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INEQUALITIES OF OPERATOR POWERS

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ABSTRACT. Duggal-Jeon-Kubrusly([2]) introduced Hilbert space operator T satisfying property $|T|^2 \leq |T^2|$, where $|T| = (T^*T)^{1/2}$. In this paper we extend this property to general version, namely property $B(n)$. In addition, we construct examples which distinguish the classes of operators with property $B(n)$ for each $n \in \mathbb{N}$.

1. INTRODUCTION

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator T in $\mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* if $(T^*T)^p - (TT^*)^p \geq 0$, $p \in (0, \infty)$. If $p = 1$, then T is *hyponormal*. The Löwner-Heinz inequality([3]) implies that every p-hyponormal operator is a q-hyponormal operator for $0 < q \leq p$. In particular, T is said to be *∞-hyponormal* if T is p-hyponormal for every $p > 0$ ([7]). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an *$A(p)$-operator* if $(T^* |T|^{2p} T)^{1/(p+1)} \geq |T|^2$ $(0 < p < \infty)$ where $|T| = (T^*T)^{1/2}$. It is well known that every p-hyponormal is $A(p)$-operator([3]).

In [2], Duggal-Jeon-Kubrusly studied operators T on $\mathcal{H}$ satisfying property

$$|T|^2 \leq |T^2|. \tag{1}$$

In this paper we extend this property to a general version, namely property $B(n)$ whose definition will be introduced in Section 3. The operator satisfying (1) will be equivalent to property $B(2)$. It follows from (1) that an operator T in $\mathcal{L}(\mathcal{H})$ has property $B(2)$ if and only if T is $A(1)$-operator. Because only a few examples for property (1) have been known, it is worthwhile to find such examples.

In this paper, we construct examples which distinguish property $B(n)$ of an operator in $\mathcal{L}(\mathcal{H})$ for each $n \geq 2$. Also, we see the relationships between property $B(n)$ for each $n \geq 2$ and hyponormality of an operator T on $\mathcal{H}$ from some simple examples. In addition, we show mutually disjoint ranges of property $B(n)$ for $n \geq 2$ of operator in the 2-dimensional space.

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2. PROPERTY $\mathbf{B(n)}$

For $n \geq 2$, an operator $T \in \mathcal{L}(\mathcal{H})$ has the *property* $B(n)$ if $|T^n| \geq |T|^n$. If T is a p-hyponormal operator for $p > 0$, then $T^{*n}T^n \geq (T^*T)^n$ for all positive integer $n < p([8])$. Hence we have the following proposition.

Proposition 2.1. *If T is ∞-hyponormal, then T has property $B(n)$ for all $n \geq 2$.*

Proof. Since T is ∞-hyponormal, T is $(n+1)$-hyponormal for all $n \in \mathbb{N}$. By the above known result, we have that $T^{*n}T^n \geq (T^*T)^n$, which implies that $|T^n|^2 \geq |T|^{2n}$. By Löwner-Heinz inequality([3]), $|T^n| \geq |T|^n$. Thus, T has property $B(n)$. □

Theorem 2.2. *Let W_α be a weighted shift with weight sequence $\alpha = \{\alpha_k\}_{k=0}^{\infty}$. Then W_α has property $B(n)$ if and only if*

$$|\alpha_{k+1}| \cdot |\alpha_{k+2}| \cdots |\alpha_{k+n-1}| \geq |\alpha_k|^{n-1}$$

for all $k = 0, 1, \cdots$.

Proof. If W_α has property $B(n)$, then we have $|{W_\alpha}^n| \geq |W_\alpha|^n$. Hence by simple computation, we have

$$|{W_\alpha}^n|^2 = (W_\alpha^*)^n {W_a}^n = \text{Diag}\{|\alpha_0\alpha_1 \cdots \alpha_{n-1}|^2, |\alpha_1\alpha_2 \cdots \alpha_n|^2, |\alpha_2\alpha_3 \cdots \alpha_{n+1}|^2, ...\}$$

and

$$|W_\alpha|^{2n} = (W_\alpha^* W_\alpha)^n = \text{Diag}\{|\alpha_0|^{2n}, |\alpha_1|^{2n}, |\alpha_2|^{2n}, ...\}.$$

Thus W_α has property $B(n)$, which is equivalent to $|\alpha_k\alpha_{k+1} \cdots \alpha_{k+n-1}| \geq |\alpha_k|^n$ $(k \geq 0)$, i.e.,

$$|\alpha_{k+1}| \cdot |\alpha_{k+2}| \cdots |\alpha_{k+n-1}| \geq |\alpha_k|^{n-1} \quad (k \geq 0).$$

This completes the proof. □

Corollary 2.3. *Let W_α be a weighted shift with weight sequence α. Then we have the following statements.*
(i) W_α has property $B(2)$ if and only if W_α is hyponormal.
(ii) If W_α is hyponormal, then W_α has property $B(n)$ for all $n \geq 2$.

