0E-2	3.7E-2
3000	1.0E-3	2.8E-4	2.2E-3	6.1E-3	2.1E-3	6.9E-3	3.4E-2	2.7E-2	3.5E-2

(b) The Fifth Element

Buffer	$U = 0.3$			$U = 0.5$			$U = 0.7$		
size [pkts]	real	t-M/G/∞	Geo^Y/G/∞	real	t-M/G/∞	Geo^Y/G/∞	real	t-M/G/∞	Geo^Y/G/∞
10	5.5E-4	2.8E-4	3.7E-3	2.2E-2	1.5E-2	2.6E-2	8.0E-2	6.8E-2	7.5E-2
50	5.4E-4	2.2E-4	3.5E-3	2.2E-2	1.3E-2	2.5E-2	7.9E-2	6.4E-2	7.4E-2
100	5.3E-4	1.9E-4	3.2E-3	2.2E-2	1.2E-2	2.4E-2	7.8E-2	6.2E-2	7.3E-2
500	4.1E-4	1.1E-4	2.3E-3	2.0E-2	9.6E-3	2.1E-2	7.4E-2	5.4E-2	6.8E-2
1000	3.1E-4	7.6E-5	1.7E-3	1.8E-2	8.2E-3	1.9E-2	7.0E-2	4.9E-2	6.4E-2
1500	2.3E-4	5.4E-5	1.4E-3	1.7E-2	7.3E-3	1.7E-2	6.6E-2	4.5E-2	6.1E-2
2000	1.8E-4	3.9E-5	1.1E-3	1.6E-2	6.6E-3	1.6E-2	6.3E-2	4.3E-2	5.8E-2
3000	8.6E-5	1.9E-5	7.9E-4	1.4E-2	5.6E-3	1.4E-2	5.9E-2	3.8E-2	5.4E-2

(c) The English Patient

Buffer	$U = 0.3$			$U = 0.5$			$U = 0.7$		
size [pkts]	real	t-M/G/∞	Geo^Y/G/∞	real	t-M/G/∞	Geo^Y/G/∞	real	t-M/G/∞	Geo^Y/G/∞
10	5.3E-4	2.8E-4	2.2E-3	2.0E-2	1.5E-2	2.2E-2	7.4E-2	6.7E-2	7.2E-2
50	5.2E-4	2.5E-4	2.2E-3	2.0E-2	1.4E-2	2.2E-2	7.3E-2	6.5E-2	7.2E-2
100	5.1E-4	2.3E-4	2.1E-3	2.0E-2	1.4E-2	2.2E-2	7.3E-2	6.4E-2	7.2E-2
500	4.4E-4	1.7E-4	1.8E-3	1.9E-2	1.2E-2	2.1E-2	7.0E-2	6.0E-2	7.0E-2
1000	3.8E-4	1.4E-4	1.5E-3	1.8E-2	1.1E-2	2.0E-2	6.8E-2	5.7E-2	6.9E-2
1500	3.5E-4	1.1E-4	1.4E-3	1.7E-2	1.0E-2	1.9E-2	6.6E-2	5.5E-2	6.7E-2
2000	3.2E-4	9.6E-5	1.3E-3	1.7E-2	1.0E-2	1.9E-2	6.4E-2	5.3E-2	6.6E-2
3000	2.6E-4	7.0E-5	1.1E-3	1.6E-2	9.2E-3	1.8E-2	6.2E-2	5.1E-2	6.5E-2

(d) Shrek

where the rv's $b_t^{(0)}$ and $b_t^{(a)}$ describe the contributions to the number of customers in the system at the beginning of time slot $[t, t+1)$ from those present at $t = 0$ and from the new arrivals, respectively. We have

$$b_t^{(a)} = \sum_{s=1}^{t} \sum_{i=1}^{Y_s} \mathbf{1}[\sigma_{s,i} > t - s] \tag{19}$$

and

$$b_t^{(0)} = \sum_{s=-\infty}^{0} \sum_{i=1}^{Y_s} \mathbf{1}[\sigma_{s,i} > t - s]. \tag{20}$$

Invoking large deviation theory, we define

$$\Lambda_t(\theta) \equiv \frac{1}{v_t} \ln \mathrm{E}\left[e^{\theta_t(\tilde{S}_t - ct)}\right],\ \theta \in \mathrm{R}. \tag{21}$$

where

$$\theta_t \equiv \theta \frac{v_t}{t}.$$

where $v_t = -\ln \mathrm{P}[\hat{\sigma} > t]$ for which it is assumed that $\lim_{t\to\infty} v_t = \infty$. The remaining of this section is for finding the limit

$$\Lambda(\theta) \equiv \lim_{t\to\infty} \Lambda_t(\theta). \tag{22}$$

Note

$$\tilde{S}_t =_{st} S_t(\text{in distribution}),\ t = 0, 1, 2, \cdots,$$

where

$$S_0 = 0, S_t = b_1 + \cdots + b_t,\ t = 1, 2, \cdots,$$

Hense

$$\Lambda_t(\theta) \equiv \frac{1}{v_t} \ln \mathrm{E}\left[e^{\theta_t(S_t - ct)}\right],\ \theta \in \mathrm{R}.$$

