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CONVERGENCE OF APPROXIMATING FIXED POINTS FOR MULTIVALUED NONSELF-MAPPINGS IN BANACH SPACES

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Abstract. Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $T : C \to \mathcal{K}(E)$ a multivalued nonself-mapping such that P_T is nonexpansive, where $P_T(x) = \{u_x \in Tx : \|x - u_x\| = d(x, Tx)\}$. For $f : C \to C$ a contraction and $t \in (0, 1)$, let x_t be a fixed point of a contraction $S_t : C \to \mathcal{K}(E)$, defined by $S_t x := tP_T(x) + (1-t)f(x),\ x \in C$. It is proved that if C is a nonexpansive retract of E and $\{x_t\}$ is bounded, then the strong $\lim_{t\to 1} x_t$ exists and belongs to the fixed point set of T. Moreover, we study the strong convergence of $\{x_t\}$ with the weak inwardness condition on T in a reflexive Banach space with a uniformly Gâteaux differentiable norm. Our results provide a partial answer to Jung's question.

1. Introduction

Let E be a Banach space and C a nonempty closed subset of E. We shall denote by $\mathcal{F}(E)$ the family of nonempty closed subsets of E, by $\mathcal{CB}(E)$ the family of nonempty closed bounded subsets of E, by $\mathcal{K}(E)$ the family of nonempty compact subsets of E, and by $\mathcal{KC}(E)$ the family of nonempty compact convex subsets of E. Let $H(\cdot, \cdot)$ be the Hausdorff distance on $\mathcal{CB}(E)$, that is,

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}$$

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for all $A,\ B \in \mathcal{CB}(E)$, where $d(a, B) = \inf\{\|a - b\| : b \in B\}$ is the distance from the point a to the subset B. Recall that a mapping $f : C \to C$ is a *contraction* on C if there exists a constant $k \in (0, 1)$ such that $\|f(x)-f(y)\| \leq k\|x-y\|,\ x,\ y \in C$. We use Σ_C to denote the collection of mappings f verifying the above inequality. That is, $\Sigma_C = \{f : C \to C \mid f$ is a contraction with constant $k\}$. Note that each $f \in \Sigma_C$ has a unique fixed point in C.

A multivalued mapping $T : C \to \mathcal{F}(E)$ is said to be a *contraction* if there exists a constant $k \in [0, 1)$ such that

$$H(Tx, Ty) \leq k\|x - y\| \tag{1}$$

for all $x,\ y \in C)$. If (1) is valid when $k = 1$, the T is called nonexpansive. A point x is a fixed point for a multi-valued mapping T if $x \in Tx$. Banach's Contraction Principle was extended to a multivalued contraction by Nadler [18] in 1969. The set of fixed points is denoted by $F(T)$.

Given a $f \in \Sigma_C$ and a $t \in (0, 1)$, we can define a contraction $G_t : C \to \mathcal{K}(C)$ by

$$G_t x := tTx + (1 - t)f(x),\ \ x \in C. \tag{2}$$

Then G_t is a multivalued and hence it has a (non-unique, in general) fixed point $x_t := x_t^f \in C$ (see [18]): that is

$$x_t \in tTx_t + (1 - t)f(x_t). \tag{3}$$

If T is single valued, we have

$$x_t = tTx_t + (1 - t)f(x_t). \tag{4}$$

A special case of (4) has been considered by Browder [2] in a Hilbert space as follows. Fix $u \in C$ and define a contraction G_t on C by

$$G_t x = tTx + (1 - t)u, \quad x \in C.$$

Let $z_t \in C$ be the unique fixed point of G_t. Thus

$$z_t = tTz_t + (1 - t)u. \tag{5}$$

(Such a sequence $\{z_t\}$ is said to be an approximating fixed point of T since it possesses the property that if $\{x_t\}$ is bounded, then $\lim_{t\to 1} \|Tz_t - z_t\| = 0$.) The strong convergence of $\{z_t\}$ as $t \to 1$ for a single-valued nonexpansive self or non-self mapping T was studied in Hilbert space or certain Banach spaces by many authors (see for instance, Browder [2], Halpern [8], Jung and Kim [11], Jung and Kim [12], Kim and Takahashi

[13], Reich [26], Singh and Waston [23], Takahashi and Kim [30], Xu [32], and Xu and Yin [36]).

