DYNAMICS OF AN IMPULSIVE FOOD CHAIN SYSTEM WITH A LOTKA-VOLTERRA FUNCTIONAL RESPONSE

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ABSTRACT. We investigate a three species food chain system with Lotka-Volterra type functional response and impulsive perturbations. We find a condition for the local stability of prey or predator free periodic solutions by applying the Floquet theory and the comparison theorems and show the boundedness of this system. Furthermore, we illustrate some examples.

1. INTRODUCTION

In ecology, one of main goals is to understand the dynamical relationship between predator and prey. Such relationship can be represented by the functional response which refers to the change in the density of prey attached per unit time per predator as the prey density changes. One of well-known functional responses is a Lotka-Volterra functional response [1, 2] . The principles of Lotka-Volterra models, conservation of mass and decomposition of the rates of change in birth and death processes, have remained valid until today and many theoretical ecologists adhere to their principles. For the reason, we need to consider a Lotka-Volterra type food chain model, which can be described by the following differential equations:

$$
\begin{cases}
x'(t) = x(t)(a - bx(t) - c_1 y(t)), \\
y'(t) = y(t)(-d_1 + c_2 x(t) - e_1 z(t)), \\
z'(t) = z(t)(-d_2 + e_2 y(t)),
\end{cases} \tag{1.1}
$$

where $x(t), y(t)$ and $z(t)$ are the densities of lowest-level prey, mid-level predator and top predator at time t, respectively. The parameters $a, b, c_1, c_2, d_1, d_2, e_1$ and e_2 are positive constants.

As Cushing [5] pointed out that it is necessary and important to consider models with periodic ecological parameters or perturbations which might be quite naturally exposed (for example, those due to seasonal effects of weather, food supply, mating habits, hunting or harvesting seasons and so on.). Such perturbations were often treated continually. But, there are still some other perturbations such as fire, flood, etc, that are not suitable to be considered continually.

These impulsive perturbations bring sudden change to the system. Let's think of the mid-level predator in (1.1) as a pest and the top predator as a natural enemy of it. There are many ways to beat pests. For examples, harvesting on pest, spreading pesticides, releasing natural enemies and so on. Such tactics are discontinuous and periodical. With the idea mentioned above, in this paper, we consider the following food chain system with a proportion periodic impulsive poisoning for all species and periodic constant impulsive immigration of the top predator at different fixed time.

$$
\begin{cases}
\left.\begin{array}{l}
x'(t) = x(t)(a - bx(t) - c_1 y(t)), \\
y'(t) = y(t)(-d_1 + c_2 x(t) - e_1 z(t)), \\
z'(t) = z(t)(-d_2 + e_2 y(t)),
\end{array}\right\} t \neq nT, t \neq (n + \tau - 1)T, \\
\left.\begin{array}{l}
x(t^+) = (1 - p_1)x(t), \\
y(t^+) = (1 - p_2)y(t), \\
z(t^+) = (1 - p_3)z(t),
\end{array}\right\} t = (n + \tau - 1)T, \\
\left.\begin{array}{l}
x(t^+) = x(t), \\
y(t^+) = y(t), \\
z(t^+) = z(t) + q,
\end{array}\right\} t = nT, \\
(x(0^+), y(0^+), z(0^+)) = (x_0, y_0, z_0).
\end{cases}
\tag{1.2}
$$

where $0 \leq \tau, p_1, p_2, p_3 < 1$ and T is the period of the impulsive immigration and q is the size of immigration. Such model is an impulsive differential equation whose theory and applications were greatly developed by the efforts of Lakshmikantham and Bainov et al. [9].

In recent years, models with sudden perturbations have been intensively researched[7, 10, 11, 12, 13, 14, 18, 19, 20, 21, 22]. The authors in [10, 11] have studied the local stability for a two species food chain system with Lotka-Volterra functional response and impulsive perturbations.

In the next section, we introduce some notations and lemmas used in this paper. In section 3, we find a condition for the local stabilities of a lower-level prey and mid-level predator free periodic solution and a mid-level predator free periodic solution by applying the Floquet theory and the comparison theorems. In section 4, we illustrate some examples. Finally, we give a conclusion.

2. PRELIMINARIES

First, we shall introduce a few notations and definitions together with a few auxiliary results relating to the comparison theorems, which will be useful for our main results.

Let $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_+^3 = \{\mathbf{x}x = (x(t), y(t), z(t)) \in \mathbb{R}^3 : x(t), y(t), z(t) \geq 0\}$. Denote $\mathbb{N}$ the set of all of nonnegative integers, $\mathbb{R}_+^* = (0, \infty)$ and $f = (f_1, f_2, f_3)^T$ the right hand of

the first three equations in (1.2). Let $V : \mathbb{R}_+ \times \mathbb{R}_+^3 \to \mathbb{R}_+$. Then V is said to be in a class V_0 if

(1) V is continuous on $(nT, (n+1)T] \times \mathbb{R}_+^3$ and $\lim\limits_{\substack{(t,\mathbf{y}) \to (nT, \mathbf{x}) \\ t > nT}} V(t, \mathbf{y}) = V(nT^+, \mathbf{x})$ exists.