Theorem 1. Assume that

(1) $\lim_{t\to\infty} v_t/t = 0$,
(2) $\mathrm{E}[\sigma^2] < \infty$ ($\Leftrightarrow \mathrm{E}[\hat{\sigma}] < \infty \Leftrightarrow \sum_{s=1}^{\infty} \mathrm{E}[(\sigma - s)^+] < \infty$),
(3) $\sum_{t=1}^{\infty} \mathrm{P}[\hat{\sigma} > t]^\alpha < \infty$ for $0 < \alpha < 1$,
(4) $\{v_t/t;\ t = 1, 2, \cdots\}$ is monotone decreasing,
(5) Y has an exponential moments, i.e., $\mathrm{E}[e^{\theta Y}] < \infty$ for $\theta > 0$,
(6) $\mathrm{E}[\hat{\sigma}] < \infty$, and
(7) $\lim_{t\to\infty} \mathrm{P}[\hat{\sigma} > t]/\mathrm{P}[\sigma > t] = \infty$.

Then, for each $\theta \neq 1$ in R, the limit $\Lambda(\theta)$ is given by

$$\Lambda(\theta) = \begin{cases} (\mathrm{E}[Y]\mathrm{E}[\sigma] - c)\theta, & \text{if } \theta < 1 \\ \infty, & \text{if } \theta > 1 \end{cases} \tag{23}$$

Proof of Theorem 1: From Lemma 1 through 9 in the Appendix, we have, for each $\theta \in \mathrm{R}$,

$$\Lambda_b(\theta) \equiv \lim_{t\to\infty} \frac{1}{v_t} \ln \mathrm{E}\left[\exp(\frac{v_t}{t}\theta S_t)\right] = \begin{cases} \mathrm{E}[Y]\mathrm{E}[\sigma]\theta, & \text{if } \theta < 1 \\ \infty, & \text{if } \theta > 1 \end{cases}$$

Since

$$\Lambda(\theta) = \Lambda_b(\theta) - c\theta,$$

(23) holds. □

Our goal is to approximate $\mathrm{P}[q_0 > n]$ in an asymptotic manner. Let

$$\Lambda^*(z) \equiv \sup_{\theta \in R}(\theta z - \Lambda(\theta)), (z \in \mathrm{R}). \tag{24}$$

Invoking [8], we have an upper bound

$$\limsup_{n\to\infty} \frac{1}{\beta n^\alpha} \ln \mathrm{P}[q_0 > n] \le -\gamma^*, \tag{25}$$

where

$$\gamma^* = \sup_{y>0} \min(f(y), g(y)) \tag{26}$$

with the notation

$$f(y) = \sup_{\theta>0} \liminf_{n\to\infty} \left[\inf_{x>y} \left\{ \frac{\beta n^\alpha}{\beta (nx)^\alpha} (\theta x - \Lambda(\theta)) \right\} \right] \tag{27}$$

and

$$g(y) = \Lambda^*(0)/y^\alpha. \tag{28}$$

Let $m = \mathrm{E}[b] = \mathrm{E}[Y]\mathrm{E}[\sigma]$. We have

$$f(y) = \begin{cases} (c-m)^{1-\alpha} \frac{\alpha^{-\alpha}}{(1-\alpha)^{1-\alpha}}, & \text{if } 0 < y \le \frac{\alpha}{1-\alpha}(c-m) \\ y^{1-\alpha} + (c-m) y^{-\alpha}, & \text{if } y > \frac{\alpha}{1-\alpha}(c-m) \end{cases} \tag{29}$$

To compute $\min(f(y), g(y))$, we have
(Case I) if $y < (c-m)\alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}}$, $\min(f(y), g(y)) = (c-m)^{1-\alpha} \frac{\alpha^{-\alpha}}{(1-\alpha)^{1-\alpha}}$,
(Case II) if $y > (c-m)\alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}}$, $\min(f(y), g(y)) = y^{-\alpha}(c-m)$.
Consequently, we obtain

$$\gamma^* = (c-m)^{1-\alpha} \frac{\alpha^{-\alpha}}{(1-\alpha)^{1-\alpha}}. \tag{30}$$

