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ON THE ASYMPTOTIC CONVERGENCE OF ORTHONORMAL CARDINAL REFINABLE FUNCTIONS

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ABSTRACT. We prove an extended version of asymptotic behavior of the orthonormal cardinal refinable functions from Blaschke products introduced by Contronei et al [2]. In fact, we show the orthonormal cardinal refinable function $\varphi_{k,q}$ converges in $L^p(\mathbb{R})$ $(2 \leq p \leq \infty)$ to the Shannon refinable function as $k \to \infty$ uniformly on a class $\mathcal{Q}_{A,B}$ of real symmetric polynomials determined by positive constants $A \leq B$.

1. INTRODUCTION

Recently, Contronei et al. [2] constructed an interesting class of orthonormal cardinal refinable functions with rational symbols using Blaschke products. The rational symbols are of the form

$$P_{k,q}(z) := \frac{(1+z)^{2k+1}q(z)}{(1+z)^{2k+1}q(z) - (1-z)^{2k+1}q(-z)}, \quad z \in \mathbb{C}, \tag{1.1}$$

where

$$q(z) = \sum_{j=0}^{N/2} \alpha_{2j}(1+z)^{N-2j}(1-z)^{2j}, \quad z \in \mathbb{C}, \tag{1.2}$$

is a real symmetric polynomial of degree $N := 2n - 2k$, and the corresponding orthonormal cardinal refinable functions $\varphi_{k,q}$ are defined by the Fourier transform:

$$\widehat{\varphi}_{k,q}(w) := \prod_{l=1}^{\infty} P_{k,q}(e^{-iw/2^l}). \tag{1.3}$$

The study of refinable functions which are both orthonormal, and cardinal was addressed in [3, 6].

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We consider the class $\mathcal{Q}_{A,B}$ $(0 < A \le B < \infty)$ of all real symmetric polynomials q of (1.2) such that

$$A \le |Q(w)| \le B \text{ for all } w,$$

where Q is defined for $w \in \mathbb{R}$ as

$$Q(w) := \sum_{j=0}^{n-k} (-1)^j \alpha_{2j} \left(\cos^2 \frac{w}{2}\right)^{n-k-j} \left(\sin^2 \frac{w}{2}\right)^j .$$

Note that $\left|q\left(e^{-iw}\right)\right| = 2^{2(n-k)} |Q(w)|$. We show that the refinable function $\varphi_{k,q}$ converges in $L^p(\mathbb{R})$ $(2 \le p \le \infty)$ to the Shannon refinable function φ_{SH} uniformly on $q \in \mathcal{Q}_{A,B}$ as $k \to \infty$, where

$$\hat{\varphi}_{SH}(w) := \chi_{[-\pi,\pi]}(w).$$

The main result here is an extended version of [2, Thoerem 4.3]. We mention that the analogous asymptotic behaviors for other families of refinable functions are treated in [2, 4, 5] with similar proofs.

2. MAIN RESULT

For a real symmetric polynomial $q(z)$ with

$$0 < A \le 2^{-2(n-k)} |q(e^{-iw})| = |Q(w)| \le B < \infty, \tag{2.1}$$

we can easily check that as $k \to \infty$ the symbol

$$P_{k,q}(e^{-iw}) = \frac{1}{1 - i^{2k+1} (\tan \frac{w}{2})^{2k+1} q(-e^{-iw})/q(e^{-iw})}$$

converges pointwise to the symbol

$$m_{SH}(w) = \begin{cases} 1, & |w| < \pi/2, \\ 0, & \pi/2 < |w| < \pi \end{cases}$$

of the Shannon refinable function φ_{SH} as $k \to \infty$. We also note that for a fixed w with $|w| < \pi/2$ or $\pi/2 < |w| < \pi$ the convergence is uniform on the class $\mathcal{Q}_{A,B}$ of all real symmetric polynomial q satisfying (2.1).

Before the statement and proof of the main result, we need some technical lemmas. Fix a positive integer K, we define an auxiliary symbol

$$m_K(w) = \begin{cases} 1, & |w| \le \frac{\pi}{2}; \\ \dfrac{B}{A} \left(\dfrac{\cos^{2(2K+1)}(w/2)}{\cos^{2(2K+1)}(w/2) + \sin^{2(2K+1)}(w/2)} \right)^{1/2}, & \frac{\pi}{2} \le |w| \le \pi \end{cases}$$

for the domination of $P_{k,q}(e^{-iw})$.