Proof. (i) By the Theorem 2.2 and the fact in [1], we may have that

$$\begin{aligned} W_\alpha \text{ has property } B(2) &\Longleftrightarrow \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \\ &\Longleftrightarrow W_\alpha \text{ is hyponormal.} \end{aligned}$$

(ii) Using the above fact (i), we can easily obtain that

$$|\alpha_{k+1}| \cdot |\alpha_{k+2}| \cdots |\alpha_{k+n-1}| \geq |\alpha_k|^{n-1} \ (k \geq 0).$$

By Theorem 2.2, W_α has property $B(n)$ for each $n \geq 2$. □

Theorem 2.4. *Let W_α be a weighted shift with weight sequence*

$$\alpha : x \equiv \alpha_0,\ y \equiv \alpha_1, \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq \cdots \leq \alpha_n \leq \cdots$$

for $x,\ y \geq 0$ and $\alpha_2 > 0$. For all $n \geq 2$, if we set

$$\mathcal{B}_n := \{(x, y) :\ W_\alpha \text{ has property } B(n)\},$$

then we have
(i) $\mathcal{B}_n = \left\{(x, y) :\ 0 \leq x \leq (\alpha_2\alpha_3 \cdots \alpha_{n-1}y)^{\frac{1}{n-1}},\ 0 \leq y \leq (\alpha_2\alpha_3 \cdots \alpha_n)^{\frac{1}{n-1}}\right\}$,
(ii) $\mathcal{B}_m \subsetneq \mathcal{B}_n$ *for* $2 \leq m < n$,
(iii) $\bigcap_{n=2}^{\infty} \mathcal{B}_n = \{(x, y) : 0 \leq x \leq y \leq \alpha_2\}$.

Proof. (i) By Theorem 2.2 and the condition of $0 < \alpha_k \leq \alpha_{k+1}$ for all $k \geq 2$, we have that W_α has property $B(n)$, which is equivalent to $\alpha_1\alpha_2 \cdots \alpha_{n-1} \geq \alpha_0^{n-1}$ and $\alpha_2\alpha_3 \cdots \alpha_n \geq \alpha_1^{n-1}$, and that is

$$\alpha_2 \cdots \alpha_{n-1}y \geq x^{n-1} \text{ and } 0 \leq y \leq (\alpha_2\alpha_3 \cdots \alpha_n)^{\frac{1}{n-1}}$$

for each $n \geq 2$.

(ii) Put $f(n, x) := \frac{x^{n-1}}{\alpha_2\alpha_3\cdots\alpha_{n-1}}$ for all $n \geq 2$ and $x > 0$. Then

$$\frac{\partial f(n, x)}{\partial x} = \frac{(n-1)x^{n-2}}{\alpha_2\alpha_3 \cdots \alpha_{n-1}} > 0 \text{ and } \frac{\partial^2 f(n, x)}{\partial x^2} = \frac{(n-1)(n-2)x^{n-3}}{\alpha_2\alpha_3 \cdots \alpha_{n-1}} \geq 0$$

for all $n \geq 2$ and $x > 0$. So the function $f(n, x)$ is strictly increasing function about $x > 0$ and for all $n \geq 2$.
Suppose $2 \leq m < n$. For $0 < x < (\alpha_m\alpha_{m+1} \cdots \alpha_{n-1})^{\frac{1}{n-m}}$, we have

$$f(n, x) - f(m, x) = \frac{x^{m-1}}{\alpha_2 \cdots \alpha_{m-1}}\left(\frac{x^{n-m}}{\alpha_m\alpha_{m+1} \cdots \alpha_{n-1}} - 1\right) < 0.$$

i.e. $f(m, x) > f(n, x)$ for $2 \leq m < n$ and $x \in (0, (\alpha_m\alpha_{m+1} \cdots \alpha_{n-1})^{\frac{1}{n-m}})$.
Let we set $a_n := (\alpha_2\alpha_3 \cdots \alpha_n)^{\frac{1}{n-1}}$ for each $n \geq 2$. Then, using the assumption $0 < \alpha_k \leq \alpha_{k+1}$ for all $k \geq 2$, we obtain that

$$\begin{aligned} a_{n+1} - a_n &= (\alpha_2\alpha_3 \cdots \alpha_{n+1})^{\frac{1}{n}} - (\alpha_2\alpha_3 \cdots \alpha_n)^{\frac{1}{n-1}} \\ &= (\alpha_2 \cdots \alpha_n)^{\frac{1}{n}}[\alpha_{n+1}^{\frac{1}{n}} - (\alpha_2 \cdots \alpha_n)^{\frac{1}{n(n-1)}}] \\ &\geq (\alpha_2 \cdots \alpha_n)^{\frac{1}{n}}[\alpha_{n+1}^{\frac{1}{n}} - \alpha_n^{\frac{1}{n}}] \geq 0. \end{aligned}$$