On the other hand, we have a lower bound

$$-\gamma_* \le \liminf_{n\to\infty} \frac{1}{\beta n^\alpha} \ln \mathrm{P}[q_\infty > n], \tag{31}$$

where $\gamma_* = \inf_{y>0} \Lambda^*(y)/y^\alpha$. It is calculated as

$$\begin{aligned} \gamma_* &= \inf_{y>0} \left[\frac{1}{y^\alpha} \cdot \sup_{\theta\in R} \{\theta y - \Lambda(\theta)\} \right] \\ &= \inf_{y>0} \left\{ \frac{1}{y^\alpha}(y + c - m) \right\} \\ &= (c-m)^{1-\alpha} \frac{\alpha^{-\alpha}}{(1-\alpha)^{1-\alpha}}. \end{aligned} \tag{32}$$

Finally, we have

$$\gamma(c) \equiv \gamma^* = \gamma_* \tag{33}$$

as a function of output link rate, and, from (25) and (31),

$$\lim_{n\to\infty} \frac{1}{\beta n^\alpha} \ln \mathrm{P}[q_\infty > n] = -\gamma(c). \tag{34}$$

The above equation can be utilized as an overestimation of packet loss ratio for a queueing system with Geo^Y/G/∞ input processes, when we implement video transmission systems. An overestimation of PLR gives a conservative result which guarantees given quality of service levels in terms of packet loss ratio.

5. Conclusion

We have proposed and analyzed a Geo^Y/G/∞ input process to model video traffic. Our proposed model effectively characterize the video traffic in the aspect of marginal distribution and autocorrelation and yet tractable in queueing analysis. With four full-length video traces including action movies, a drama, and an animation, we evaluate the modeling performance in terms of packet loss ratio. Geo^Y/G/∞ model is observed to consistently provide conservative performance predictions within acceptable error at various traffic loads. For real-time implementation of our model, we analyze G/D/1/K queueing systems with Geo^Y/G/∞ input process to upper estimate the packet loss probabilities.

References

[1] F. Fitzek and M. Reisslein, "MPEG-4 and H.263 video traces for network performance evaluation," *IEEE Network,* pp. 40 – 54, Nov./Dec. 2001. Traces available at http://www-tkn.ee.tu-berlin.de/research/trace/trace.html and http://trace.eas.asu.edu

[2] J. Beran, R. Sherman, M. S. Taqqu, and W. Willinger, "Long-range dependence in variable-bit-rate video traffic," *IEEE Trans. on Commun.,* vol.43, no.2/3/4, pp. 1566 – 1579, Feb/Mar/Apr. 1995.

[3] M. M. Krunz and A. M. Makowski, "Modeling video traffic using M/G/∞ input processes: a compromise between Markovian and LRD Models," *IEEE Journal on Selected Areas in Communications,* vol. 16, no. 5, pp. 733 – 748, June 1998.

[4] K. Nagarajan and G. T. Zhou, "Self-similar traffic sources: modeling and real-time resources allocation," *Statistical Sig. Processing 2001,* pp. 74 – 76, 2001.

[5] K. Nagarajan and G. T. Zhou, "A new resource allocation scheme for VBR video traffic source," *Proc. 34th Asilomar Conference on Signals, Systems, and Computers,* pp. 1245 – 1249, Pacific Grove, CA, Oct. 2000.

[6] ETSI, "GSM 03.64 Overall description of GPRS radio interface, Stage 2," vol. 2.1.1, May 1997.

[7] M. Parulekar and A. M. Makowski, "Tail probabilities for M/G/∞ input process (I): Preliminary asymptotics," Queueing Systems, vol. 27, pp. 271 – 296, 1997.

[8] M. Parulekar and A. M. Makowski, "M/G/∞ input processes: A versatile class of models for network traffic," *Proc. IEEE INFOCOM '97,* vol.2, pp. 419 – 426, April 1997.

Appendix A. APPENDIX

Lemma 1. For $t = 1, 2, \cdots,$
(i) $S_t = S_t^{(0)} + S_t^{(a)}$ where $S_t^{(0)} \equiv \sum_{i=1}^{t} b_i^{(0)}$ and $S_t^{(a)} \equiv \sum_{i=1}^{t} b_i^{(a)}$
(ii)

$$\ln \mathrm{E}\left[e^{\theta S_t^{(0)}}\right] = \sum_{s=1}^{\infty} \ln \mathrm{E}\left[\mathrm{E}\left[e^{\theta((\sigma-s)^+ \wedge t)}\right]^Y\right] \tag{35}$$

