Korean J. Math. **16** (2008), No. 3, pp. 389–402

MULTIPLE SOLUTIONS FOR A CLASS OF THE SYSTEMS OF THE CRITICAL GROWTH SUSPENSION BRIDGE EQUATIONS

Tacksun Jung and Q-Heung Choi*

Abstract. We show the existence of at least two solutions for a class of systems of the critical growth nonlinear suspension bridge equations with Dirichlet boundary condition and periodic condition. We first show that the system has a positive solution under suitable conditions, and next show that the system has another solution under the same conditions by the linking arguments.

1. Introduction

In this paper we consider the multiplicity of the solutions for the following class of systems of the critical growth nonlinear suspension bridge equations with Dirichlet boundary condition and periodic condition

$$\begin{cases} u_{tt}+u_{xxxx}+av_+=\dfrac{2\alpha}{\alpha+\beta}u_-^{\alpha-1}v_-^{\beta}+\phi_{00}+\epsilon_1 h_1(x,t) \\ \qquad\qquad \text{in } \ (-\pi/2,\pi/2)\times R, \\ v_{tt}+v_{xxxx}+bu_+=\dfrac{2\beta}{\alpha+\beta}u_-^{\alpha}v_-^{\beta-1}+\phi_{00}+\epsilon_2 h_2(x,t) \\ \qquad\qquad \text{in } \ (-\pi/2,\pi/2)\times R, \\ u(\pm\frac{\pi}{2},t)=u_{xx}(\pm\frac{\pi}{2},t)=v(\pm\frac{\pi}{2},t)=v_{xx}(\pm\frac{\pi}{2},t)=0, \\ u(x,t+\pi)=u(x,t)=u(-x,t)=u(x,-t), \\ v(x,t+\pi)=v(x,t)=v(-x,t)=v(x,-t), \end{cases} \tag{1.1}$$

Received August 8, 2008. Revised August 18, 2008.
2000 Mathematics Subject Classification: 35Q72.
Key words and phrases: System of the critical growth suspension bridge equations, linking arguments, eigenvalue of a matrix, boundary value problem.
*Corresponding author.

where α, $\beta > 1$ are real constants, $u_+ = \max\{u, 0\}$, $u_- = \min\{-u, 0\}$, ϵ_1, ϵ_2 are small numbers and $h_1(x,t)$, $h_2(x,t)$ are bounded, π-periodic in t and even in x and t and $\|h_1\| = \|h_2\| = 1$. Here ϕ_{00} is the eigenfunction corresponding to the positive eigenvalue $\lambda_{00} = 1$ of the eigenvalue problem $u_{tt} + u_{xxxx} = \lambda_{mn} u$ with $u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0$, $u(x, t+\pi) = u(x,t) = u(-x,t) = u(x,-t)$.

McKenna and Walter([6]) found the physical model of jumping problem from a bridge suspended by cables under a load. The nonlinear suspension bridge equation with length π is as follows

$$u_{tt} + u_{xxxx} + bu^+ = f(x,t) \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times R,$$

$$u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0.$$

This equation represents a bending beam supported by cables under a load f. The constant b represents the restoring force if the cables stretch. The nonlinearity u^+ models the fact that cables resist expansion but do not resist compression. Choi and Jung ([3], [4], [5]) investigate the existence and multiplicity of solutions for the single nonlinear suspension bridge equation with Dirichlet boundary condition. The system (1.1) can be rewritten by

$$\begin{cases} U_{tt} + U_{xxxx} & +\nabla(\frac{1}{2}(AU^+, U)) \\ & = \nabla(\frac{2}{\alpha+\beta} u_-^{\alpha} v_-^{\beta}) + \binom{\phi_{00}+\epsilon_1 h_1(x,t)}{\phi_{00}+\epsilon_2 h_2(x,t)}, \\ U(\pm\frac{\pi}{2}, t) & = U_{xx}(\pm\frac{\pi}{2}, t) = \binom{0}{0}, \\ U(x, t+\pi) & = U(x,t) = U(-x,t) = U(x,-t), \end{cases} \tag{1.2}$$

where $U = \binom{u}{v}$, $U^+ = \binom{u^+}{v^+}$, $U_{tt} + U_{xxxx} = \binom{u_{tt}+u_{xxxx}}{v_{tt}+v_{xxxx}}$, $A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \in M_{2\times 2}(R)$.