In 1967, Borwder [2] proved the following.

Theorem B. *([2]). In a Hilbert space, as $t \to 1$, z_t defined by* (5) *converges strongly to a fixed point of T that is closest to u, that is, the nearest point projection of u onto $F(T)$.*

However, Pietramala [19] (see also Jung [10]) provided an example showing that Browder's theorem [2] cannot be extended to the multivalued case without adding an extra assumption even if E is Euclidean. López Acedo and Xu [15] gave the strong convergence of $\{x_t\}$ defined by $x_t \in tTx_t + (1-t)u$, $u \in C$ under the restriction $F(T) = \{z\}$ in Hilbert space. Kim and Jung [14] extended the result of López Acedo and Xu [15] to a Banach space with a weakly sequentially continuous duality mapping. Sahu [20] also studied the multi-valued case in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Recently, Jung [10] gave the strong convergence of $\{x_t\}$ defined by $x_t \in tTx_t + (1-t)u$, $u \in C$ for the multivalued nonexpansvie nonself-mapping T in a uniformly convex or reflexive Banach space having a uniformly Gâteaux differentiable norm and mentioned that the condition $F(T) = \{z\}$ should be added in the main results of Sahu [20]. More precisely, he established the following extensions of Browder's theorem [2].

Theorem J1. ([10]). *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $T : C \to \mathcal{K}(E)$ a nonexpansive nonself-mapping. Suppose that C is a nonexpansive retract of E. Suppose that $T(y) = \{y\}$ for any fixed point y of T and that for each $u \in C$ and $t \in (0,1)$, the contraction G_t defined by $G_t x := tTx + (1-t)u$, $x \in C$. has a fixed point $x_t \in C$. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 1$ and in this case, $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T.*

Theorem J2. ([10]). *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $T : C \to \mathcal{KC}(E)$ a nonexpansive nonself-mapping satisfying the inwardness condition. Assume that every closed bounded* *convex subset of C is compact. If the fixed point set $F(T)$ of T is nonempty and $Ty = \{y\}$ for any $y \in F(T)$, then the sequence $\{x_t\}$ defined by $x_t \in tTx_t + (1-t)u$, $u \in C$ converges strongly as $t \to 1$ to a fixed point of T.*

Very recently, in order to give a partial answer to Jung's open question [10]: *Can the assumption $Tz = \{z\}$ in Theorem J1 and J2 be omitted ?*, Shahzad and Zegeye [21] considered a class of multivalued mapping under some mild conditions as follows.

Let C be a closed convex subset of a Banach space E. Let $T : C \to \mathcal{K}(E)$ be a multivalued nonself-mapping and

$$P_T x = \{u_x \in Tx : \|x - u_x\| = d(x, Tx)\}.$$

Then $P_T : C \to \mathcal{K}(E)$ is multivalued and $P_T x$ is nonempty and compact for every $x \in C$. Instead of

$$G_t x = tTx + (1-t)u, \quad u \in C, \tag{6}$$

we consider for $t \in (0, 1)$,

$$S_t x = tP_T x + (1-t)u, \quad u \in C, \tag{7}$$

It is clear that $S_t x \subseteq G_t x$ and if P_T is nonexpansive and T is weakly inward, then S_t is weakly inward contraction. Theorem 1 of Lim [16] guarantees that S_t has a fixed point point x_t, that is,

$$x_t \in tP_T x_t + (1-t)u \subseteq tTx_t + (1-t)u. \tag{8}$$

It T is single-valued, then (8) is reduced to (5).