(iii)

$$\ln \mathrm{E}\left[e^{\theta S_t^{(a)}}\right] = \sum_{s=1}^{t} \ln \mathrm{E}\left[\mathrm{E}\left[e^{\theta(\sigma\wedge s)}\right]^{Y}\right]. \tag{36}$$

where $(x \wedge y) = \min(x, y)$.
Proof of Lemma 1: (i) is trivial. For (iii), refer to [7]. (ii) is shown as follows:

$$\begin{aligned}
S_t^{(0)} &= \sum_{r=1}^{t}\sum_{s=-\infty}^{0}\sum_{i=1}^{Y_s} \mathbf{1}[\sigma_{s,i} > r-s] \\
&= \sum_{s=-\infty}^{0}\sum_{i=1}^{Y_s}\sum_{r=1}^{t} \mathbf{1}[\sigma_{s,i} > r-s] \\
&= \sum_{s=-\infty}^{0}\sum_{i=1}^{Y_s}\sum_{l=1-s}^{t-s} \mathbf{1}[\sigma_{s,i} > l] \\
&= \sum_{s=-\infty}^{0}\sum_{i=1}^{Y_s} \min\{(\sigma+s-1)^+, t\} \\
&= \sum_{m=1}^{\infty}\sum_{i=1}^{Y_m} \left((\sigma-m)^+ \wedge t\right). \quad \square
\end{aligned}$$

Lemma 2. If $\theta \le 0$,

$$\lim_{t\to\infty} \frac{1}{v_t} \ln \mathrm{E}[e^{\theta_t S_t^{(a)}}] = \mathrm{E}[Y]\mathrm{E}[\sigma]\theta. \tag{37}$$

Proof of Lemma 2: For each $t = 1, 2, \cdots$, we have $\theta_t \le 0$ and

$$\begin{aligned}
\frac{1}{v_t} \ln \mathrm{E}[e^{\theta_t S_t^{(a)}}] &= \frac{1}{v_t}\sum_{s=1}^{t} \ln \mathrm{E}\left[\mathrm{E}\left[e^{\theta_t(\sigma\wedge s)}\right]^{Y}\right] \\
&\ge \frac{t}{v_t} \ln \mathrm{E}\left[\mathrm{E}\left[e^{\theta_t \sigma}\right]^{Y}\right] \\
&= \theta \cdot \frac{\ln \mathrm{E}[\mathrm{E}[e^{\theta_t\sigma}]^Y]}{\theta_t}.
\end{aligned}$$

For a nonnegative rv X, it is known that $\lim_{\theta\uparrow 0} \ln \mathrm{E}[e^{\theta X}]/\theta = \mathrm{E}[X]$. Thus we have

$$\begin{aligned}
\liminf_{t\to\infty} \frac{1}{v_t} \ln \mathrm{E}\left[e^{\theta_t S_t^{(a)}}\right] &\ge \theta \mathrm{E}[\sigma_1 + \cdots + \sigma_Y] \\
&= \theta \mathrm{E}[Y]\mathrm{E}[\sigma].
\end{aligned}$$

Let M be a positive integer. For $t > M$,

$$
\begin{aligned}
\frac{1}{v_t} \ln \mathrm{E}[e^{\theta_t S_t^{(a)}}] &= \frac{1}{v_t} \sum_{s=1}^{t} \ln \mathrm{E}\left[\mathrm{E}\left[e^{\theta_t(\sigma \wedge s)}\right]^Y\right] \\
&\leq \frac{1}{v_t} \sum_{s=M}^{t} \ln \mathrm{E}\left[\mathrm{E}\left[e^{\theta_t(\sigma \wedge M)}\right]^Y\right] \\
&= \frac{t-M+1}{v_t} \ln \mathrm{E}\left[\mathrm{E}\left[e^{\theta_t(\sigma \wedge M)}\right]^Y\right] \\
&= \frac{t-M+1}{t} \cdot \theta \cdot \frac{\ln \mathrm{E}[\mathrm{E}[e^{\theta_t(\sigma \wedge M)}]^Y]}{\theta_t} \\
&= \frac{t-M+1}{t} \cdot \theta \cdot \frac{\ln \mathrm{E}[e^{\theta_t[(\sigma_1 \wedge M)+\cdots+(\sigma_Y \wedge M)]}]}{\theta_t}
\end{aligned}
$$

Thus we have

$$
\limsup_{t\to\infty} \frac{1}{v_t} \ln \mathrm{E}[e^{\theta_t S_t^{(a)}}] \leq \theta \mathrm{E}[Y]\mathrm{E}[\sigma \wedge M].
$$

Letting $M \to \infty$ leads to

$$
\limsup_{t\to\infty} \frac{1}{v_t} \ln \mathrm{E}[e^{\theta_t S_t^{(a)}}] \leq \theta \mathrm{E}[Y]\mathrm{E}[\sigma]. \quad \square
$$

Lemma 3. If (i) $\theta \leq 0$, (ii) $\lim_{t\to\infty} v_t/t = 0$, and (iii) $\mathrm{E}[\sigma^2] < \infty$, then

$$
\lim_{t\to\infty} \frac{1}{v_t} \ln \mathrm{E}[e^{\theta_t S_t^{(0)}}] = 0. \tag{38}
$$