The eigenvalue problem for $u(x,t)$,

$$u_{tt} + u_{xxxx} = \lambda u \qquad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R,$$

$$u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0,$$

$$u(x, t+\pi) = u(x,t) = u(-x,t) = u(x,-t)$$

has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^4 - 4m^2 \qquad (m, n = 0, 1, 2, \ldots)$$

and corresponding normalized eigenfunctions ϕ_{mn} $(m, n \geq 0)$ given by

$$\phi_{0n} = \frac{\sqrt{2}}{\pi} \cos(2n+1)x \qquad \text{for} \quad n \geq 0,$$

$$\phi_{mn} = \frac{2}{\pi} \cos 2mt \cdot \cos(2n+1)x \qquad \text{for} \quad m > 0, n \geq 0.$$

We can check easily that the eigenvalues in the interval (-19,45) are given by

$$\lambda_{20} = -15 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{41} = 17$$

We assume that

$$\lambda_{mn}^2 + ab \neq 0 \text{ for all } m, n \text{ with } (m, n) \neq (0, 0), \tag{1.3}$$

$$a < 0, \qquad b < 0, \tag{1.4}$$

$$\sqrt{ab} < 1. \tag{1.5}$$

Our main result is the following:

Theorem 1. *Assume that the conditions (1.3), (1.4) and (1.5) hold. Then, for each $h_1(x,t)$, $h_2(x,t) \in H_0$ with $\|h_1(x,t)\| = 1$, $\|h_2(x,t)\| = 1$, there exist small numbers $\bar{\epsilon}_1 > 0$ and $\bar{\epsilon}_2 > 0$ such that for any (ϵ_1, ϵ_2) with $\epsilon_1 < \bar{\epsilon}_1$ and $\epsilon_2 < \bar{\epsilon}_2$, system (1.1) has at least two nontrivial solutions, one of which is a positive $U_0 = \binom{u_0}{v_0}$ with $u_0 > 0$ and $v_0 > 0$, where H_0 is introduced in section 2.*

In section 2, we show that system (1.1) has a positive solution by direct computation and operator theory. In section 3, we approach the variational method and recall the critical point theorem which is the linking theorem for the strongly indefinite functional to find the second solution. In section 4, we prove the existence of the second solution of (1.1).

2. Existence of a positive solution

Let Q be the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and H_0 the Hilbert space defined by

$$H_0 = \{u \in L^2(Q) |\ u \text{ is even in } x \text{ and } t \text{ and } \int_Q u = 0\}.$$

The set of functions $\{\phi_{mn}\}$ is an orthonormal basis in H_0. Let us denote an element u, in H_0, by

$$u = \sum h_{mn}\phi_{mn}.$$

We define a Hilbert space H as follows

$$H = \{u \in H_0 : \sum_{mn} |\lambda_{mn}| h_{mn}^2 < \infty\}.$$

Then this space is a Banach space with norm

$$\|u\|^2 = [\sum |\lambda_{mn}| h_{mn}^2]^{\frac{1}{2}}.$$

Let us set $E = H \times H$. We endow the Hilbert space E the norm

$$\|(u,v)\|_E^2 = \|u\|^2 + \|v\|^2 \qquad \forall (u,v) \in E.$$

We are looking for the weak solutions of (1.1) in E, that is, (u,v) such that $u \in H$, $v \in H$, $u_{tt} + u_{xxxx} + av^+ = \frac{2\alpha}{\alpha+\beta} u_-^{\alpha-1} v_-^{\beta} + \psi_{00} + \epsilon_1 h_1(x,t)$, $v_{tt} + v_{xxxx} + bu^+ = \frac{2\beta}{\alpha+\beta} u_-^{\alpha} v_-^{\beta-1} + \psi_{00} + \epsilon_2 h_2(x,t)$.

