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MULTIGRID CONVERGENCE THEORY FOR FINITE ELEMENT/FINITE VOLUME METHOD FOR ELLIPTIC PROBLEMS:A SURVEY

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ABSTRACT. Multigrid methods finite element/finite volume methods and their convergence properties are reviewed in a general setting. Some early theoretical results in simple finite element methods in variational setting method are given and extension to nonnested-noninherited forms are presented. Finally, the parallel theory for nonconforming element[13] and for cell centered finite difference methods [15, 23] are discussed.

1. INTRODUCTION

In this survey article, we discuss various multigrid methods finite element/finite volume methods and their convergence propoerties. Multigrid methods have been very active area of research since it was introduced in 1960's[19]. It is one of the most efficient algorithm for solving system of linear equations; especially for elliptic problems. In this article, we review the methods in a more general setting. Some early theoretical results in simple finite difference method are given in [2, 18, 20] and for a finite element setting Bank [1] has given a nice proof and Brandt gave an extensive experiment including nonlinear problem, eigenvalue problem, non elliptic problems and/or applications such as image processing. For other cases, we refer to [11, 4, 16, 17, 21, 23, 24]. An excellent theory for conforming finite element method is arranged in the series of paper by Bramble et. al.[5, 6, 7, 9, 8, 3]. Finally, the parallel theory for nonconforming elements[13] and for cell centered finite difference methods [15, 23] are discussed.

2. MULTIGRID ALGORITHM

We consider the following problem:

$$\begin{aligned} Lu &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a polygonal domain and L is a uniformly elliptic partial differential operator given by $Lu = -\nabla \cdot K\nabla u$ and $f \in L^2(\Omega)$. To discretize it we assume we have a sequence of finite

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dimensional space $M_k, k = 1, \cdots, J$ with the equipped inner product $(\cdot,\cdot)_k$ which may or may not be a subspace of $H_0^1(\Omega)$. As a result, we obtain the following algebraic equation:

$$A_k x_k = f_k. \tag{2}$$

We introduce a discrete Galerkin form

$$(A_k x_k, v)_k = (f_k, v)_k, \quad v \in M_k. \tag{3}$$

To accomplish the philosophy of multigrid idea, we also assume prolongation operator $I_k : M_{k-1} \to M_k$ is given and its adjoint P^0_{k-1} is defined by

$$(P^0_{k-1} x_k, \phi)_{k-1} = (x_k, I_k \phi)_k, \quad x_k \in M_k,\ \phi \in M_{k-1}. \tag{4}$$

For each level, we need to relax the residual by a smoother $R_k : M_k \to M_k$, for $k = 2, \cdots, J$. We take $R_1 = A_1^{-1}$ and we only consider the case R_k is symmetric; for nonsymmetric case, see [7].

MULTIGRID ALGORITHM.

Let y_k be an approximation to the solution of $A_k x_k = f_k$, we first define $B_1(y_1, f_1) = A_1^{-1} f_1$. For $k > 1$ define $B_k(y_k, f_k)$ recursively

(1) Set $x_k^0 = 0, q^0 = 0$.
(2) $x_k^\ell = x_k^{\ell-1} + R_k(f_k - A_k x_k^{\ell-1}), \quad \ell = 1, 2, \cdots, m$.
(3) $x_k^{m+1} = x_k^m + I_k q^p, \quad p = 1, 2$ where
$q^i = q^{i-1} + B_{k-1}(0, P^0_{k-1}[f_k - A_k x^m - A_k q^{i-1}])$.
(4) $x_k^\ell = x_k^{\ell-1} + R_k(f_k - A_k x_k^{\ell-1}), \quad \ell = m+2, \cdots, 2m+1$
(5) Set $B_k(y_k, f_k) = x_k^{2m+1}$.