Therefore the sequence $\{(\alpha_2\alpha_3 \cdots \alpha_n)^{\frac{1}{n-1}} : n = 2, 3, ...\}$ is an increasing sequence. Since $\mathcal{B}_n = \{(x, y) : 0 \leq f(n, x) \leq y,\ 0 \leq y \leq a_n\}$ for each $n \geq 2$, we completes the proof of (ii).

(iii) From the facts (i) and (ii), the assertion (iii) is obvious. $\square$

Remark 2.5. For the weighted shift W_α in Theorem 2.4, we note the following facts:

$$\begin{aligned} W_\alpha \text{ is } \infty\text{-hyponormal} &\iff W_\alpha \text{ is hyponormal} \\ &\iff 0 \le x \le y \text{ and } 0 \le y \le \alpha_2 \\ &\iff W_\alpha \text{ has the property } B(n) \text{ for all } n \ge 2. \end{aligned}$$

In general, but the converse of Proposition 2.1 is not true (see Example 3.3).

3. Examples

The following example explains that for a weighted shift W_α with weight sequence α, there is no relation with the property $B(n)$ and $B(m)$ for $m,\ n > 2$ with $m \neq n$.

Example 3.1. Consider a positive bounded sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$,

$$\alpha_0 = \frac{2}{3},\ \alpha_1 = \frac{40}{81},\ \alpha_2 = \frac{9}{10},\ \alpha_3 = \frac{16000}{59049},\ \alpha_4 = \frac{4782969}{1600000},\ \alpha_{n+1} = \alpha_n + \frac{1}{n^2}\ (n \ge 4).$$

Let W_α be the weighted shift with the above weight sequence α. Then W_α has property $B(3)$ but not property $B(4)$. In fact, from simple calculations, we have $\alpha_k^2 = \alpha_{k+1}\alpha_{k+2}$ $(k = 0, 1, 2)$ and $\alpha_k^2 \le \alpha_{k+1}\alpha_{k+2}$ for all $k \ge 3$. So W_α satisfies property $B(3)$. But $\alpha_0^3 = \frac{8}{27} > \alpha_1\alpha_2\alpha_3 = \frac{64000}{531441}$. Therefore W_α does not satisfy property $B(4)$.

For the distinction of property $B(n)$, we introduce the following example which classify them clearly for each $n \ge 2$.

Example 3.2. Let W_α be the Bergman shift with weight sequence

$$\alpha : \sqrt{x}, \sqrt{y}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \sqrt{\frac{5}{6}}, \ldots, \sqrt{\frac{k+1}{k+2}}, \ldots\ (k \ge 2).$$

Then by Theorem 2.2 we may obtain the following assertion:

$$W_\alpha \text{ has property } B(n) \Leftrightarrow 0 \le x \le \left(\frac{3y}{n+1}\right)^{\frac{1}{n-1}} \text{ and } 0 \le y \le \left(\frac{3}{n+2}\right)^{\frac{1}{n-1}}$$

for each $n \ge 2$. Hence

$$\mathcal{B}_n = \left\{ (x, y) |\ 0 \le x \le (\frac{3y}{n+1})^{\frac{1}{n-1}},\ 0 \le y \le (\frac{3}{n+2})^{\frac{1}{n-1}} \right\}.$$

Now, we claim that $\mathcal{B}_m \subsetneq \mathcal{B}_n$ for $2 \le m < n$. First, we write $f(n, x) := \frac{n+1}{3}x^{n-1}$ for all $n \ge 2$ and $x > 0$. By the derivative of $f(n, x)$ about x, we can see that the function $f(n, x)$ is strictly increasing function about x for all $n \ge 2$. Suppose $2 \le m < n$. For $0 < x < (\frac{m+1}{n+1})^{\frac{1}{n-m}}$, we have

$$f(n, x) - f(m, x) = \frac{1}{3}x^{m-1}\Big(x^{n-m}(n+1) - (m+1)\Big) < 0,$$

which is that, for $2 \le m < n$, $f(m, x) > f(n, x)$ on $(0, (\frac{m+1}{n+1})^{\frac{1}{n-m}})$.