(2) V is a local Lipschitzian in $\mathbf{x}$.

Definition 2.1. For $V \in V_0$, we define the upper right Dini derivative of V with respect to the impulsive differential system (1.2) at $(t, \mathbf{x}) \in (nT, (n+1)T] \times \mathbb{R}_+^3$ by

$$D^+ V(t, \mathbf{x}) = \limsup_{h \to 0+} \frac{1}{h} [V(t+h, \mathbf{x} + hf(t, \mathbf{x})) - V(t, \mathbf{x})].$$

The solution of the system (1.2) is a piecewise continuous function $\mathbf{x} : \mathbb{R}_+ \to \mathbb{R}_+^3$, $\mathbf{x}(t)$ is continuous on $(nT, (n+1)T], n \in \mathbb{N}$ and $\mathbf{x}(nT^+) = \lim_{t \to nT^+} \mathbf{x}(t)$ exists. The smoothness properties of f guarantee the global existence and uniqueness of solutions of the system (1.2). (See [9] for the details).

Now, we give the basic properties of two impulsive differential equations. First, we consider the following impulsive differential equation:

$$\begin{cases} x'(t) = x(t)(a - bx(t)), \ t \neq nT, t \neq (n + \tau - 1)T, \\ x(t^+) = (1 - p_1)x(t), \ t = (n + \tau - 1)T, \\ x(t^+) = x(t), \ t = nT, \\ x(0^+) = x_0. \end{cases} \quad (2.1)$$

The system (2.1) is a periodically forced system. It is easily obtain that

$$x^*(t) = \frac{a\eta \exp(a(t - (n + \tau - 1)T))}{b(1 - \eta + \eta \exp(a(t - (n + \tau - 1)T)))}, (n + \tau - 1)T < t \leq (n + \tau)T, \quad (2.2)$$

is a positive periodic solution of (2.1), where $\eta = \frac{(1-p_1)\exp(aT)-1}{\exp(aT)-1}$. Thus, we obtain the following Lemma from [15].

Lemma 2.2. [15] *The following statements hold.*
(1) If $aT + \ln(1 - p_1) > 0$, then $\lim_{t \to \infty} |x(t) - x^*(t)| = 0$ for all solutions $x(t)$ of (2.1) with $x_0 > 0$.
(2) If $aT + \ln(1 - p_1) \leq 0$, then $x(t) \to 0$ as $t \to \infty$ for all solutions $x(t)$ of (2.1).

Next, we consider the impulsive differential equation as follows:

$$\begin{cases} z'(t) = -d_2 z(t), \ t \neq nT, t \neq (n + \tau - 1)T, \\ z(t^+) = (1 - p_3)z(t), \ t = (n + \tau - 1)T, \\ z(t^+) = z(t) + q, \ t = nT, \\ z(0^+) = z_0. \end{cases} \quad (2.3)$$

The system (2.3) is a periodically forced linear system. It is easy to obtain that

$$
z^*(t) = \begin{cases}
\dfrac{q\exp(-d_2(t-(n-1)T))}{1-(1-p_3)\exp(-d_2T)}, (n-1)T < t \le (n+\tau-1)T, \\[3mm]
\dfrac{q(1-p_3)\exp(-d_2(t-(n-1)T))}{1-(1-p_3)\exp(-d_2T)}, (n+\tau-1)T < t \le nT,
\end{cases}
\tag{2.4}
$$

$z^*(0^+) = z^*(nT^+) = \frac{q}{1-(1-p_3)\exp(-d_2T)}$, $z^*((n+\tau-1)T^+) = \frac{q(1-p_3)\exp(-d_2\tau T)}{1-(1-p_3)\exp(-d_2T)}$ is a positive periodic solution of (2.3). Moreover, we can obtain that

$$
z(t) = \begin{cases}
(1-p_3)^{n-1}\left(z(0^+) - \dfrac{q(1-p_3)e^{-T}}{1-(1-p_3)\exp(-d_2T)}\right)\exp(-d_2t) + z^*(t), \\[2mm]
(n-1)T < t \le (n+\tau-1)T, \\[3mm]
(1-p_3)^{n}\left(z(0^+) - \dfrac{q(1-p_3)e^{-T}}{1-(1-p_3)\exp(-d_2T)}\right)\exp(-d_2t) + z^*(t), \\[2mm]
(n+\tau-1)T < t \le nT,
\end{cases}
\tag{2.5}
$$

is a solution of (2.3). From (2.4) and (2.5), we get easily the following result.