When $p = 1$ it is a $V-$cycle, and when $p = 2$ it is a $W-$cycle. Then it is easy to see that

$$q^p = (I - I_k(I - B_{k-1}A_{k-1})^p) A_{k-1}^{-1} P^0_{k-1} A_k (x_k - x_k^m). \tag{5}$$

We shall derive a recurrence relation for $I - B_k A_k$. Let $K_k = I - R_k A_k$. First letting $P_{k-1} = A_{k-1}^{-1} P^0_{k-1} A_k$, we have

$$\begin{aligned} x_k - x_k^{m+1} &= x_k - x_k^m - q^p \\ &= (I - I_k B_{k-1} A_{k-1})^p A_{k-1}^{-1} P^0_{k-1} A_k (x_k - x_k^m) \\ &= (I - I_k B_{k-1} A_{k-1})^p P_{k-1} K_k^m x_k \end{aligned}$$

on M_k. Then

$$\begin{aligned} x_k - x_k^{2m+1} &= K_k^m (x - x_k^{m+1}) = K_k^m (I - I_k B_{k-1} A_{k-1})^p P_{k-1} K_k^m x_k \\ &= K_k^m [I - I_k P_{k-1} + I_k (I - B_{k-1} A_{k-1})^p] P_{k-1} K_k^m x_k \end{aligned}$$

so that

$$I - B_k A_k = K_k^m [I - I_k P_{k-1} + I_k (I - B_{k-1} A_{k-1})^p] P_{k-1} K_k^m.$$

3. CONVERGENCE THEORY FOR THE CONFORMING FINITE ELEMENT METHOD

In this section, we assume the space M_k consists of continuous functions which are piecewise linear on each element(triangles or rectangles) where each element in τ_k is divided by connecting mid point of its side to produce elements in τ_{k+1} so that all the spaces M_k are nested. Hence the transfer operator I_k is the natural injection operator. Also, let

$$a(u,v) = \int_\Omega K\nabla u \cdot \nabla v \, dx = (f,v), \quad v \in H_0^1(\Omega)$$

be the usual variational formulation. Obviously the following is satisfied:

(A.0): $(A_k I_k u, I_k u) = (A_{k-1}u, u), \ u \in M_{k-1}$.

The convergence proof in this case is based on the following assumptions.

(A.1): There exists some $0 < \alpha \leq 1$ such that

$$A((I - P_{k-1})u, u) \leq C_\alpha^2 \left(\frac{\|A_k u\|_k^2}{\lambda_k} \right)^\alpha A(u,u)^{1-\alpha} \text{ for all } u \in M_k,$$

(A.2): $\frac{\|u\|_k^2}{\lambda_k} \leq C_R(R_k u, u)_k$, for all $u \in M_k$,

where λ_k is the largest eigenvalue of A_k. The following result is in [5].

Theorem 3.1. *Let $p = 1$ and $m =$ constant. Then*

$$A((I - B_k^s A_k)u, u) \leq \delta_k A(u,u),$$

with $\delta_k = \frac{C_{\alpha,k}}{C_{\alpha,k} + m^\alpha}$.

This result can be improved if we use a product form of multigrid algorithm: We first consider presmoothing only with $p = 1$. Then

$$(I - B_k^n A_k) = [(I - P_{k-1}) + (I - B_{k-1}^n A_{k-1})P_{k-1}]K_k^m \text{ on } M_k.$$

For the improvement of result, we derive a recursive relation: Let $T_k = (I - K_k^m)P_k$. Then we have

$$I - B_k^n A_k P_k = [I - B_{k-1}^n A_{k-1} P_{k-1}](I - T_k).$$

The corresponding relation for symmetric smoother(with postsmoothing also) is given by

$$(I - B_k^s A_k) = K_k^m[(I - P_{k-1}) + (I - B_{k-1}^n A_{k-1})^p P_{k-1}]K_k^m \text{ on } M_k$$

so that

$$(I - B_J^s A_J) = (I - B_J^n A_J)^*(I - B_J^n A_J). \tag{6}$$

Then we have the following result.