On the other hand, Xu [35] studied the strong convergence of x_t defined by (4) as $t \to 1$ in either a Hilbert space or a uniformly smooth Banach space and showed that the strong $\lim_{t\to 1} x_t$ is the unique solution of certain variational inequality. This result of Xu [35] also improved Theorem 2.1 of Moudafi [17] as the continuous version. In 2006, Jung [9] also established the strong convergence of x_t defined by (4) for finite nonexpansive mappings in a reflexive Banach space Banach space having a uniformly Gâteaux differentiable norm with the condition that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings.

In this paper, motivated by [10, 21, 35], we establish the strong convergence of $\{x_t\}$ defined by

$$x_t \in tP_T x_t + (1-t)f(x_t), \; f \in \Sigma_C,$$

for the multivalued nonself-mapping T in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. We also study the strong convergence of $\{x_t\}$ for the multivalued nonself-mapping T satisfying the inwardness condition in a reflexive Banach space with a uniformly Gâteaux differentiable norm. Our results improve and extend the results in [2, 10, 11, 12, 20, 21, 32, 36] to the viscosity approximation method for multivalued nonself-mapping case. We also point out that our results give a partial answer to Jung's question [10].

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^*\rangle$.

A Banach space E is called *uniformly convex* if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$, where the modulus $\delta(\varepsilon)$ of convexity of E is defined by

$$\delta(\varepsilon) = \inf\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \le 1, \quad \|y\| \le 1, \quad \|x-y\| \ge \varepsilon\}$$

for every ε with $0 \le \varepsilon \le 2$. It is well-known that if E is uniformly convex, then E is reflexive and strictly convex (cf. [5]).

The norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{9}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be *uniformly Gâteaux differentiable* if for each $y \in U$, this limit is attained uniformly for $x \in U$. Finally, the norm is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth* if the limit in (9) is attained uniformly for $(x, y) \in U \times U$. A discussion of these and related concepts may be found in [3].

The *normalized duality mapping* J from E into the family of nonempty (by Hahn-Banach theorem) weak-star compact subsets of its dual E^* is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}.$$

for each $x \in E$. It is single valued if and only if E is smooth.

Let D be a subset of C. Then a mapping $Q : C \to D$ is said to be *retraction* if $Qx = x$ for all $x \in D$. A retraction $Q : C \to D$ is said to be *sunny* if each point on the ray $\{Qx + t(x - Qx) \ : t > 0\}$ is mapped by Q back onto Qx, in other words, $Q(Qx + t(x - Qx)) = Qx$ for all $t \geq 0$ and $x \in C$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D (cf. [5, 25]). In a smooth Banach space E, it is known (cf. [5, p. 48]) that Q is a sunny nonexpansive retraction from C onto D if and only if the following inequality holds:

$$(10) \qquad \langle x - Qx, J(z - Qx)\rangle \leq 0, \quad x \in C, \quad z \in D.$$

A mapping $T : C \to \mathcal{CB}(E)$ is $*$-*nonexpansive* ([7]) if for all $x,\ y \in C$ and $u_x \in Tx$ with $\|x-u_x\| = \inf\{\|x-z\| : z \in Tx\}$, there exists $u_y \in Ty$ with $\|y - u_y\| = \inf\{\|y - w\| : w \in Ty\}$ such that

$$\|u_x - u_y\| \leq \|x - y\|.$$

It is known that $*$-nonexpansiveness is different from nonexpansiveness for multivalued mappings. There are some $*$-nonexpansiveness multivalued mappings which are not nonexpansive and some nonexpansive multivalued mappings which are not $*$-nonexpansive [31].

Let μ be a linear continuous functional on ℓ^∞ and let $a = (a_1, a_2, ...) \in \ell^\infty$. We will sometimes write $\mu_n(a_n)$ in place of the value $\mu(a)$. A linear continuous functional μ such that $\|\mu\| = 1 = \mu(1)$ and $\mu_n(a_n) = \mu_n(a_{n+1})$ for every $a = (a_1, a_2, ...) \in \ell^\infty$ is called a *Banach limit*. We know that if μ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$$

for every $a = (a_1, a_2, ...) \in \ell^\infty$. Let $\{x_n\}$ be a bounded sequence in E. Then we can define the real valued continuous convex function ϕ on E by

$$\phi(z) = \mu_n\|x_n - z\|^2$$

for each $z \in E$.