Lemma 2.3. *All solutions $z(t)$ of (1.2) with $z_0 \ge 0$ tend to $z^*(t)$. i.e., $|z(t) - z^*(t)| \to 0$ as $t \to \infty$.*

Thus, there are at least two periodic solutions $(0, 0, z^*(t))$ and $(x^*(t), 0, z^*(t))$ of the system (1.2). It is important to investigate the stabilities of these two periodic solutions because they play a major role in the impulsive system (1.2) like the equilibrium points of a system with no impulsiveness. Moreover, if we regard the mid-level predator $y(t)$ as a pest, then the periodic solutions represent a kind of pest-free solutions.

We will use a comparison result of impulsive differential inequalities. We suppose that $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfies the following hypotheses:
(H) g is continuous on $(nT, (n+1)T] \times \mathbb{R}_+$ and the limit $\lim_{(t,y)\to(nT^+,x)} g(t,y) = g(nT^+, x)$ exists and is finite for $x \in \mathbb{R}_+$ and $n \in \mathbb{N}$.

Lemma 2.4. [9] *Suppose $V \in V_0$ and*

$$
\begin{cases}
D^+V(t,\mathbf{x}) \le g(t, V(t,\mathbf{x})), \ t \ne (n+\tau-1)T, t \ne nT, \\
V(t,\mathbf{x}(t^+)) \le \psi_n^1(V(t,\mathbf{x})), \ t = (n+\tau-1)T, \\
V(t,\mathbf{x}(t^+)) \le \psi_n^2(V(t,\mathbf{x})), \ t = nT,
\end{cases}
\tag{2.6}
$$

where $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfies (H) and $\psi_n^1, \psi_+^2 : \mathbb{R}_+ \to \mathbb{R}_+$ are non-decreasing for all $n \in \mathbb{N}$. Let $r(t)$ be the maximal solution for the impulsive Cauchy problem

$$
\begin{cases}
u'(t) = g(t, u(t)), \ t \ne (n+\tau-1)T, t \ne nT, \\
u(t^+) = \psi_n^1(u(t)), \ t = (n+\tau-1)T, \\
u(t^+) = \psi_n^2(u(t)), \ t = nT, \\
u(0^+) = u_0,
\end{cases}
\tag{2.7}
$$

defined on $[0, \infty)$. *Then* $V(0^+, \mathbf{x}_0) \leq u_0$ *implies that* $V(t, \mathbf{x}(t)) \leq r(t), t \geq 0$, *where* $\mathbf{x}(t)$ *is any solution of (2.6).*

We now indicate a special case of Lemma 2.4 which provides estimations for the solution of a system of differential inequalities. For this, we let $PC(\mathbb{R}_+, \mathbb{R})(PC^1(\mathbb{R}_+, \mathbb{R}))$ denote the class of real piecewise continuous(real piecewise continuously differentiable) functions defined on $\mathbb{R}_+$.

Lemma 2.5. [9] *Let the function* $u(t) \in PC^1(\mathbb{R}^+, \mathbb{R})$ *satisfy the inequalities*

$$
\begin{cases}
\dfrac{du}{dt} \leq f(t)u(t) + h(t), t \neq \tau_k, t > 0, \\
u(\tau_k^+) \leq \alpha_k u(\tau_k) + \beta_k, k \geq 0, \\
u(0^+) \leq u_0,
\end{cases}
\tag{2.8}
$$

where $f, h \in PC(\mathbb{R}_+, \mathbb{R})$ *and* $\alpha_k \geq 0$, β_k *and* u_0 *are constants and* $(\tau_k)_{k \geq 0}$ *is a strictly increasing sequence of positive real numbers. Then, for* $t > 0$,

$$
u(t) \leq u_0 \left(\prod_{0 < \tau_k < t} \alpha_k \right) \exp\left(\int_0^t f(s)ds \right) + \int_0^t \left(\prod_{0 \leq \tau_k < t} d_k \right) \exp\left(\int_s^t f(\gamma)d\gamma \right) h(s)ds
$$

$$
+ \sum_{0 < \tau_k < t} \left(\prod_{\tau_k < \tau_j < t} d_j \right) \exp\left(\int_{\tau_k}^t f(\gamma)d\gamma \right) \beta_k.
$$

Similar result can be obtained when all conditions of the inequalities in the Lemma 2.4 and 2.5 are reversed. Using Lemma 2.5, it is proven that all solution of (1.2) with $x_0 > 0$ remain strictly positive as follows:

Lemma 2.6. *The positive orthant* $(\mathbb{R}_+^*)^3$ *is an invariant region for the system (1.2).*