Since $|\lambda_{mn}| \geq 1$ for all m, n, we have that

LEMMA 1. *(i)* $\|u\| \geq \|u\|_{L^2(Q)}$, *where* $\|u\|_{L^2(Q)}$ *denotes the* L^2 *norm of* u.
(ii) $\|u\| = 0$ *if and only if* $\|u\|_{L^2(Q)} = 0$.
(iii) $u_{tt} + u_{xxxx} \in H$ *implies* $u \in H$.

LEMMA 2. *Suppose that* c *is not an eigenvalue of* L, $Lu = u_{tt} + u_{xxxx}$, *and let* $f \in H_0$. *Then we have* $(L-c)^{-1} f \in H$.

Proof. When n is fixed, we define

$$\lambda_n^+ = \inf_m \{\lambda_{mn} : \ \lambda_{mn} > 0\} = 8n^2 + 8n + 1,$$

$$\lambda_n^- = \sup_m \{\lambda_{mn} : \lambda_{mn} < 0\} = -8n^2 - 8n - 3.$$

We see that $\lambda_n^+ \to +\infty$ and $\lambda_n^- \to -\infty$ as $n \to \infty$. Hence the number of elements in the set $\{\lambda_{mn} : \ |\lambda_{mn}| < |c|\}$ is finite, where λ_{mn} is an eigenvalue of L. Let

$$f = \sum h_{mn}\phi_{mn}.$$

Then

$$(L-c)^{-1} f = \sum \frac{1}{\lambda_{mn} - c} h_{mn}\phi_{mn}.$$

Hence we have the inequality

$$\|(L-c)^{-1}f\| = \sum |\lambda_{mn}| \frac{1}{(\lambda_{mn}-c)^2} h_{mn}^2 \le C \sum h_{mn}^2$$

for some C, which means that

$$\|(L-c)^{-1}f\| \le C_1 \|f\|_{L^2(Q)}, \qquad C_1 = \sqrt{C}.$$

□

LEMMA 3. *Assume that the conditions (1.3) and (1.4) hold. Then the system*

$$\begin{cases} u_{tt} + u_{xxxx} & +av = \phi_{00} \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}), \\ v_{tt} + v_{xxxx} & +bu = \phi_{00} \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}), \\ u(\pm\frac{\pi}{2}, t) & = u_{xx}(\pm\frac{\pi}{2}, t) = v(\pm\frac{\pi}{2}, t) = v_{xx}(\pm\frac{\pi}{2}, t) = 0, \\ u(x, t+\pi) & = u(x,t) = u(-x,t) = u(x,-t), \\ v(x, t+\pi) & = v(x,t) = v(-x,t) = v(x,-t), \end{cases} \tag{2.1}$$

has a positive solution $(u_*, v_*) \in E$, which is of the form

$$u_* = [\frac{-a}{\lambda_{00}} \frac{-b+\lambda_{00}}{\lambda_{00}^2 - ab} + \frac{1}{\lambda_{00}}]\phi_{00}, \tag{2.2}$$

$$v_* = [\frac{-b+\lambda_{00}}{\lambda_{00}^2 - ab}]\phi_{00}.$$

Proof. We note that (u_*, v_*) is a solution of the system (2.1) with $u_* > 0$ and $v_* > 0$. □

Define $\mathcal{L}U = (Lu, Lv)$, $Lu = u_{tt} + u_{xxxx}$. We need to find a spectral analysis for the linear operator $\mathcal{L}U + AU$. The following lemma need a simple 'Fourier Series' argument.