The following lemma which was given in [6, 28] is, in fact, a variant of Lemma 1.3 in [25].

LEMMA 1. *Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm and let $\{x_n\}$ be a bounded sequence in E. Let μ be a Banach limit and $u \in C$. Then*

$$\mu_n\|x_n - u\|^2 = \min_{y\in C} \mu_n\|x_n - y\|^2$$

if and only if

$$\mu_n\langle x-u, J(x_n-u)\rangle \le 0 \tag{11}$$

for all $x \in C$.

We also need the following result, which was essentially given by Reich [27, pp. 314-315] and was also proved by Takahashi and Jeong [29].

LEMMA 2. *Let E be a uniformly convex Banach space, C a nonempty closed convex subset of E, and $\{x_n\}$ a bounded sequence of E. Then the set*

$$M = \{u \in C : \mu_n\|x_n - u\|^2 = \min_{z\in C} \mu_n\|x_n - z\|^2\}$$

consists of one point.

We introduce some terminology for boundary conditions for non-self mappings. The *inward set* of C at x is defined by

$$I_C(x) = \{z \in E : z = x + \lambda(y-x) : \ y \in C, \ \lambda \ge 0\}.$$

Let $\bar{I}_C(x) = x + T_C(x)$ with

$$T_C(x) = \left\{ y \in E : \ \liminf_{\lambda\to 0^+} \frac{d(x+\lambda y, C)}{\lambda} = 0 \right\}$$

for any $x \in C$. Note that for a convex set C, we have $\bar{I}_C(x) = \overline{I_C(x)}$, the closure of $I_C(x)$. A multivalued mapping $T : C \to \mathcal{F}(E)$ is said to satisfy the *inwardness condition* if $Tx \subset I_C(x)$ for all $x \in C$ and respectively, to satisfy the *weak inwardness condition* if $Tx \subset \bar{I}_C(x)$ for all $x \in C$. We notice that a fixed point theorem for nonexpansive mappings satisfying the inwardness condition is given in Corollary 3.5 of Reich [24]. A fixed point theorem for multi-valued strict contractions was given in Theorem 3.4 of Reich [24], too. It is also well-known that if C is a nonempty closed subset of a Banach space E, $T : C \to \mathcal{F}(E)$ ia a contraction satisfying the weak inwardness condition, and $x \in E$ has a nearest point in Tx, then T has a fixed point ([Theorem 11.4 of Deimling [4]).

Finally, the following lemmas were given by Xu [34] (also see Lemma 2.3.2 of Xu [33] for Lemma 4).

LEMMA 3. *If C is a closed bounded convex subset of a uniformly convex Banach space E and $T : C \to \mathcal{K}(E)$ is a nonexpansive mapping satisfying the weak inwardness condition, then T has a fixed point.*

Lemma 4. *If C is a compact convex subset of a Banach space E and $T : C \to \mathcal{KC}(E)$ is a nonexpansive mapping satisfying the boundary condition:*

$$Tx \cap \bar{I}_C(x) \neq \emptyset, \quad x \in C,$$

then T has a fixed point.

3. Main results

Now, we first prove a strong convergence theorem.