Proof. Let $(x(t), y(t), z(t)) : [0, t_0) \to \mathbb{R}^2$ be a saturated solution of the system (1.2) with a strictly positive initial value (x_0, y_0, z_0). By Lemma 2.5, we can obtain that, for $0 \leq t < t_0$,

$$
\begin{cases}
x(t) \geq x(0)(1 - p_1)^{[\frac{t}{T}]} \exp\left(\int_0^t f_1(s)ds \right), \\
y(t) \geq y(0)(1 - p_2)^{[\frac{t}{T}]} \exp\left(\int_0^t f_2(s)ds \right), \\
z(t) \geq z(0)(1 - p_3)^{[\frac{t}{T}]} \exp\left(\int_0^t f_3(s)ds \right)q,
\end{cases}
\tag{2.9}
$$

where $f_1(s) = a - bx(s) - cy(s)$, $f_2(s) = -d_1 - e_1 z(s)$ and $f_3(s) = -d_2$. Thus, $x(t), y(t)$ and $z(t)$ remain strictly positive on $[0, t_0)$. $\qquad \square$

3. Main Theorems

In this section, we study the stability of the lowest-level prey and mid-level predator free periodic solution $(0, 0, z^*(t))$ and of the mid-level predator free periodic solution $(x^*(t), 0, z^*(t))$. First, we show that all solutions of (1.2) are uniformly ultimately bounded.

Theorem 3.1. *There is an $M > 0$ such that $x(t) \leq M, y(t) \leq M$ and $z(t) \leq M$ for all t large enough, where $(x(t), y(t), z(t))$ is a solution of the system (1.2).*

Proof. Let $(x(t), y(t), z(t))$ be a solution of (1.2) and define $u(t) = \frac{c_2}{c_1}x(t) + y(t) + \frac{e_1}{e_2}z(t)$. It is easily shown that, for $t \neq nT, t \neq (n + \tau - 1)T$,

$$\frac{du(t)}{dt} = -\frac{c_2 b}{c_1}x^2(t) + \frac{e_1 a}{e_2}x(t) - d_1 y(t) - \frac{c_3 d_2}{c_4}z(t). \tag{3.1}$$

Also, it follows from choosing $0 < \beta_0 < \min\{d_1, d_2\}$ that

$$\frac{du(t)}{dt} + \beta_0 u(t) \leq -\frac{c_2 b}{c_1}x^2(t) + \frac{c_2}{c_1}(a + \beta_0)x(t), \ t \neq nT, t \neq (n + \tau - 1)T. \tag{3.2}$$

Since the right-hand side of (3.2) is bounded from above by $M_0 = \frac{c_2(a+\beta_0)^2}{4bc_1}$, we obtain that

$$\frac{du(t)}{dt} + \beta_0 u(t) \leq M_0, t \neq nT, n \neq (n + \tau - 1)T.$$

If $t = nT$, then $u(t^+) = u(t) + \frac{e_1}{e_2}q$ and if $t = (n + \tau - 1)T$, then $u(t^+) \leq (1 - p)u(t)$, where $p = \min\{p_1, p_2, p_3\}$. It is from Lemma 2.5 that

$$
\begin{aligned}
u(t) \leq{} & u_0 \left(\prod_{0 < kT < t} (1 - p) \right) \exp\left(\int_0^t -\beta_0 ds \right) \\
& + \int_0^t \left(\prod_{0 \leq kT < t} (1 - p) \right) \exp\left(\int_s^t -\beta_0 d\gamma \right) M_0 ds \\
& + \sum_{0 < kT < t} \left(\prod_{kT < jT < t} (1 - p) \right) \exp\left(\int_{kT}^t -\beta_0 d\gamma \right) \frac{e_1}{e_2}q \\
\leq{} & u(0^+) \exp(-\beta_0 t) + \frac{M_0}{\beta_0}(1 - \exp(-\beta_0 t)) + \frac{c_3 q \exp(\beta_0 T)}{c_4 \exp(\beta_0 T) - 1}.
\end{aligned}
\tag{3.3}
$$

Since the limit of the right-hand side of (3.3) as $t \to \infty$ is

$$\frac{M_0}{\beta_0} + \frac{e_1 q \exp(\beta_0 T)}{e_2 \exp(\beta_0 T) - 1} < \infty,$$

it easily follows that $u(t)$ is bounded for sufficiently large t. □

Theorem 3.2. *(1)The periodic solution $(0, 0, z^*(t))$ is locally stable if $aT + \ln(1 - p_1) \leq 0$. (2)The periodic solution $(0, 0, z^*(t))$ is unstable if $aT + \ln(1 - p_1) > 0$.*