Lemma 2.1. (a) $|P_{k,q}(e^{-iw})| \le m_K(w),\ k \ge K,\ q \in \mathcal{Q}_{A,B}$.
(b) $\widehat{\varphi}_K(w) := \prod_{l\in\mathbb{N}} m_K(w/2^l)$ *has the decay* $|\widehat{\varphi}_K(w)| \le C(1+|w|)^{-K-1/2+\log_2 B/A}$.
(c) $|P_{k,q}(e^{-iw}) - 1| \le \begin{cases} 1, & \textit{for all } w, \\ \dfrac{2B}{\pi A}|w|, & |w| \le \pi/2, \end{cases} \quad q \in \mathcal{Q}_{A,B}$.

Proof. (a) It is obtained by direct computation that

$$|P_{k,q}(e^{-iw})| = \left(\frac{(\cos^2 \frac{w}{2})^{2k+1} Q(w)^2}{(\cos^2 \frac{w}{2})^{2k+1} Q(w)^2 + (\sin^2 \frac{w}{2})^{2k+1} Q(w+\pi)^2} \right)^{1/2} \le 1, \text{ all } w.$$

For $\pi/2 \le |w| \le \pi$, $|P_{k,q}(e^{-iw})| \le m_K(w)$ by (2.1) if $k \ge K$ since $|P_{k,q}(e^{-iw})|$ is decreasing as k increases.
(b) We note that $m_K(w) = \cos^{2K+1}(w/2) S_K(w)$, where

$$S_K(w) = \begin{cases} \dfrac{1}{\cos^{2K+1}(w/2)}, & |w| \le \frac{\pi}{2} \\ \dfrac{B}{A} \left(\dfrac{1}{\cos^{2(2K+1)}(w/2) + \sin^{2(2K+1)}(w/2)} \right)^{1/2}, & \frac{\pi}{2} \le |w| \le \pi, \end{cases}$$

and note that $\sup_w |S_K(w)| = 2^K \max\{B/A, \sqrt{2}\} \le 2^{K+1/2} B/A$. Therefore, the decay of $\widehat{\varphi}_K(w)$ follows, for example, from [1, Theorem 5.5].
(c) We note that

$$|P(e^{-iw}) - 1|^2 = \frac{(\sin^2 \frac{w}{2})^{2k+1} Q(w+\pi)^2}{(\cos^2 \frac{w}{2})^{2k+1} Q(w)^2 + (\sin^2 \frac{w}{2})^{2k+1} Q(w+\pi)^2}.$$

The first estimate of (c) is obvious. For the second estimate of (c), we let $|w| \le \pi/2$ and note

$$\begin{aligned} |P(e^{-iw}) - 1|^2 &\le \frac{(\sin^2 \frac{w}{2})^{2k+1} Q(w+\pi)^2}{(\cos^2 \frac{w}{2})^{2k+1} Q(w)^2} \\ &\le \left(\frac{B}{A}\right)^2 (\tan^2 \frac{w}{2})^{2k+1} \le \left(\frac{B}{A}\right)^2 \tan^2 \frac{w}{2} \le \left(\frac{B}{A}\right)^2 \left(\frac{2}{\pi}|w|\right)^2. \end{aligned}$$

□

Lemma 2.2. (a) *For each fixed* w, $\widehat{\varphi}_{k,q}(w) = \prod_{l=1}^{\infty} P_{k,q}(e^{-iw/2^l})$ *converges uniformly on* k *and on* $q \in \mathcal{Q}_{A,B}$.
(b) *For* a.e. w, $\widehat{\varphi}_{k,q}(w) \to \widehat{\varphi}_{SH}(w)$ *uniformly on* $q \in \mathcal{Q}_{A,B}$ *as* $k \to \infty$.

Proof. (a) Fix w and choose l_0 so that $|w/2^{l_0}| \le \pi/2$. By Lemma 2.1 (c),

$$\sum_{l=1}^{\infty} |P_{k,q}(e^{-iw/2^l}) - 1| = \sum_{l=1}^{l_0} |P_{k,q}(e^{-iw/2^l}) - 1| + \sum_{l=l_0+1}^{\infty} |P_{k,q}(e^{-iw/2^l}) - 1|$$
$$\le l_0 + \sum_{l=l_0+1}^{\infty} \frac{2B}{\pi A} \frac{|w|}{2^l} = l_0 + \frac{2B}{\pi A} \frac{|w|}{2^{l_0}},$$

uniformly on k and on $q \in \mathcal{Q}_{A,B}$. Therefore, the product $\widehat{\varphi}_{k,q}(w)$ converges uniformly on k and on $q \in \mathcal{Q}_{A,B}$ for a fixed w.