Theorem 1. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $T : C \to \mathcal{K}(E)$ a multivalued nonself-mapping such that P_T is nonexpansive. Suppose that C is a nonexpansive retract of E. Suppose that for $f \in \Sigma_C$ and $t \in (0,1)$, the contraction S_t defined by $S_t x = tP_T x + (1-t)f(x)$ has a fixed point $x_t \in C$. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 1$ and in this case, $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T.*

If we define $Q : \Sigma_C \to F(T)$ by $Q(f) := \lim_{t\to 1} x_t$ for $f \in \Sigma_C$, then $Q(f)$ solves the variational inequality

$$\langle (I-f)(Q(f)), J(Q(f)-z)\rangle \leq 0, \quad f \in \Sigma_C, \quad z \in F(T). \tag{12}$$

Proof. For given any $x_t \in C$, we can find some $y_t \in P_T x_t$ such that

$$x_t = ty_t + (1-t)f(x_t).$$

Let $z \in F(T)$. Then $\{x_t\}$ is uniformly bounded. In fact, noting that $P_T y = \{y\}$ whenever y is a fixed point of T, we have $z \in P_T z$ and

$$\|y_t - z\| = d(y_t, P_T z) \leq H(P_T x_t, P_T z) \leq \|x_t - z\| \tag{13}$$

for all $t \in (0,1)$. Thus we have

$$\begin{aligned}\|x_t - z\| &\leq t\|y_t - z\| + (1-t)\|f(x_t) - z\| \\ &\leq t\|x_t - z\| + (1-t)(\|f(x_t) - f(z)\| + \|f(z) - z\|) \\ &\leq t\|x_t - z\| + (1-t)(k\|x_t - z\| + \|f(z) - z\|.\end{aligned}$$

This implies that

$$\|x_t - z\| \leq \frac{1}{1-k}\|f(z) - z\|$$

and so $\{x_t\}$ is uniformly bounded. Also $\{f(x_t)\}$ is bounded.

Suppose conversely that $\{x_t\}$ remains bounded as $t \to 1$. We now show that T has a fixed point z and that $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T. To this end, let $t_n \to 1$ and $x_n = x_{t_n}$. Define $\phi : E \to [0, \infty)$ by $\phi(z) = \mu_n \|x_n - z\|^2$. Since ϕ is continuous and convex, $\phi(z) \to \infty$ as $\|z\| \to \infty$, and E is reflexive, ϕ attains its infimum over C (cf. [1, p. 79]). Let $z \in C$ be such that

$$\mu_n \|x_n - z\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2$$

and let

$$M = \{x \in C : \mu_n \|x_n - x\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2\}.$$

Then M is a nonempty bounded closed convex subset of C. Since C is a nonexpansive retract of E, the point z is the unique global minimum (over all of E). In fact, let Q be a nonexpansive retraction of E onto C. Then for any $y \in E$, we have

$$\mu_n \|x_n - z\|^2 \leq \mu_n \|x_n - Qy\|^2 = \mu_n \|Qx_n - Qy\|^2 \leq \mu_n \|x_n - y\|^2$$

and hence

$$\mu_n \|x_n - z\|^2 = \min_{y \in E} \mu_n \|x_n - y\|^2.$$

This global minimum point z is also unique by Lemma 2.

On the other hand, since $x_t = ty_t + (1-t)f(x_t)$ for some $y_t \in P_T x_t$, it follows that

$$\|x_t - y_t\| = (1-t)\|f(x_t) - y_t\| \to 0 \tag{14}$$

as $t \to 1$. Since P_T is compact valued, we have for each $n \geq 1$, some $w_n \in P_T z$ for $z \in M$ such that

$$\|y_n - w_n\| = d(y_n, P_T z) \leq H(P_T x_n, P_T z) \leq \|x_n - z\|. \tag{15}$$

Let $w = \lim_{n \to \infty} w_n \in P_T z$. It follows from (14) and (15) that

$$\mu_n \|x_n - w\|^2 \leq \mu_n \|y_n - w_n\|^2 \leq \mu_n \|x_n - z\|^2.$$

Since z is the unique global minimum, we have $w = z \in P_T z \subset Tz$ and hence $F(T) \neq \emptyset$. We have also that $P_T z = \{z\}$,