Proof. The local stability of the periodic solution $(0, 0, z^*(t))$ of the system (1.2) may be determined by considering the behavior of small amplitude perturbations of the solution. Let $(x(t), y(t), z(t))$ be a solution of the system (1.2), where $x(t) = u(t), y(t) = v(t)$ and $z(t) = w(t) + z^*(t)$. Then they may be written as

$$\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} u(0) \\ v(0) \\ w(0) \end{pmatrix}$$

where $\Phi(t)$ satisfies

$$\frac{d\Phi}{dt} = \begin{pmatrix} a & 0 & 0 \\ 0 & -d_1 - e_1 z^*(t) & 0 \\ 0 & e_2 z^*(t) & -d_2 \end{pmatrix} \Phi(t)$$

and $\Phi(0) = I$ is the identity matrix. So the fundamental solution matrix is

$$\Phi(t) = \begin{pmatrix} \exp(at) & 0 & 0 \\ 0 & \exp(\int_0^t -d_1 - e_1 z^*(s)ds) & 0 \\ 0 & \exp(\int_0^t e_2 z^*(s)ds) & \exp(-d_2 t) \end{pmatrix}.$$

The resetting impulsive conditions of the system (1.2) become

$$\begin{pmatrix} u((n+\tau-1)T^+) \\ v((n+\tau-1)T^+) \\ u((n+\tau-1)T^+) \end{pmatrix} = \begin{pmatrix} 1-p_1 & 0 & 0 \\ 0 & 1-p_2 & 0 \\ 0 & 0 & 1-p_3 \end{pmatrix} \begin{pmatrix} u((n+\tau-1)T) \\ v((n+\tau-1)T) \\ w((n+\tau-1)T) \end{pmatrix}$$

and

$$\begin{pmatrix} u(nT^+) \\ v(nT^+) \\ w(nT^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u(nT) \\ v(nT) \\ w(nT) \end{pmatrix}.$$

Note that the eigenvalues of

$$S = \begin{pmatrix} 1-p_1 & 0 & 0 \\ 0 & 1-p_2 & 0 \\ 0 & 0 & 1-p_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Phi(T)$$

are $\mu_1 = (1 - p_1) \exp(aT)$, $\mu_2 = (1 - p_2) \exp(-\int_0^T d_1 + e_1 z^*(s)ds) < 1$ and $\mu_3 = \exp(-d_2 T) < 1$. The condition $\mu_1 \leq 1 (> 1)$ is equivalent to the equation $aT + \ln(1 - p_1) \leq 0 (> 0)$. Therefore, by Floquet theory of impulsive differential equations, the periodic solution $(0, 0, z^*(t))$ is locally stable if $aT + \ln(1 - p_1) \leq 0$ and is unstable if $aT + \ln(1 - p_1) > 0$. □

Theorem 3.3. *The periodic solution* $(x^*(t), 0, z^*(t))$ *is locally asymptotically stable if* $aT + \ln(1 - p_1) > 0$ *and*

$$\frac{c_2}{b}(aT + \ln(1 - p_1)) + \ln(1 - p_2) < d_1 T + \frac{e_2 q(1 - (1 - p_3) \exp(-d_2 T) - p_3 \exp(-d_2 \tau T))}{d_2(1 - (1 - p_3) \exp(-d_2 T))}.$$

$$(3.4)$$

Proof. Assume that $aT + \ln(1 - p_1) > 0$ and the equation (3.4) hold. We apply the same method as Theorem 3.2 to the periodic solution $(x^*(t), 0, z^*(t))$ to determine its stability. So, we define $x(t) = u(t) + x^*(t), y(t) = v(t), z(t) = w(t) + z^*(t)$. Then they may be written as

$$
\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} u(0) \\ v(0) \\ w(0) \end{pmatrix}
$$

where $\Phi(t)$ satisfies

$$
\frac{d\Phi}{dt} = \begin{pmatrix} a - 2bx^*(t) & -c_1 x^*(t) & 0 \\ 0 & -d_1 + c_2 x^*(t) - e_1 z^*(t) & 0 \\ 0 & e_2 z^*(t) & -d_2 \end{pmatrix} \Phi(t)
$$

and $\Phi(0) = I$ is the identity matrix. The resetting impulsive conditions of the system (1.2) become

$$
\begin{pmatrix} u((n+\tau-1)T^+) \\ v((n+\tau-1)T^+) \\ u((n+\tau-1)T^+) \end{pmatrix} = \begin{pmatrix} 1-p_1 & 0 & 0 \\ 0 & 1-p_2 & 0 \\ 0 & 0 & 1-p_3 \end{pmatrix} \begin{pmatrix} u((n+\tau-1)T) \\ v((n+\tau-1)T) \\ w((n+\tau-1)T) \end{pmatrix}
$$

and

$$
\begin{pmatrix} u(nT^+) \\ v(nT^+) \\ w(nT^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u(nT) \\ v(nT) \\ w(nT) \end{pmatrix}.
$$