Theorem 3.2. [8]*The V-cycle converges with $\delta_J = 1 - \frac{1}{CJ}$, with no regularity assumptions other than H^1.*

Proof. For the proof, we use

$$(I - B_J^n A_J) = (I - T_J)\cdots(I - T_0)$$

$$E_k = (I - T_k)E_{k-1},\ E_J = (I - B_J A_J)$$

and the following assumptions:

(A.3): There exists $Q_k : M_J \to M_k$ such that

$$\|(Q_k - Q_{k-1}u\|_k^2 \le C_1 \lambda_k^{-1} A((u,u), \quad k = 1, \cdots, J$$

(A.4): $A(Q_k u, Q_k u) \le C_2 A(u,u)_k, \quad k = 1, \cdots, J.$

□

With a more ingenuous analysis shows the convergence can be further improved where the convergence rate is independent of the number of levels:

Theorem 3.3. [3] *Let $u \in H^{1+\alpha}(\Omega)$ with $\alpha > 0$. Then the V-cycle with one smoothing converges with $\delta_J = 1 - \frac{1}{C}$.*

4. NONNESTED SPACE OR NONINHERITED FORMS.

In the case of nonconforming finite element where $M_k \not\subset H_0^1(\Omega)$ or finite difference methods we usually do not have 'variational equality', i.e, the following

$$A_k(I_k u, I_k v) = A_{k-1}(u,v), \quad u, v \in M_{k-1}$$

do not hold. This also happens when treating curved boundaries with straight edged element. The recurrence relation becomes

$$I - B_k A_k = K_k^m[(I - I_k P_{k-1}) + I_k(I - B_{k-1}A_{k-1})^p P_{k-1}]K_k^m.$$

Now the condition (A.0), (A.1) have to be replaced by

(B.0): $A_k(I_k u, I_k u) \le C_* A_{k-1}(u,u), \quad u \in M_{k-1}.$

(B.1): $A((I - I_k P_{k-1})u, u) \le C_\alpha^2 \left(\frac{\|A_k u\|_k^2}{\lambda_k}\right)^\alpha A(u,u)^{1-\alpha},\ u \in M_k.$

The typical finite difference method applied to the Laplace operator with linear or bilinear element give rise to a bilinear form which satisfy (B.0) with $C_* = 1$. In this case, we have V-cycle result.

Theorem 4.1. *If $C_* \le 1$, then the multigrid V-cycle algorithm converges with $\delta_k = \frac{C(\alpha,k)}{C(\alpha,k)+m^\alpha}$.*

W-cycle result

For general elliptic problem, however, FDM does not yields $C_* = 1$. In most cases C_* is strictly greater than 1. The same is true for nonconforming finite element. For these cases, we need to use $p = 2$(W-cycle).

Theorem 4.2. *Assume (B.0) and (B.1) hold and* $p = 2$. *Then if* m *is sufficiently large we have* $\delta_k = \frac{C}{C+m(k)^\alpha}$. *If in addition,*

$$A_k(I_k u, I_k u) \leq 2A_{k-1}(u, u)$$

holds then the W-cycle algorithm converge with the same δ *with* $m = 1$.

Nonconforming Elements. For various nonconforming elements even $C_* \leq 2$ do not holds generally. Thus, we investigate specific case in detail. Let $V_k, k = 1, \cdots, J$ denote the nonconforming finite element spaces. The variational for usually satisfies

$$a_k(I_k v, I_k v) \leq C_* a_{k-1}(v, v), \qquad \forall\, v \in V_{k-1}, \tag{7}$$

for some constant C. For triangular nonconforming element, it is known that $C_* > 2$. Hence W–cycle with large m only converges. However, for rectangular case, we have convergence for $m = 1$.

Theorem 4.3. [12] *The following estimate holds*

$$a_k(I_k v, I_k v) \leq 2a_{k-1}(v, v), \quad \forall v \in V_{k-1}$$

and thus the W-cycle converges with $m = 1$.

Cell Centered Method. Similar framework can be used to prove convergence of multigrid applied to the cell centered finite difference. First consider the following model problem:

$$-\nabla \cdot p\nabla\tilde{u} = f \text{ in } \Omega, \tag{8}$$

$$\tilde{u} = 0 \text{ on } \partial\Omega. \tag{9}$$

Integrating by parts on each cell, we obtain

$$-\int_{\partial E_{ij}^k} p\frac{\partial\tilde{u}}{\partial n} ds = \int_{E_{ij}^k} f dx \tag{10}$$

for $i, j = 1, \ldots, n$. We approximate (10) by

$$p\partial\tilde{u}/\partial n \approx p_{i,j+1/2}\frac{u_{i,j+1} - u_{i,j}}{h}$$

to obtain a system of linear equation of the form

$$\bar{A}_k\bar{u} = \bar{f}, \tag{11}$$

whose multgrid method was first considered in [4]. However, they used a natural operator whose $V-$convergence is very slow. In fact, they show