Proof of Lemma 3: For each $t = 1, 2, \cdots$,

$$\begin{aligned}
-\frac{1}{v_t}\ln \mathrm{E}\left[e^{\theta_t S_t^{(0)}}\right] &= -\frac{1}{v_t}\sum_{s=1}^{\infty}\ln \mathrm{E}\left[\mathrm{E}\left[e^{\theta_t((\sigma-s)^+\wedge t)}\right]^Y\right] \\
&\le -\frac{1}{v_t}\sum_{s=1}^{\infty}\mathrm{E}\left[\ln \mathrm{E}\left[e^{\theta_t((\sigma-s)^+\wedge t)}\right]^Y\right] \\
&= -\frac{1}{v_t}\sum_{s=1}^{\infty}\mathrm{E}[Y]\ln \mathrm{E}\left[e^{\theta_t((\sigma-s)^+\wedge t)}\right] \\
&\le -\frac{1}{v_t}\sum_{s=1}^{\infty}\mathrm{E}[Y]\mathrm{E}\left[\theta_t((\sigma-s)^+\wedge t)\right] \\
&= \frac{\mathrm{E}[Y](-\theta_t)}{v_t}\sum_{s=1}^{\infty}\mathrm{E}\left[(\sigma-s)^+\wedge t\right] \\
&\le \frac{\mathrm{E}[Y](-\theta_t)}{v_t}\sum_{s=1}^{\infty}\mathrm{E}\left[(\sigma-s)^+\right] \\
&\to 0 \text{ (as } t\to\infty). \qquad (39)
\end{aligned}$$

from the fact that $\mathrm{E}[\sigma^2] < \infty \Longleftrightarrow \mathrm{E}\left[(\sigma-s)^+\right] < \infty$. □

Lemma 4. If (i) $0 < \theta < 1$, (ii) $\sum_{t=1}^{\infty}\mathrm{P}[\hat{\sigma} > t]^{1-\theta} < \infty$, and (iii) $\{v_t/t, t = 1, 2, \cdots\}$ is monotone decreasing, then

$$\lim_{t\to\infty}\frac{\mathrm{E}[e^{\theta_t(\sigma\wedge t)}]-1}{\theta_t} = \mathrm{E}[\sigma]. \qquad (40)$$

Proof of Lemma 4: Let $M > 1$. Since $\mathrm{P}[\sigma > x] \ge \mathrm{E}[\sigma]\mathrm{P}[\hat{\sigma} > \lfloor x\rfloor]$, for $x \ge 0$ and t such that $e^{\theta_t} \le M$,

$$\begin{aligned}
\mathrm{P}[\sigma > x]e^{\theta_t x}\mathbf{1}[x < t] &\le \mathrm{E}[\sigma]\mathrm{P}[\hat{\sigma} > \lfloor x\rfloor]e^{\theta_t\lfloor x\rfloor}e^{\theta_t}\mathbf{1}[x<t] \\
&= \mathrm{E}[\sigma]e^{-v_{\lfloor x\rfloor}}e^{\lfloor x\rfloor\theta_t}e^{\theta_t}\mathbf{1}[x<t] \\
&= \mathrm{E}[\sigma]e^{-v_{\lfloor x\rfloor}}e^{\lfloor x\rfloor\theta v_t/t}e^{\theta_t}\mathbf{1}[x<t] \\
&\le \mathrm{E}[\sigma]e^{-v_{\lfloor x\rfloor}}e^{\theta v_{\lfloor x\rfloor}}e^{\theta_t}\mathbf{1}[x<t] \\
&\le \mathrm{E}[\sigma]\left(\mathrm{P}[\hat{\sigma} > \lfloor x\rfloor]\right)^{1-\theta}M.
\end{aligned}$$

Since $\int_0^{\infty}\mathrm{E}[\sigma]\left(\mathrm{P}[\hat{\sigma} > \lfloor x\rfloor]\right)^{1-\theta}M dx < \infty$, by Lebesgue's dominated convergence theorem (LDCT)

$$\begin{aligned}
\lim_{t\to\infty}\int_0^{\infty}\mathrm{P}[\sigma > x]e^{\theta_t x}\mathbf{1}[x<t]dx &= \int_0^{\infty}\mathrm{P}[\sigma > x]dx \\
&= \mathrm{E}[\sigma]. \qquad (41)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{\mathrm{E}[e^{\theta_t(\sigma\wedge t)}]-1}{\theta_t} &= \mathrm{E}\left[\int_0^{\sigma\wedge t} e^{\theta_t x}dx\right] \\
&= \mathrm{E}\left[\int_0^{t} \mathbf{1}[\sigma > x] e^{\theta_t x}dx\right] \\
&= \int_0^{t} \mathrm{P}[\sigma > x] e^{\theta_t x}dx. \qquad (42)
\end{aligned}$$