LEMMA 4. *Let a, $b \in R$ and let $\mathcal{L}_{ab} : H \times H \to H_0 \times H_0$ be defined by $\mathcal{L}_{ab}(u,v) = (Lu + av, Lv + bu)$. For $\mu \in R$ we have*
(a) if $(\lambda_{mn} - \mu)^2 \ne ab$ for every m, n, then

$$(\mathcal{L}_{ab} - \mu I)^{-1} : H_0 \times H_0 \to H_0 \times H_0$$

is well defined and continuous;
(b) if $(\lambda_{mn} - \mu)^2 = ab$ for some m, n, then

$$\mathrm{Ker}(\mathcal{L}_{ab} - \mu I) = \mathrm{span}\{\phi_{mn} : \ (\lambda_{mn} - \mu)^2 = ab\};$$

moreover if $X_\mu = \overline{\mathrm{span}\{\phi_{mn} : \ (\lambda_{mn} - \mu)^2 \ne ab\}}$, then

$$(\mathcal{L}_{ab} - \mu I)^{-1} : X_\mu \times X_\mu \to X_\mu \times X_\mu$$

is well defined and continuous.
Notice that if $ab < 0$, the second alternative can never occur.

Proof. To prove (a) we take (f,g) in $H_0 \times H_0$. We can write $f = \sum_{mn} f_{mn}\phi_{mn}$ with $\sum_{mn} f_{mn}^2 < +\infty$ and $g = \sum_{mn} g_{mn}\phi_{mn}$ with $\sum_{mn} g_{mn}^2 < +\infty$. We define, for m, n integers,

$$u_{mn} = \frac{(\lambda_{mn} - \mu)f_{mn} - ag_{mn}}{(\lambda_{mn} - \mu)^2 - ab}, \qquad v_{mn} = \frac{(\lambda_{mn} - \mu)g_{mn} - bf_{mn}}{(\lambda_{mn} - \mu)^2 - ab},$$

which make sense since $(\lambda_{mn} - \mu)^2 \neq ab$ for every m, n. We have

$$|u_{mn}| \leq \frac{C}{|\lambda_{mn}|}(|f_{mn}| + |g_{mn}|) \Longrightarrow \lambda_{mn}^2 u_{mn}^2 \leq C_1(f_{mn}^2 + g_{mn}^2)$$

for suitable constants C, C_1 not depending on mn. The same inequality applies for v_{mn}. So if $u = \sum_{mn} u_{mn}\phi_{mn}$, $v = \sum_{mn} v_{mn}\phi_{mn}$, then $(u,v) \in H \times H$. Arguing componentwise it is simple to check that $\mathcal{L}_{ab}(u,v) - \mu I(u,v) = (f,g)$. So $(\mathcal{L}_{ab} - \mu I)^{-1} : H_0 \times H_0 \to H_0 \times H_0$ is well defined. To prove (b) we first observe that if $(\lambda_{mn} - \mu)^2 = ab$, then $(\mathcal{L}_{ab} - \mu I)\phi_{mn} = 0$, as one can easily check. Secondly given (f,g) in X_μ we can argue as in the first case since $f_{mn} = g_{mn} = 0$ for all mn such that $(\lambda_{mn} - \mu)^2 = ab$. This allows to define u_{mn} and v_{mn} as before for all mn such that $(\lambda_{mn} - \mu)^2 \neq ab$ and $u_{mn} = v_{mn} = 0$ for all mn such that $(\lambda_{mn} - \mu)^2 = ab$. □

Using Lemma 2.4 with the case $\mu = 0$ we can easily derive Lemma 2.5

LEMMA 5. *Assume that the conditions (1.3) and (1.4) hold. Then for each $h_1(x,t)$, $h_2(x,t) \in H_0$ with $\|h_1\| = 1$ and $\|h_2\| = 1$, there exist small numbers ϵ_1 and ϵ_2 such that system*

the system

$$\begin{cases} u_{tt} + u_{xxxx} & +av = \epsilon_1 h_1(x,t), \\ v_{tt} + v_{xxxx} & +bu = \epsilon_2 h_2(x,t), \\ u(\pm\frac{\pi}{2},t) & = u_{xx}(\pm\frac{\pi}{2},t) = v(\pm\frac{\pi}{2},t) = v_{xx}(\pm\frac{\pi}{2},t) = 0, \\ u(x,t+\pi) & = u(x,t) = u(-x,t) = u(x,-t), \\ v(x,t+\pi) & = v(x,t) = v(-x,t) = v(x,-t), \end{cases} \tag{2.3}$$

has a unique solution $(u_{\epsilon_1\epsilon_2}, v_{\epsilon_1\epsilon_2}) \in E = H \times H$.