$$A(I_k^n v, I_k^n v) = 2A(v, v)$$

and, hence, only W-cycle works. However, we can design a new prolongation operator and improve the energy norm. Define a weighted prolongation operator $\mathcal{I}_{k-1}^k$ as follows:

$$\begin{aligned}(\mathcal{I}_{k-1}^k v)_{I,J} &= \frac{1}{4}(2v_{i,j} + v_{i,j+1} + v_{i+1,j}),\\ (\mathcal{I}_{k-1}^k v)_{I-1,J} &= \frac{1}{4}(2v_{i,j} + v_{i,j+1} + v_{i-1,j}),\\ (\mathcal{I}_{k-1}^k v)_{I,J-1} &= \frac{1}{4}(2v_{i,j} + v_{i+1,j} + v_{i,j-1}),\\ (\mathcal{I}_{k-1}^k v)_{I-1,J-1} &= \frac{1}{4}(2v_{i,j} + v_{i-1,j} + v_{i,j-1}).\end{aligned}$$

We can now prove the following crucial energy norm estimate.

Proposition 1. *For all $v \in V_{k-1}$, we have*

$$A_k(\mathcal{I}_{k-1}^k v, \mathcal{I}_{k-1}^k v) \le C(p) A_{k-1}(v,v), \tag{12}$$

where $C(p) = 1$ if the coefficient p is constant and $C(p) = 1 + O(h_k)$ if p is a general Lipschitz continuous function.

Lemma 4.1. *Regularity and approximation property holds for $\alpha = \frac{1}{2}$.*

With these preparations, we prove the following: [15]

Theorem 4.4. *With $E_k = I - B_k A_k$, we have the following: If p is constant, then $V-$cycle converges and we have*

$$0 \le A_k(E_k u, u) \le \delta_k A_k(u,u) \quad \forall u \in V_k \tag{13}$$

where $\delta_k = \frac{Ck}{Ck+\sqrt{m}}$.

Triangular case. Similar idea can be used to design the prolongation operator for triangular element: Observe

$$A_{k-1}(v,v) = \theta \sum_{i \ne j} (v_i - v_j)^2, \tag{14}$$

where the sum is taken for all pairs of adjacent triangles i and j. Let $u = I_k^t v$. Then

$$A_k(u,u) = \theta \sum_{I \ne J} (u_I - u_J)^2. \tag{15}$$

If we define I_k to be certain average, it can be shown that

$$A_k(I_k^t v, I_k^t v) = 2\theta \sum_{i \ne j} (v_i - v_j)^2.$$

We find that the result does not change even if we change the weight as long as the sum of coefficients is 1. For details, see [25].

5. SOME $V-$CYCLE THEORY FOR THE NONCONFORMING CASE

The $V-$cycle convergence with the nonconforming element has been computationally observed but the proof seemed to be an open problem for a while. It was partially resolved by Chen and later Chen and Kwak[13] with no regularity assumption.

The obstacle of the convergence proof is that we have $C_* > 1$ in (7). The idea is to judiciously introduce a new bilinear form so that (7) holds with $C_* = 1$.

Given finite element spaces

$$V_0,\, V_1, \ldots, V_J,$$

and coarse-to-fine grid operators $I_k : V_{k-1} \to V_k$ for $k = 1, \ldots, J$.

Assume a quadratic forms $a_J(\cdot,\cdot)$ on $V_J \times V_J$ and define the iterates $H_k^J = I_J \cdots I_{k+1} : V_k \to V_J$, of I_k and their adjoint Λ_J^k by

$$H_k^J = I_J \cdots I_{k+1} : V_k \to V_J, \quad k = 0, \ldots, J-1, \tag{16}$$

$$a_k(\Lambda_J^k v, w) = a_J(v, H_k^J w), \quad k = 0, \ldots, J. \tag{17}$$

Using a_J form, we define new quadratic forms $b_k(\cdot,\cdot)$ on $V_k \times V_k$ by

$$b_k(v,w) = a_J(H_k^J v, H_k^J w), \qquad \forall v,\, w \in V_k,\, k = 0, \ldots, J.$$