Further, the eigenvalues of

$$
S = \begin{pmatrix} 1-p_1 & 0 & 0 \\ 0 & 1-p_2 & 0 \\ 0 & 0 & 1-p_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Phi(T)
$$

are $\mu_1 = (1 - p_1) \exp(\int_0^T a - 2bx^*(t)dt)$, $\mu_2 = (1 - p_2) \exp(\int_0^T -d_1 + c_2 x^*(t) - e_1 z^*(t)dt)$ and $\mu_3 = (1 - p_3) \exp(-d_2 T) < 1$. Since $x^*(t) = \dfrac{a\eta \exp(at)}{b(1 - \eta + \eta \exp(at))}$ for $0 < t \leq T$, we get

$$
\begin{aligned}
\int_0^T x^*(t)dt &= \frac{a\eta}{b} \int_0^T \frac{\exp(at)}{1 - \eta + \eta \exp(at)} dt \\
&= \frac{1}{b}(\ln(1 - p_1) + aT),
\end{aligned} \tag{3.5}
$$

where $\eta = \dfrac{(1 - p_1) \exp(aT) - 1}{\exp(aT) - 1}$. And since $z^*(t) = \dfrac{q \exp(-d_2 t)}{1 - (1 - p_3) \exp(-d_2 T)}$ for $0 < t \leq$

τT and $z^*(t) = \dfrac{q(1 - p_3) \exp(-d_2 t)}{1 - (1 - p_3) \exp(-d_2 T)}$ for $\tau T < t \leq T$, we obtain that

$$
\begin{aligned}
\int_0^T z^*(t) dt &= \int_0^{\tau T} z^*(t) dt + \int_{\tau T}^T z^*(t) dt \\
&= \frac{q(1 - (1 - p_3) \exp(-d_2 T) - p_3 \exp(-d_2 \tau T))}{d_2(1 - (1 - p_3) \exp(-d_2 T))}.
\end{aligned}
\tag{3.6}
$$

From (3.5) and (3.6), we get that the conditions $|\mu_1| < 1$ and $|\mu_2| < 1$ are equivalent to the equations $aT + \ln(1 - p_1) > 0$ and (3.4), respectively. Therefore, it follows from Floquet Theory of impulsive differential equations that $(x^*(t), 0, z^*(t))$ is locally asymptotically stable. $\square$
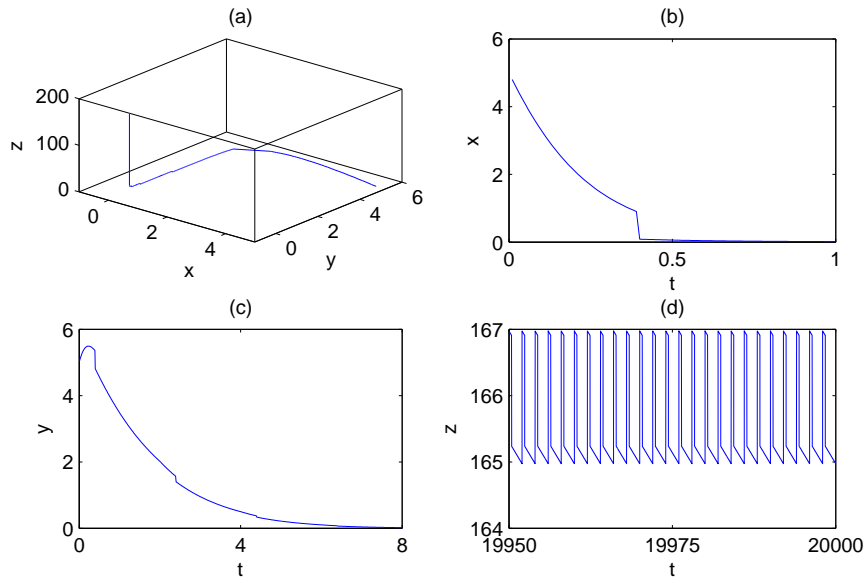

FIGURE 1. $a = 1.0, b = 0.0002, c_1 = 1.0, c_2 = 0.3, d_1 = 0.3, d_2 = 0.001, e_1 = 0.05, e_2 = 0.0005, p_1 = 0.9, p_2 = 0.1, p_3 = 0.01, \tau = 0.2, q = 2$ and $T = 2$. (a) The trajectory of the system (1.2). (b-d) Time series.

4. NUMERICAL EXAMPLES

In this section, we investigate numerical examples for the system (1.2).