By (41) and (42), the proof is completed. □

Lemma 5. If (i) $0 < \theta < 1$, (ii) $\{v_t/t, t = 1, 2, \cdots\}$ is monotone decreasing, and (iii) Y has an exponential moment, i.e., $\mathrm{E}[e^{\theta Y}] < \infty$ for $\theta > 0$, then

$$\limsup_{t\to\infty} \frac{1}{v_t} \ln \mathrm{E}\left[e^{\theta_t S_t^{(a)}}\right] \leq \mathrm{E}[Y]\mathrm{E}[\sigma]\theta. \qquad (43)$$

Proof of Lemma 5: For each $t = 1, 2, \cdots$,

$$\begin{aligned}
\frac{1}{v_t} \ln \mathrm{E}\left[e^{\theta_t S_t^{(a)}}\right] &= \frac{1}{v_t}\sum_{s=1}^{t} \ln \mathrm{E}\left[\mathrm{E}\left[e^{\theta_t(\sigma\wedge s)}\right]^Y\right] \\
&\leq \frac{t}{v_t} \ln \mathrm{E}\left[\mathrm{E}\left[e^{\theta_t(\sigma\wedge t)}\right]^Y\right] \\
&= \theta \frac{1}{\theta_t} \ln \mathrm{E}\left[\mathrm{E}\left[e^{\theta_t(\sigma\wedge t)}\right]^Y\right] \\
&= \theta \cdot \frac{\mathrm{E}[e^{\theta_t(\sigma\wedge t)}]-1}{\theta_t} \cdot \frac{\ln \mathrm{E}\left[\mathrm{E}[e^{\theta_t(\sigma\wedge t)}]^Y\right]}{\mathrm{E}[e^{\theta_t(\sigma\wedge t)}]-1}. \qquad (44)
\end{aligned}$$

By Lemma 4, $\mathrm{E}[e^{\theta_t(\sigma\wedge t)}] \to 1$ as $t \to \infty$. Hence

$$\begin{aligned}
\lim_{t\to\infty} \frac{\ln \mathrm{E}\left[\mathrm{E}[e^{\theta_t(\sigma\wedge t)}]^Y\right]}{\mathrm{E}[e^{\theta_t(\sigma\wedge t)}]-1} &= \lim_{x\to 1} \frac{\ln \mathrm{E}[x^Y]}{x-1} \\
&= \mathrm{E}[Y]. \qquad (45)
\end{aligned}$$

By Lemma 4, (44) and (45), the proof is completed. □

Lemma 6. If $\theta > 0$,

$$\liminf_{t\to\infty} \frac{1}{v_t} \ln \mathrm{E}\left[e^{\theta_t S_t^{(a)}}\right] \geq \mathrm{E}[Y]\mathrm{E}[\sigma]\theta. \qquad (46)$$

Proof of Lemma 6: Let M be a positive integer. For $t \geq M$,

$$\begin{aligned}
\frac{1}{v_t} \ln \mathrm{E}\left[e^{\theta_t S_t^{(a)}}\right] &= \frac{1}{v_t} \sum_{s=1}^{t} \ln \mathrm{E}\left[\mathrm{E}\left[e^{\theta_t(\sigma \wedge s)}\right]^Y\right] \\
&\geq \frac{1}{v_t} \sum_{s=M}^{t} \ln \mathrm{E}\left[\mathrm{E}\left[e^{\theta_t(\sigma \wedge M)}\right]^Y\right] \\
&= \frac{t-M+1}{v_t} \ln \mathrm{E}\left[\mathrm{E}\left[e^{\theta_t(\sigma \wedge M)}\right]^Y\right] \\
&\geq \frac{t-M+1}{v_t} \mathrm{E}[Y] \ln \mathrm{E}\left[e^{\theta_t(\sigma \wedge M)}\right] \\
&= \frac{t-M+1}{t} \mathrm{E}[Y] \cdot \theta \cdot \frac{\ln \mathrm{E}\left[e^{\theta_t(\sigma \wedge M)}\right]}{\theta_t}.
\end{aligned}$$

Hence

$$\liminf_{t \to \infty} \frac{1}{v_t} \ln \mathrm{E}\left[e^{\theta_t S_t^{(a)}}\right] \geq \theta \mathrm{E}[Y] \mathrm{E}[\sigma \wedge M].$$

Letting $M \to \infty$ completes the proof. □

Lemma 7. If (i) $0 < \theta < 1$, (ii) $\{v_t/t, t = 1, 2, \cdots\}$ is monotone decreasing in the limit, (iii) Y has an exponential moment, and (iv) $\mathrm{E}[\hat{\sigma}] < \infty$, then

$$\lim_{t \to \infty} \frac{1}{v_t} \ln \mathrm{E}\left[e^{\theta_t S_t^{(0)}}\right] = 0. \tag{47}$$