If we let

$$(A_k v, w)_k = b_k(v,w), \qquad \forall\, w \in V_k,\, k = 0, \ldots, J,$$

then we have

$$\begin{aligned}(A_k I_k v, I_k v) &= b_k(I_k v, I_k v)\\ &= a_J(H_k^J I_k v, H_k^J I_k v)\\ &= a_J(H_{k-1}^J v, H_{k-1}^J v)\\ &= b_{k-1}(v,v)\\ &= (A_{k-1} v, v).\end{aligned}$$

Theorem 5.1 (Chen 96)**.** *With certain regularity V-cycle MG for nonconforming element converges with* $\delta_k = \frac{C_\alpha}{C_\alpha + m^\alpha}$.

Sketch of proof: We define the operators $P_{k-1} : V_k \to V_{k-1}$ and $P_{k-1}^0 : V_k \to V_{k-1}$ by

$$b_{k-1}(P_{k-1} v, w) = b_k(v, I_k w), \qquad \forall\, w \in V_{k-1},\, k = 1, \ldots, J,$$

and

$$\left(P_{k-1}^0 v, w\right)_{k-1} = (v, I_k w)_k, \qquad \forall\, w \in V_{k-1},\, k = 1, \ldots, J.$$

As usual, we need "approx. Regularity" for $b_k(\cdot,\cdot)$ form:

$$|(A_k(I - I_k P_{k-1})v, v)_k| \le C h_k \|A_k v\|_k \|v\|_{1,k} \tag{18}$$

For a proof, we need properties of $a(\bar{P}_{k-1} v, w) = b_k(v, I_k w)$ and the following lemma.

Lemma 5.1.

$$\begin{aligned} C_1\|v\|_{\mathcal{E},k} &\leq \|v\|_{1,k} \leq C_2\|v\|_{\mathcal{E},k} \\ \|\Lambda_J^k v\|_{\mathcal{E},k} &\leq C\|v\|_{\mathcal{E},k} \\ \|v - H_k^H v\| &\leq Ch_k\|v\|_{\mathcal{E},k}. \end{aligned}$$

This result works for smooth problem. The case of rough coefficients with only one smoothing is provided below.

Theorem 5.2 (Chen, Kwak 96)**.** *(No regularity) For problems with large jump coefficients, V-cycle multigrid with one smoothing converges with* $\delta = 1 - \frac{1}{CJ}$.

For the proof, we have to resort to the product algorithm introduced in section 3. For that purpose we need several new operators. First define $\Pi_J^k : V_J \to V_k$ by

$$b_k(\Pi_J^k v, w) = b_J(v, H_k^J w), v \in V_j, w \in V_k.$$

Then the following properties hold.

Lemma 5.2. *It holds that*

$$P_{k-1}^0 A_k = A_{k-1} P_{k-1}, \tag{19}$$

$$P_{k-1} I_k = I \quad \text{on } V_{k-1}. \tag{20}$$

Lemma 5.3. *We have*

$$\Pi_J^{k-1} = P_{k-1}\Pi_J^k.$$

$$\Pi_J^k = P_k P_{k+1} \dots P_{J-1};$$

$$\Pi_J^k H_k^J = I \quad \text{on } V_k.$$

With the iterated transfer operators H_k^J and Π_J^k let $S^k = H_k^J(I - J_k^{m(k)})\Pi_J^k$. Then we have

$$I - H_k^J B_k^n A_k \Pi_J^k = (I - H_{k-1}^J B_{k-1}^n A_{k-1} \Pi_J^{k-1})(I - S^k).$$

Therefore, by induction we obtain

$$I - B_J^n A_J = (I - S^0)\dots(I - S^J).$$

Denote the norm induced by the a_k form by $\|\cdot\|_{\mathcal{E},k}$. It has been shown [12] that the norms $\|\cdot\|_{\mathcal{E},k}$ and $\|\cdot\|_{1,k}$ are equivalent:

$$C_5\|v\|_{\mathcal{E},k} \leq \|v\|_{1,k} \leq C_6\|v\|_{\mathcal{E},k}, \quad \forall v \in V_k,\ k = 0, \dots, J, \tag{21}$$

with C_5 and C_6 independent of k . Assume the existence of $Q_J^k : V_J \to V_k$ such that $Q_J^k H_k^J v = v$ and

$$\|Q_J^k v\|_{1,k} \leq C\|v\|_{1,J}.$$

Later, Q_J^k will be given as a product of certain projection operators T_k.