On the another hand, for $P_T z = \{z\} \in C$, we have from (13)

$$\begin{aligned} \langle x_n - y_n, J(x_n - z)\rangle &= \langle (x_n - z) + (z - y_n), J(x_n - z)\rangle \\ &\geq \|x_n - z\|^2 - \|y_n - z\|\|x_n - z\| \\ &\geq \|x_n - z\|^2 - \|x_n - z\|^2 = 0, \end{aligned}$$

and it follows that

$$(16)\qquad 0 \le \langle x_n - y_n, J(x_n - z)\rangle = (1 - t_n)\langle f(x_n) - y_n, J(x_n - z)\rangle.$$

Hence from (14) and (16), we obtain

$$(17)\qquad \mu_n\langle x_n - f(x_n), J(x_n - z)\rangle \le 0$$

for $P_T z = \{z\} = M$. But, from (11) in Lemma 1, we have

$$\mu_n\langle x - z, J(x_n - z)\rangle \le 0$$

for all $x \in C$. In particular, we have

$$(18)\qquad \mu_n\langle f(z) - z, J(x_n - z)\rangle \le 0.$$

Combining (17) and (18), we get

$$\begin{aligned} \mu_n\|x_n - z\|^2 &= \mu_n\langle x_n - z, J(x_n - z)\rangle \\ &\le \mu_n\langle f(x_n) - f(z), J(x_n - z)\rangle + \mu_n\langle f(z) - zJ(x_n - z)\rangle \\ &\le k\mu_n\|x_n - z\|^2 \end{aligned}$$

and hence $\mu_n\|x_n - z\|^2 \le 0$. Therefore, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges strongly to z. To complete the proof, suppose that there is another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges strongly to (say) y. Since

$$d(x_{n_k}, P_T x_{n_k}) \le \|x_{n_k} - y_{n_k}\| = (1 - t_{n_k})\|f(x_{n_k}) - y_{n_k}\| \to 0$$

as $k \to \infty$, we have $d(y, Ty) = 0$ and hence $y \in P_T y \subset Ty$. Noting that $P_T y = \{y\}$, from (17) we have

$$\langle z - f(z), J(z - y)\rangle \le 0 \quad \text{and} \quad \langle y - f(y), J(y - z)\rangle \le 0.$$

Adding these two inequalities yields

$$\|z - y\|^2 \le \langle f(z) - f(y), J(z - y)\rangle = k\|z - y\|^2$$

and thus $z = y$. This proves the strong convergence of $\{x_t\}$ to z.

Define $Q : \Sigma_C \to F(T)$ by $Q(f) := \lim_{t\to 1} x_t$. Since $x_t = ty_t + (1 - t)f(x_t)$ for some $y_t \in P_T x_t$,

$$(I - f)(x_t) = -\frac{t}{1-t}(x_t - y_t).$$

From (13), we have for $z \in F(T)$

$$\begin{aligned}\langle (I-f)(x_t), J(x_t - z)\rangle &= -\frac{t}{1-t}\langle (x_t - z) + (z - y_t), J(x_t - z)\rangle \\ &\leq -\frac{t}{1-t}(\|x_t - z\|^2 - \|y_t - z\|\|x_t - z\|) \\ &\leq -\frac{t}{1-t}(\|x_t - z\|^2 - \|x_t - z\|^2) = 0.\end{aligned}$$

Letting $t \to 1$ yields

$$\langle (I-f)(Q(f)), J(Q(f) - z)\rangle \leq 0, \quad f \in \Sigma_C, \quad z \in F(T).$$

□

REMARK 1. In Theorem 1, if $f(x) = u, \ x \in C$, is a constant mapping, then it follows from (12) that

$$\langle Q(u) - u, J(Q(u) - z)\rangle \leq 0, \quad u \in C, \quad z \in F(T).$$

Hence by (10), Q reduces to the sunny nonexpansive retraction from C onto $F(T)$.

By definition of the Hausdorff metric, we obtain that if T is $*$-nonexpansive, then P_T is nonexpansive. Hence, as a direct consequence of Theorem 1, we have the following result.