Proof of Lemma 7: Since, for each $t = 1, 2, \cdots$, $\frac{1}{v_t} \ln \mathrm{E}[e^{\theta_t S_t^{(0)}}] \geq 0$, it suffices to show

$$\limsup_{t \to \infty} \frac{1}{v_t} \ln \mathrm{E}\left[e^{\theta_t S_t^{(0)}}\right] = \limsup_{t \to \infty} \frac{1}{v_t} \sum_{s=1}^{\infty} \ln \mathrm{E}\left[\mathrm{E}\left[e^{\theta_t((\sigma-s)^+ \wedge t)}\right]^Y\right] = 0.$$

First we note that

$$\mathrm{E}\left[e^{\theta_t((\sigma-s)^+ \wedge t)}\right] \geq 1$$

and

$$\mathrm{E}\left[e^{\theta_t((\sigma-s)^+ \wedge t)}\right] \leq \mathrm{E}\left[e^{\theta_t(\sigma \wedge t)}\right] \to 1 \text{ (as } t \to \infty).$$

Since

$$\lim_{x \downarrow 1} \frac{\ln \mathrm{E}[x^Y]}{x-1} = \mathrm{E}[Y],$$

for $1 < x < 1 + \delta$ with sufficiently small δ, we have $\epsilon > 0$ such that

$$\ln \mathrm{E}[x^Y] < (\mathrm{E}[Y] + \epsilon)(x-1).$$

Thus, for sufficiently large t,

$$\ln \mathrm{E}\left[\mathrm{E}\left[e^{\theta_t((\sigma-s)^+ \wedge t)}\right]^Y\right] < (\mathrm{E}[Y] + \epsilon)\left(\mathrm{E}\left[e^{\theta_t((\sigma-s)^+ \wedge t)}\right] - 1\right),$$

and it suffices to show

$$\frac{1}{v_t}\sum_{s=1}^{\infty}\left(\mathrm{E}\left[e^{\theta_t((\sigma-s)^+\wedge t)}\right]-1\right)\to 0 \text{ as } t\to\infty.$$

We have

$$\begin{aligned}
\frac{1}{v_t}\sum_{s=1}^{\infty}\mathrm{E}\left[e^{\theta_t((\sigma-s)^+\wedge t)}-1\right] &= \frac{1}{v_t}\sum_{s=1}^{\infty}\theta_t\mathrm{E}\left[\int_0^{(\sigma-s)^+\wedge t}e^{\theta_t x}\right]dx \\
&= \frac{1}{v_t}\sum_{s=1}^{\infty}\theta_t\int_0^t \mathrm{P}[\sigma-s>x]e^{\theta_t x}dx \\
&= \frac{\theta_t}{v_t}\int_0^t\sum_{s=1}^{\infty}\mathrm{P}[\sigma>x+s]e^{\theta_t x}dx \\
&= \frac{\theta_t}{v_t}\int_0^t\left(\sum_{s=0}^{\infty}\mathrm{P}[\sigma>x+s]-\mathrm{P}[\sigma>x]\right)e^{\theta_t x}dx \\
&= \frac{\theta}{t}\left(\int_0^t \mathrm{P}[\hat{\sigma}>x]e^{\theta_t x}dx-\int_0^t\mathrm{P}[\sigma>x]e^{\theta_t x}dx\right) \\
&= \frac{\theta}{t}\left(\int_0^\infty \mathrm{P}[\hat{\sigma}>x]\mathbf{1}[x<t]e^{\theta_t x}dx-\int_0^\infty\mathrm{P}[\sigma>x]\mathbf{1}[x<t]e^{\theta_t x}dx\right) \\
&\to 0 \text{ as } t\to\infty,
\end{aligned} \tag{48}$$

because, by LDCT,

$$\lim_{t\to\infty}\int_0^\infty \mathrm{P}[\hat{\sigma}>x]\mathbf{1}[x<t]e^{\theta_t x}dx=\mathrm{E}[\sigma]\mathrm{E}[\hat{\sigma}]<\infty$$

and

$$\lim_{t\to\infty}\int_0^\infty \mathrm{P}[\sigma>x]\mathbf{1}[x<t]e^{\theta_t x}dx=\mathrm{E}[\sigma]<\infty. \quad \square$$

Lemma 8. If (i) $\theta>1$ and (ii)

$$\lim_{t\to\infty}\frac{\mathrm{P}[\hat{\sigma}>t]}{\mathrm{P}[\sigma>t]}=\infty,$$

then

$$\lim_{t\to\infty}\frac{1}{v_t}\ln\mathrm{E}\left[e^{\theta_t S_t}\right]=\infty. \tag{49}$$