Also, by simple calculations, we have that the sequence $\{(\frac{3}{n+2})^{\frac{1}{n-1}} : n = 2, 3, ...\}$ is an increasing sequence and converges to 1. Therefore we have $\mathcal{B}_m \subsetneq \mathcal{B}_n$. In fact, we can easily show the disjoint ranges of properties $B(n)$ for each $n \geq 2$ of W_α by usual way.

For each integer $n \geq 2$, we consider the following block matrix of operators in [6] and [5].

Example 3.3. Let $C = (c_{ij})$ be an $m \times m$ matrix with $c_{ij} = 1/m$ $(1 \leq i, j \leq m)$ and let $D \equiv D(x_1, x_2, ..., x_m) := \text{Diag}\{x_1, x_2, ..., x_m\}$ with $x_i \geq 0$, $i = 1, ..., m$. We define an operator $T(x_1, x_2, ..., x_m)$ on $\mathcal{H} \equiv \mathbb{C}^m \otimes \ell_2(\mathbb{Z})$ by

$$T := T(x_1, x_2, ..., x_m) = \begin{pmatrix} \ddots & & & & & & \\ \ddots & O & & & & & \\ & C & O & & & & \\ & & C & \boxed{O} & & & \\ & & & D & O & & \\ & & & & D & O & \\ & & & & & \ddots & \ddots \end{pmatrix},$$

where $\boxed{\cdot}$ denotes the center of the two sided infinite matrix. We note that $C^p = C$ for every $p > 0$. By simple calculations, we have that

$$(CD^kC)^{\frac{1}{2}} = \sqrt{\frac{x_1^k + x_2^k + ... + x_m^k}{m}}\, C$$

and

$$(CD^kC)^{\frac{1}{2}} - C \geq 0 \Leftrightarrow x_1^k + x_2^k + ... + x_m^k \geq m,$$

for $x_i \geq 0$ $(i = 1, 2, ..., m)$ and $k \geq 1$. Hence T has property $B(n) \Leftrightarrow (T^{n*}T^n)^{\frac{1}{2}} \geq (T^*T)^{\frac{n}{2}}$, which is equivalent to

$$(CD^2C)^{\frac{1}{2}} \geq C,\ (CD^4C)^{\frac{1}{2}} \geq C, ..., (CD^{2(n-1)}C)^{\frac{1}{2}} \geq C.$$

For an integer $n \geq 2$, we denote

$$\mathcal{E}_n = \{(x_1, x_2, ..., x_m) : T \text{ has property } B(n) \text{ for } x_i \geq 0\}.$$

If x_i satisfy $x_i \geq m^{\frac{1}{2(l-1)}}$ for some i, then $x_1^{2(l-1)} + x_2^{2(l-1)} + ... + x_m^{2(l-1)} \geq m$. So we have that $x_1^{2(n-1)} + x_2^{2(n-1)} + ... + x_m^{2(n-1)} \geq m$ for $2 \leq l < n$. Suppose $0 < x_i < m^{\frac{1}{2(n-1)}}$ for all $i = 1, 2, ..., m-1$ for all $n \geq 2$. Then we obtain that the function

$$\phi_m(n, x_1, x_2, ..., x_{m-1}) := (m - x_1^{2(n-1)} - x_2^{2(n-1)} - \cdots - x_{m-1}^{2(n-1)})^{\frac{1}{2(n-1)}}$$

is strictly decreasing with respect to all $n \geq 2$ on $(0, m^{\frac{1}{2(n-1)}}) \times \cdots \times (0, m^{\frac{1}{2(n-1)}})$ (see [6] and [5] for the detail methods). Therefore we have

$$\begin{aligned}\mathcal{E}_n &= \{(x_1, ..., x_m) : (CD^{2(j-1)}C)^{\frac{1}{2}} \geq C,\ 2 \leq j \leq n,\ x_i \geq 0,\ 1 \leq i \leq m\} \\ &= \bigcap_{2 \leq j \leq n} \{(x_1, ..., x_m) : x_1^{2(j-1)} + x_2^{2(j-1)} + ... + x_m^{2(j-1)} \geq m,\ x_i \geq 0,\ 1 \leq i \leq m\} \\ &= \mathcal{E}_2.\end{aligned}$$

Moreover, we have that

$$\begin{aligned}\mathcal{E}_2 &= \{(x_1, ..., x_m) : x_1^2 + x_2^2 + ... + x_m^2 \geq m,\ x_i \geq 0,\ 1 \leq i \leq m\} \\ &= \{(x_1, ..., x_m) : T \text{ is } A(1)\text{-operator}\}\end{aligned}$$

and T is ∞-hyponormal (see [5]). Therefore we have this implication: T is ∞-hyponormal $\Rightarrow$ T is hyponormal $\Rightarrow$ T has property $B(2)$, and the converse is not true.

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