Proof of the existence of a positive solution

By Lemma 2.3 and Lemma 2.5, $(u_* + u_{\epsilon_1\epsilon_2}, v_* + v_{\epsilon_1\epsilon_2})$ is a solution of the system

$$\begin{cases} u_{tt} + u_{xxxx} & +av = \phi_{00} + \epsilon_1 h_1(x,t), \\ v_{tt} + v_{xxxx} & +bv = \phi_{00} + \epsilon_2 h_2(x,t), \\ u(\pm\frac{\pi}{2}, t) & = u_{xx}(\pm\frac{\pi}{2}, t) = v(\pm\frac{\pi}{2}, t) = v_{xx}(\pm\frac{\pi}{2}, t) = 0, \\ u(x, t+\pi) & = u(x,t) = u(-x,t) = u(x,-t), \\ v(x, t+\pi) & = v(x,t) = v(-x,t) = v(x,-t), \end{cases} \tag{2.4}$$

where $u_* = [\frac{-a}{\lambda_{00}} \frac{-b+\lambda_{00}}{\lambda_{00}^2 - ab} + \frac{1}{\lambda_{00}}]\phi_{00} > 0$, $v_* = [\frac{-b+\lambda_{00}}{\lambda_{00}^2 - ab}]\phi_{00} > 0$. By Lemma 2.4, $u_{\epsilon_1\epsilon_2} \in H$ and $v_{\epsilon_1\epsilon_2} \in H$. Since the elements of H lies in C^1, the elements $u_{\epsilon_1\epsilon_2}$, $v_{\epsilon_1\epsilon_2} \in C^1$. Thus we can find small numbers $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$ such that for any (ϵ_1, ϵ_2) with $\epsilon_1 < \bar{\epsilon}_1$ and $\epsilon_2 < \bar{\epsilon}_2$, $u_* + u_{\epsilon_1\epsilon_2} > 0$ and $v_* + v_{\epsilon_1\epsilon_2} > 0$, which is also a positive solution of system (1.1).

3. Variational approach

Now we are looking for the other weak solutions of system (1.1). To find the other nontrivial weak solutions of system (1.1) we approach the variational method and recall the linking theorem for the strongly indefinite functional. We observe that the weak solutions of (1.1) coincide with the critical points of the corresponding functional

$$I : E \to R \in C^{1,1},$$

$$I(U) = \frac{1}{2}\int_Q \mathcal{L}U \cdot U dxdt + \frac{1}{2}\int_Q (AU^+, U)_{R^2} dxdt - \frac{2}{\alpha+\beta}\int_Q u_-^{\alpha} v_-^{\beta} dxdt$$

$$- \int_Q (\phi_{00} + \epsilon_1 h_1(x,t)) u dxdt - \int_Q (\phi_{00} + \epsilon_2 h_2(x,t)) v dxdt. \tag{3.1}$$

We notice that the solution (u, v) of system (1.1) is of the form $(u, v) = (\bar{u}, \bar{v}) + (u_0, v_0)$, where (u_0, v_0) is a positive solution with $u_0 = u_* + u_{\epsilon_1\epsilon_2} >$

0 and $v_0 = v_* + v_{\epsilon_1\epsilon_2} > 0$, and $(\bar{u}, \bar{v})$ is a nontrivial solution of the system

$$\begin{cases} u_{tt} + u_{xxxx} + a(v+v_0)_+ - av_0 & = \frac{2\alpha}{\alpha+\beta}(u+u_0)_-^{\alpha-1}(v+v_0)_-^{\beta} \\ & \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2}), \\ v_{tt} + v_{xxxx} + b(u+u_0)_+ - bu_0 & = \frac{2\beta}{\alpha+\beta}(u+u_0)_-^{\alpha}(v+v_0)_-^{\beta-1} \\ & \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2}), \\ u(\pm\frac{\pi}{2}, t) & = v(\pm\frac{\pi}{2}, t) = 0, \\ u(x, t+\pi) & = u(x,t) = u(-x,t) = u(x,-t), \\ v(x, t+\pi) & = v(x,t) = v(-x,t) = v(x,-t), \end{cases} \tag{3.2}$$