Proof of Lemma 8: From Lemma 1,

$$\begin{aligned}
\frac{1}{v_t}\ln \mathrm{E}\left[e^{\theta_t S_t}\right] &\geq \frac{1}{v_t}\sum_{s=1}^{\infty}\ln \mathrm{E}\left[\mathrm{E}\left[e^{\theta_t((\sigma-s)^+\wedge t)}\right]^Y\right]+\frac{1}{v_t}\ln \mathrm{E}\left[\mathrm{E}\left[e^{\theta_t(\sigma\wedge t)}\right]^Y\right]\\
&= \frac{1}{v_t}\sum_{s=0}^{\infty}\ln \mathrm{E}\left[\mathrm{E}\left[e^{\theta_t((\sigma-s)^+\wedge t)}\right]^Y\right]\\
&\geq \frac{\mathrm{E}[Y]}{v_t}\sum_{s=0}^{\infty}\ln \mathrm{E}\left[e^{\theta_t((\sigma-s)^+\wedge t)}\right]\\
&= \frac{\mathrm{E}[Y]}{v_t}\sum_{s=0}^{\infty}\ln\left(1+\theta_t\int_0^t \mathrm{P}[\sigma>s+x]e^{\theta_t x}dx\right)\\
&\geq \frac{\mathrm{E}[Y]}{v_t}\sum_{s=0}^{\infty}\ln\left(1+\theta_t \mathrm{P}[\sigma>s+t]\int_0^t e^{\theta_t x}dx\right)\\
&= \frac{\mathrm{E}[Y]}{v_t}\sum_{s=0}^{\infty}\ln\left(1+\mathrm{P}[\sigma>s+t]e^{\theta_t t}-\mathrm{P}[\sigma>s+t]\right)\\
&\geq \frac{\mathrm{E}[Y]}{v_t}\sum_{s=0}^{\infty}\left[\ln\left(1+\mathrm{P}[\sigma>s+t]e^{\theta_t t}\right)-\mathrm{P}[\sigma>s+t]\right]\\
&\geq \frac{\mathrm{E}[Y]}{v_t}\sum_{s=0}^{\infty}\frac{\ln(1+\mathrm{P}[\sigma>t]e^{\theta_t t})}{\mathrm{P}[\sigma>t]e^{\theta_t t}}\mathrm{P}[\sigma<s+t]e^{\theta_t t}-\frac{\mathrm{E}[\sigma]}{v_t}\mathrm{P}[\hat{\sigma}>t]\\
&= \mathrm{E}[Y]\frac{\mathrm{E}[\sigma]}{v_t}\frac{\ln(1+\mathrm{P}[\sigma>t]e^{\theta_t t})}{\mathrm{P}[\sigma>t]e^{\theta_t t}}\mathrm{P}[\hat{\sigma}>t]e^{\theta_t t}-\frac{\mathrm{E}[\sigma]}{v_t}\mathrm{P}[\hat{\sigma}>t]\\
&= \mathrm{E}[Y](\theta-1)\mathrm{E}[\sigma]\frac{\mathrm{P}[\hat{\sigma}>t]e^{\theta_t t}}{\ln(\mathrm{P}[\hat{\sigma}>t]e^{\theta_t t})}\frac{\ln(1+\mathrm{P}[\sigma>t]e^{\theta_t t})}{\mathrm{P}[\sigma>t]e^{\theta_t t}}-\frac{\mathrm{E}[\sigma]}{v_t}\mathrm{P}[\hat{\sigma}>t]\\
&\to \infty \text{ as } t\to\infty, \qquad (50)
\end{aligned}$$

by the following Lemma 9. □

Lemma 9. If we have sequence a_n and b_n such that $b_n > 0$, $a_n \to \infty$, and $a_n/b_n \to \infty$ as $n \to \infty$, then

$$\lim_{n\to\infty}\frac{a_n}{b_n}\frac{\ln(1+b_n)}{\ln(1+a_n)}=\infty,$$

and, hence,

$$\lim_{n\to\infty}\frac{a_n}{b_n}\frac{\ln(1+b_n)}{\ln a_n}=\infty.$$

Proof of Lemma 9: If $b_n \leq 1$, then

$$\frac{a_n}{b_n}\frac{\ln(1+b_n)}{\ln(1+a_n)}\geq \ln 2\frac{a_n}{\ln(1+a_n)}.$$

Since $\lim_{n\to\infty} a_n/\ln(1+a_n) = \infty$, we may assume that $b_n > 1$ for all n. It is easy to see that, for each $z > 1$, $\ln(1+x)/\ln(1+zx)$ is an increasing function of x. Hence, if $a_n > 1$

$$\frac{a_n}{b_n}\frac{\ln(1+b_n)}{\ln(1+a_n)} \geq \frac{a_n}{b_n}\frac{\ln(1+1)}{\ln(1+a_n/b_n)}$$

and, hence,

$$\liminf_{n\to\infty} \frac{a_n}{b_n}\frac{\ln(1+b_n)}{\ln(1+a_n)} \geq \ln 2 \frac{a_n/b_n}{\ln(1+a_n/b_n)} \to \infty. \quad \square$$