(b) Fix $w \notin \cup_{l=1}^{\infty} 2^l(\pm\pi/2 + 2\pi\mathbb{Z})$ and let $\epsilon > 0$. By (a) we can choose l_1 (independent of k and $q \in \mathcal{Q}_{A,B}$) so that

$$|\widehat{\varphi}_{k,q}(w) - \prod_{l=1}^{l_1} P_{k,q}(e^{-iw/2^l})| < \epsilon$$

and

$$|\widehat{\varphi}_{SH}(w) - \prod_{l=1}^{l_1} m_{SH}(w/2^l)| < \epsilon.$$

Therefore, we have

$$|\widehat{\varphi}_{k,q}(w) - \widehat{\varphi}_{SH}(w)| \le \left|\widehat{\varphi}_{k,q}(w) - \prod_{l=1}^{l_1} P_{k,q}(e^{-iw/2^l})\right|$$
$$+ \left|\prod_{l=1}^{l_1} P_{k,q}(e^{-iw/2^l}) - \prod_{l=1}^{l_1} m_{SH}(w/2^l)\right|$$
$$+ \left|\prod_{l=1}^{l_1} m_{SH}(w/2^l) - \widehat{\varphi}_{SH}(w)\right|$$
$$< 2\epsilon + \left|\prod_{l=1}^{l_1} P_{k,q}(e^{-iw/2^l}) - \prod_{l=1}^{l_1} m_{SH}(w/2^l)\right|.$$

Note that $w/2^l \notin \pm\pi/2 + 2\pi\mathbb{Z}$ for any $l \ge 1$. Since $P_{k,q}(e^{-iw/2^l}) \to m_{SH}(w/2^l)$ as $k \to \infty$ for $l = 1, 2, \cdots, l_1$, we can choose k_0 so that

$$\left|\prod_{l=1}^{l_1} P_{k,q}(e^{-iw/2^l}) - \prod_{l=1}^{l_1} m_{SH}(w/2^l)\right| < \epsilon,\ k \ge k_0.$$

Therefore, for $a.e.\ w$, $\widehat{\varphi}_{k,q}(w) \to \widehat{\varphi}_{SH}(w)$ uniformly on $q \in \mathcal{Q}_{A,B}$ as $k \to \infty$.

□

Now, we state and prove our main result. The case $A = B = 1$ reduces to Theorem 4.3 in [2].

Theorem 2.3. *Let* $0 < A < B < \infty$.

(a) *For* $1 \leq p < \infty$, $||\widehat{\varphi}_{k,q} - \widehat{\varphi}_{SH}||_{L^p(\mathbb{R})} \to 0$ $(k \to \infty)$ *uniformly on* $q \in \mathcal{Q}_{A,B}$.

(b) *For* $2 \leq p' \leq \infty$, $||\varphi_{k,q} - \varphi_{SH}||_{L^{p'}(\mathbb{R})} \to 0$ $(k \to \infty)$ *uniformly on* $q \in \mathcal{Q}_{A,B}$ *. In particular,* $\varphi_{k,q} \to \varphi_{SH}$ *uniformly on* $\mathbb{R}$ *and on* $q \in \mathcal{Q}_{A,B}$

Proof. Choose K so large that $K + 1/2 - \log_2 B/A > 1$. We estimate the decay of $\widehat{\varphi}_{k,q}$ for $k \geq K$:

$$
\begin{aligned}
|\widehat{\varphi}_{k,q}(w)| &= \prod_{l\in\mathbb{N}} \left|P_{k,q}(e^{-iw/2^l})\right| \leq \prod_{l\in\mathbb{N}} m_K(w/2^l) \\
&= |\widehat{\varphi}_K(w)| \leq C(1+|w|)^{-K-1/2+\log_2 B/A} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}),
\end{aligned}
$$

where we used Lemma 2.1. We now apply the Lebesgue dominated convergence theorem to $\sup_{q\in Q_{A,B}} |\widehat{\varphi}_{k,q} - \widehat{\varphi}_{SH}|^p$ to get

$$
\sup_{q\in Q_{A,B}} ||\widehat{\varphi}_{k,q} - \widehat{\varphi}_{SH}||_{L^p(\mathbb{R})} \leq || \sup_{q\in Q_{A,B}} |\widehat{\varphi}_{k,q} - \widehat{\varphi}_{SH}| \, ||_{L^p(\mathbb{R})} \to 0
$$

as $k \to \infty$. Therefore, $\widehat{\varphi}_{k,q} \to \widehat{\varphi}_{SH}$ in $L^p(\mathbb{R})$ uniformly on $q \in \mathcal{Q}_{A,B}$. The claim (b) follows from (a) by the Hausdorff-Young inequality:

$$
||f||_{L^{p'}(\mathbb{R})} \leq ||\widehat{f}||_{L^p(\mathbb{R})}, \text{ for } 1 \leq p \leq 2,
$$

where p' is the exponent conjugate to p. □

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