COROLLARY 1. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $T : C \to \mathcal{K}(E)$ a multivalued $*$-nonexpansive nonself-mapping. Suppose that C is a nonexpansive retract of E. Suppose that for $f \in \Sigma_C$ and $t \in (0,1)$, the contraction S_t defined by $S_t x = tP_T x + (1-t)f(x)$ has a fixed point $x_t \in C$. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 1$ and in this case, $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T.*

It is well-known that every nonempty closed convex subset C of a strictly convex reflexive Banach space E is Chebyshev, that is, for any $x \in E$, there is a unique element $u \in C$ such that $\|x - u\| = \inf\{\|x - v\| : v \in C\}$. Thus, we have the following corollary.

Corollary 2. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $T : C \to \mathcal{KC}(E)$ a multivalued nonself-mapping such that P_T is nonexpansive. Suppose that C is a nonexpansive retract of E. Suppose that for $f \in \Sigma_C$ and $t \in (0,1)$, the contraction S_t defined by $S_t x = tP_T x + (1-t)f(x)$ has a fixed point $x_t \in C$. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 1$ and in this case, $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T.*

Proof. In this case, Tx is Chebyshev for each $x \in C$. So P_T is a selector of T and P_T is single valued. Thus the result follows from Theorem 1. □

Corollary 3. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $T : C \to \mathcal{KC}(E)$ a multivalued $*$-nonexpansive nonself-mapping. Suppose that C is a nonexpansive retract of E. Suppose that for $f \in \Sigma_C$ and $t \in (0,1)$, the contraction S_t defined by $S_t x = tP_T x + (1-t)f(x)$ has a fixed point $x_t \in C$. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 1$ and in this case, $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T.*

Corollary 4. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed bounded convex subset of E, and $T : C \to \mathcal{K}(E)$ a multivalued nonself-mapping satisfying the weak inwardness condition such that P_T is nonexpansive. Suppose that C is a nonexpansive retract of E. Let $f \in \Sigma_C$ and $t \in (0,1)$. Then $\{x_t\}$ defined by $x_t \in tP_T x_t + (1-t)f(x_t)$ converges strongly as $t \to 1$ to a fixed point of T.*

Proof. Fix $f \in \Sigma_C$ and define for each $t \in (0,1)$, the contraction $S_t : C \to \mathcal{K}(E)$ by

$$S_t x := tP_T x + (1-t)f(x), \quad x \in C.$$

As it is easily seen that S_t also satisfies the weak inwardness condition: $S_t x \subset \bar{I}_C(x)$ for all $x \in C$, it follows from Lemma 3 that S_t has a fixed point denoted by x_t. Thus the result follows from Theorem 1. □

REMARK 2. (1) As in [31], Shahzad and Zegeye [21] gave the following example of a multivalued T such that P_T is nonexpansive: Let $C = [0, \infty)$ and T be defined by $Tx = [x, 2x]$ for $x \in C$. Then $P_T x = \{x\}$ for $x \in C$. Also T is $*$-nonexpansive but not nonexpansive (see [31]).

(2) Theorem 1 (and Corollaries 1-4) generalizes Theorem 3.1 (and Corollaries 3.3-3.5) of Shahzad and Zegeye [21] to the viscosity approximation method.

(3) Theorem 1 also improves and complements the corresponding results of Jung [10], Kim and Jung [14] and Sahu [20]. Theorem 1 extends the corresponding results of Jung and Kim [11], Jung and Kim of [12] and Xu and Yin [36], to the multivalued mapping case, too.

(4) Our results apply to all L^p spaces or ℓ^p spaces for $1 < p < \infty$.

THEOREM 2. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $T : C \to \mathcal{KC}(E)$ a multivalued nonself-mapping satisfying the inwardness condition such that P_T is nonexpansive. Let $f \in \Sigma_C$ and $t \in (0, 1)$. Assume that every closed bounded convex subset of C is compact. If P_T has a fixed point, then the sequence $\{x_t\}$ defined by*

$$x_t \in tP_T x_t + (1 - t)f(x_t), \tag{19}$$

converges strongly as $t \to 1$ to a fixed point of T.