Firstly, we consider the unperturbed system (1.1) which has no control terms. It is easy to see that the unperturbed three species food chain system (1.1) has four non-negative equilibria:
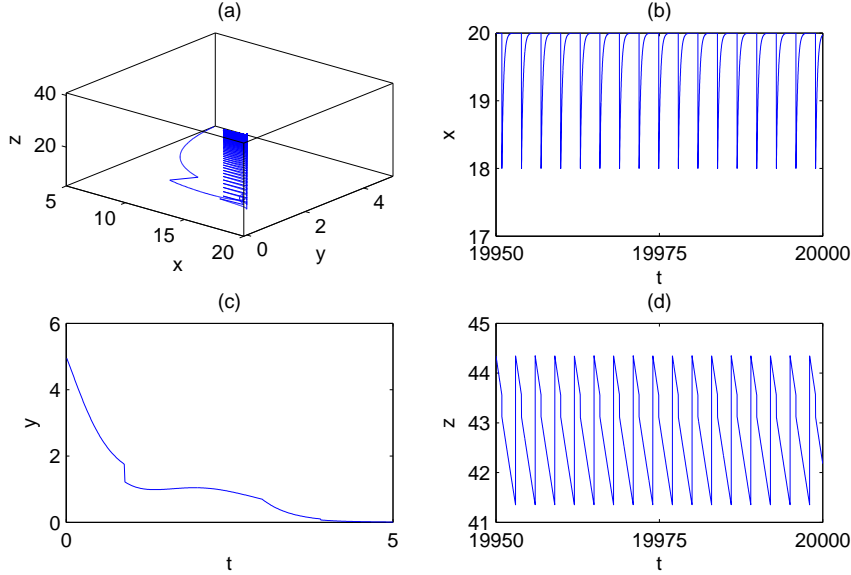

FIGURE 2. $a = 1.0, b = 0.2, c_1 = 0.5, c_2 = 0.1, d_1 = 0.4, d_2 = 0.02, e_1 = 0.5, e_2 = 0.002, p_1 = 0.1, p_2 = 0.3, p_3 = 0.01, \tau = 0.3, T = 3, q = 3.$ (a) The trajectory of the system (1.2). (b-d) Time series.

(1) The trivial equilibrium $A(0,0,0)$.
(2) The mid-level predator and top predator free equilibrium $B(\frac{d_1}{b}, 0, 0)$.
(3) The top predator free equilibrium $C(\frac{d_1}{c_2}, \frac{ac_2 - d_1 b}{c_1 c_2}, 0)$ if $ac_2 - d_1 b > 0$.
(4) The positive equilibrium $E^* = (x^*, y^*, z^*)$, where

$$x^* = \frac{ae_2 - d_2 c_1}{be_2}, y^* = \frac{d_2}{e_2}, z^* = \frac{ae_2 c_2 - d_2 c_1 c_2 - d_1 be_2}{be_1 e_2} \text{ and } ae_2 c_2 - d_2 c_1 c_2 - d_1 be_2 > 0.$$

Stabilities for the equilibria of the system (1.1) have been studied by Zhang and Chen[19].

Lemma 4.1. [19] *(1) If positive equilibrium E^* exists, then E^* is globally stable.*
(2) If positive equilibrium E^* dose not exist and C exists, then C is globally stable.
(3) If positive equilibrium E^* and C do not exist, then B is globally stable.

Throughout this section, we chose $(x_0, y_0, z_0) = (5, 5, 5)$ as an initial point. We consider the following three cases:
(A) $a = 1.0, b = 0.0002, c_1 = 1.0, c_2 = 0.3, d_1 = 0.3, d_2 = 0.001, e_1 = 0.05, e_2 = 0.0005, p_1 = 0.9, p_2 = 0.1, p_3 = 0.01, \tau = 0.2, T = 2, q = 2.$
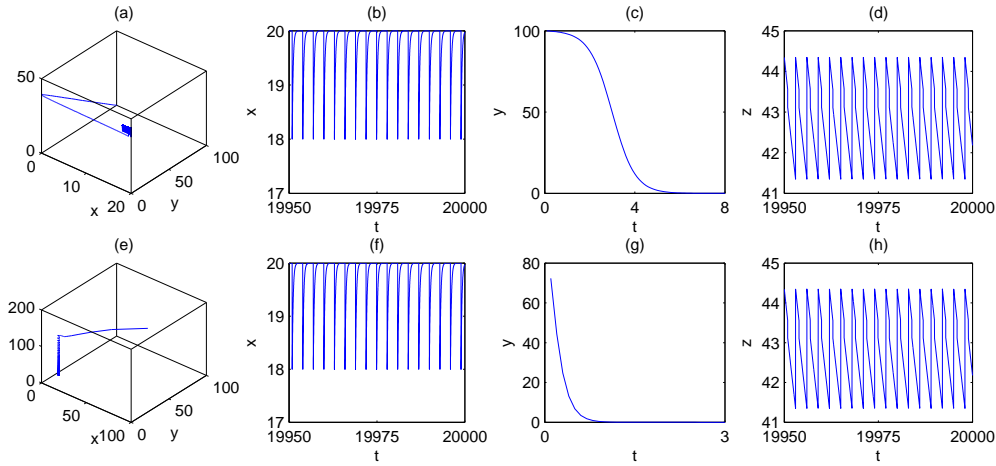
FIGURE 3. $a = 1.0, b = 0.2, c_1 = 0.5, c_2 = 0.1, d_1 = 0.4, d_2 = 0.02, e_1 = 0.5, e_2 = 0.002, p_1 = 0.1, p_2 = 0.3, p_3 = 0.01, \tau = 0.3, T = 3, q = 3$. (a-d) The trajectory of the system (1.2) with the initial value $(0.1, 100, 0.1)$. (e-h) The trajectory of the system (1.2) with the initial value $(100, 100, 100)$.