Lemma 5.4. *There exist constants C independent of k such that*

$$\begin{aligned} \|Q_J^k v\|_{\mathcal{E},k} &\leq C\|v\|_{\mathcal{E},J}, \quad \forall v \in V_J, &(22)\\ \|T_{k-1} v\|_{\mathcal{E},k-1} &\leq C\|v\|_{\mathcal{E},k}, \quad \forall v \in V_k. &(23)\end{aligned}$$

Now we apply product form of multigrid algorithm: For that purpose, we need to verify two conditions:

$$\begin{aligned} \|(Q_J^k - I_k Q_J^{k-1})v\|_k^2 &\leq C_1 \lambda_k^{-1} b_J(v,v), \quad k = 1, \cdots, J, &(24)\\ b_k(Q_J^k v, Q_J^k v) &\leq C_2 b_J(v,v), \quad k = 0, \cdots, J-1. &(25)\end{aligned}$$

(25) follows from (22); To prove (24), observe that

$$Q_J^k - I_k Q_J^{k-1} = (I - I_k T_{k-1}) Q_J^k.$$

By approximation properties of I_k and T_{k-1}

$$\begin{aligned} \|(Q_J^k - I_k Q_J^{k-1})v\|_k &= \|(I - I_k T_{k-1}) Q_J^k v\| \\ &\leq C\big(\|(I - T_{k-1}) Q_J^k v\| + \|(I - I_k) T_{k-1} Q_J^k v\|\big) \\ &\leq C h_k \|v\|_{\mathcal{E},J}, \quad v \in V_J. \end{aligned}$$

This, together with (21), implies the assumption (24). With these preparations, we can prove the result using similar frame in section 3.

Remark 5.1. *Similar result holds for nonsymmetric problem*[14].

Properties of transfer operator. First we consider triangular nonconforming element. We consider two sets of intergrid transfer operators $I_k : V_{k-1} \to V_k$ and $T_{k-1} : V_k \to V_{k-1}$ as follows.

$$(I_k v)(q) = \begin{cases} 0 & \text{if } q \in \Gamma, \\ v(q) & \text{if } q \notin \partial E \text{ for any } E \in \mathcal{E}_{k-1}, \\ \frac{1}{2}\{v|_{E_1}(q) + v|_{E_2}(q)\} & \text{if } q \in \partial E_1 \cap \partial E_2. \end{cases}$$

If $v \in V_k$ and q is the midpoint of an edge e of a triangle in $\mathcal{E}_{k-1}$, following [12], we define $T_{k-1}v \in V_{k-1}$ by

$$(T_{k-1} v)(q) = \frac{1}{2}(v(q_1) + v(q_2)),$$

where q_1 and q_2 are the respective midpoints of the edges e_1 and e_2 in $\mathcal{E}_k$, which form the edge e in $\mathcal{E}_{k-1}$. Note that the definition of T_{k-1} automatically preserves the zero nodal values on boundary edges. Also, it can be seen that

$$T_{k-1} I_k v = v, \qquad v \in V_{k-1},\ k = 1, \ldots, J. \tag{26}$$

As in [12], the iterated intergrid transfer operators

$$\begin{aligned} H_k^J &= I_J \cdots I_{k+1} : V_k \to V_J, \quad k = 0, \ldots, J-1, &(27)\\ Q_J^k &= T_k \cdots T_{J-1} : V_J \to V_k, \quad k = 0, \ldots, J. &(28)\end{aligned}$$

satisfies

$$Q_J^k H_k^J v = v, \qquad v \in V_k.$$

Similar result holds in the case of the Q_1-nonconforming element.

Remark 5.2. *Finally, one can consider a perturbed problem. Perturbation comes from a variety of sources: Nonsymmetric term, numerical integration, finite difference methods, treating curved boundaries, nonconforming element, etc. These cases can be handled almost universally by introducing certain perturbed bilinear form and estimating the energy norm appropriately and $W-$ cycle can be shown to converge if the unperturbed problem converge. See* [22] *for details.*

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