Proof. Let $z \in P_T z$. As in proof of Theorem 1, we have $\|x_t - z\| \leq \frac{1}{1-k}\|f(z) - z\|$ for all $t \in (0, 1)$ and hence $\{x_t\}$ is uniformly bounded.

We now show that $\{x_t\}$ converges strongly as $t \to 1^-$ to a fixed point of T. To this end, let $t_n \to 1$ and $x_n = x_{t_n}$. As in the proof of Theorem 1, we define the same function $\phi : E \to [0, \infty)$ by $\phi(z) = \mu_n\|x_n - z\|^2$ and let

$$M = \{x \in C : \mu_n\|x_n - x\|^2 = \min_{y \in C} \mu_n\|x_n - y\|^2\}.$$

Then M is a nonempty closed bounded convex subset of C and by assumption, M is compact convex. Clearly, P_T satisfies the inwardness condition. By using the same argument as in Theorem 2 of Jung [10], we can prove that the inwardness condition of P_T on C implies a weaker inwardness of P_T on M, that is,

$$P_T z \cap I_M(z) \neq \emptyset, \quad z \in M.$$

So, by Lemma 4, there exists $z \in M$ such that $z \in P_T z \subseteq Tz$ and so $P_T z = \{z\}$. The strong convergence of $\{x_t\}$ to z is the same as given in the proof of Theorem 1. □

Corollary 5. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $T : C \to \mathcal{KC}(E)$ a multivalued $*$-nonexpansive nonself-mapping satisfying the inwardness condition such that P_T is nonexpansive. Let $f \in \Sigma_C$ and $t \in (0,1)$. Assume that every closed bounded convex subset of C is compact. If P_T has a fixed point, then the sequence $\{x_t\}$ defined by (19) converges strongly as $t \to 1$ to a fixed point of T.*

Corollary 6. *Let E be a uniformly smooth Banach space, C a nonempty closed convex subset of E, and $T : C \to \mathcal{KC}(E)$ a multivalued nonself-mapping satisfying the inwardness condition such that P_T is nonexpansive. Let $f \in \Sigma_C$ and $t \in (0,1)$. Assume that every closed bounded convex subset of C is compact. If P_T has a fixed point, then the sequence $\{x_t\}$ defined by (19) converges strongly as $t \to 1$ to a fixed point of T.*

Corollary 7. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty compact convex subset of E, and $T : C \to \mathcal{KC}(E)$ a multivalued nonself-mapping satisfying the inwardness condition such that P_T is nonexpansive. Let $f \in \Sigma_C$ and $t \in (0,1)$. If P_T has a fixed point, then the sequence $\{x_t\}$ defined by (19) converges strongly as $t \to 1$ to a fixed point of T.*

Corollary 8. *Let E be a uniformly smooth Banach space, C a nonempty compact convex subset of E, and $T : C \to \mathcal{KC}(E)$ a multivalued nonself-mapping satisfying the inwardness condition such that P_T is nonexpansive. Let $f \in \Sigma_C$ and $t \in (0,1)$. If P_T has a fixed point, then the sequence $\{x_t\}$ defined by (19) converges strongly as $t \to 1$ to a fixed point of T.*

Remark 3. (1) Theorem 2 (and Corollaries 5-8) also improves Theorem 3.9 (and Corollaries 3.10-3.12) of Shahzad and Zegeye [21] to the viscosity approximation method. Theorem 2 (and Corollaries 6-7) complements Theorem 2 (and Corollaries 4-5) of Jung [10], too.

(2) Theorem 2 is also a multivalued version of Theorem 1 and Corollary 1 of Jung and Kim [12] and Theorem 1 of Xu [32].

(3) A fixed point theorem for $T : C \to \mathcal{KC}(E)$ a $*$-nonexpansive, 1-χ-contractive multivalued mapping satisfying the inwardness condition in a special Banach space was recently given by Shahzad and Lone [22]. In this case, one can relax the assumption that $F(T) \neq \emptyset$.

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