(B) $a = 1.0, b = 0.2, c_1 = 0.5, c_2 = 0.1, d_1 = 0.4, d_2 = 0.02, e_1 = 0.5, e_2 = 0.002, p_1 = 0.1, p_2 = 0.3, p_3 = 0.01, \tau = 0.3, T = 3, q = 3$.

(C) $a = 2, b = 0.0001, c_1 = 1.0, c_2 = 0.3, d_1 = 0.3, d_2 = 0.01, e_1 = 0.05, e_2 = 0.0025, p_1 = 0.01, p_2 = 0.02, p_3 = 0.001, \tau = 0.3, T = 5, q = 4$.

For (A), it follows from Theorem 3.2 that the periodic solution $(0, 0, z^*(t))$ is locally stable since $aT + \ln(1 - p_1) = -0.3026 \leq 0$. Also, we infer from Lemma 4.1 that the unperturbed system (1.1) has a globally stable top predator free equilibrium $C(1, 0.9997, 0)$, but no positive equilibria. The behavior of the trajectories of the system (1.2) exhibited in Figure 1 shows that $z(t)$ is synchronizing with the periodic solution $z^*(t)$ for sufficiently large t.

For (B), it is easy to see from Theorem 3.3 that the periodic solution $(x^*(t), 0, z^*(t))$ is locally asymptotically stable. Further, from Lemma 4.1, we see that the unperturbed system (1.1) has a global stable positive equilibrium $E^* = (0.79, 0.1, 11.77)$. The trajectory of the system (1.2) is illustrated in Figure 4. In this case, we conjecture that the periodic solution $(x^*(t), 0, z^*(t))$ may be globally stable under the same condition of Theorem 3.3 because many trajectories with different initial values show stable phenomena. (See Figure 3)

For (C), it is from Lemma 4.1 that the unperturbed system (1.1) also has a globally stable top predator free equilibrium $C(1, 2, 0)$. However, Figure 4, which is the phase portraits of the system (1.2) for the case (C), suggests that the system (1.2) may be permanent though we cannot determine whether the system (1.2) is stable or not because it does not satisfy the condition (3.4).
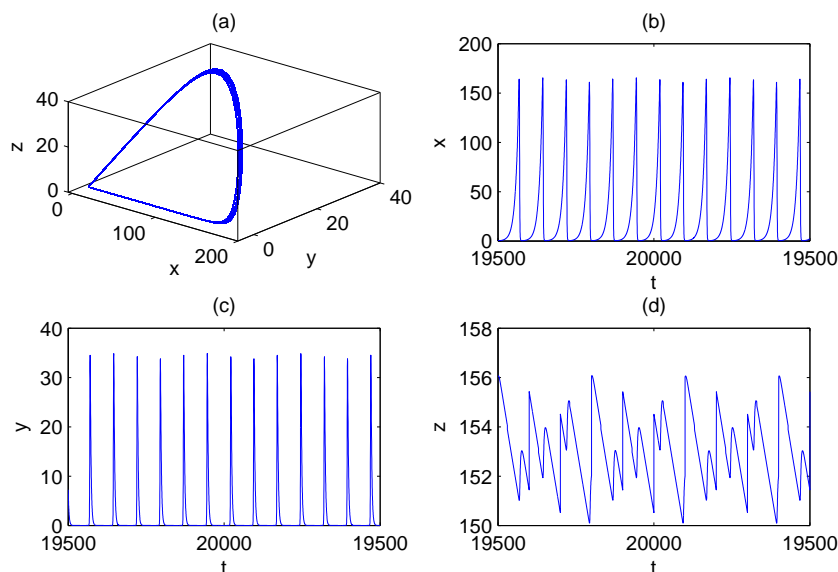
FIGURE 4. $a = 2, b = 0.0001, c_1 = 1.0, c_2 = 0.3, d_1 = 0.3, d_2 = 0.01, e_1 = 0.05, e_2 = 0.0025, p_1 = 0.01, p_2 = 0.02, p_3 = 0.001, \tau = 0.3, T = 5, q = 4.$
(a) The trajectory of the system (1.2). (b-d) Time series.

5. CONCLUSION

In this paper, we have studied a three species food chain system with Lokta-Volterra functional response and impulsive perturbations. We have found a condition for the local stabilities of a lower-level prey and mid-level predator free periodic solution and a mid-level predator free periodic solution by applying the Floquet theory and the comparison theorems and have proven the boundedness of this system. In addition, we have given numerical examples and suggested that the prey or predator free solutions may be globally stable and the system (1.2) may be permanent.

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