Thus it suffices to find the nontrivial solution of system (3.2). We observe that the weak solutions of (3.2) are the critical points of the functional

$$J : E \to R \in C^{1,1},$$

$$J(U) = \frac{1}{2}\int_Q \mathcal{L}U \cdot U dxdt + \frac{1}{2}\int_Q (A(U+U_0)_+, U)_{R^2} dxdt$$

$$- \int_Q (AU_0, U)_{R^2} dxdt - \frac{2}{\alpha+\beta}\int_Q (u+u_0)_-^{\alpha}(v+v_0)_-^{\beta} dxdt, \tag{3.3}$$

where $(U+U_0)_+ = \binom{(u+u_0)_+}{(v+v_0)_+}$. Thus we shall find the critical points for J. Now we recall the linking theorem for strongly indefinite functional (cf. [8]).

Lemma 6. *(Linking Theorem)*
Let E be a real Hilbert space with $E = E_1 \oplus E_2$ and $E_2 = E_1^{\perp}$. We suppose that
(J1) $J \in C^1(E, R)$, satisfies $(P.S.)^*$ condition, and
(J2) $J(u) = \frac{1}{2}(Lu, u) + bu$, where $Lu = L_1P_1u + L_2P_2u$ and $L_i : E_i \to E_i$ is bounded and selfadjoint, $i = 1, 2$,
(J3) b' is compact, and
(J4) there exists a subspace $\tilde{E} \subset E$ and sets $S \subset E$, $T \subset \tilde{E}$ and constants $\gamma > w$ such that,
(i) $S \subset E_1$ and $J|_S \geq \gamma$,
(ii) T is bounded and $J|_{\partial T} \leq w$,
(iii) S and ∂T link.
Then J possesses a critical value $c \geq \gamma$.

Let E^-, E^0, E^+ be the subspaces of E on which the functional $U \mapsto \frac{1}{2}\int_Q \mathcal{L}U \cdot U$ is negative definite, null, positive definite and E^-, E^0 and E^+ are mutually orthogonal. Let P^+ be the projection for E onto E^+,

P^0 the one from E onto E^0 and P^- the one from E onto E^-. Let $(E_n)_n$ be a sequence of closed subspaces of E with the conditions:

$$E_n = E_n^- \oplus E^0 \oplus E_n^+, \text{ where } E_n^+ \subset E^+,\ E_n^- \subset E^- \text{ for all } n, \tag{3.4}$$

$(E_n^+$ and E_n^- are subspaces of $E), \dim E_n < +\infty, E_n \subset E_{n+1}, \cup_{n\in N} E_n$ is dense in E. Let P_{E_n} be the orthogonal projections from E onto E_n.

Let us define

$$C_{\alpha,\beta}(Q) = \inf_{(u,v)\in E\setminus(0,0)} \frac{\int_Q \mathcal{L}U \cdot U dxdt}{(\int_Q |u|^\alpha |v|^\beta dxdt)^{\frac{2}{\alpha+\beta}}}, \quad \text{for } U = (u,v). \tag{3.5}$$

Let us prove that the functional J satisfies the linking geometry.

Lemma 7. *Assume that the conditions (1.3), (1.4) and (1.5) hold. Then*
(i) there exist a number $\rho > 0$ and a small ball $B_\rho \subset E^+$ with radius ρ such that if $U \in \partial B_\rho$, then

$$\inf_{U\in\partial B_\rho} J(U) > 0,$$

(ii) there is an $e \in E^+$, $R > \rho$ and a large ball $D_R \subset E^0 \oplus E^-$ with radius $R > 0$ such that if

$$W = (\bar{D_R} \cap (E^0 \oplus E^-)) \oplus \{re|\ , e \in E^+,\ 0 < r < R\},$$

then

$$\sup_{U\in\partial W} J(U) \le 0.$$

Proof. (i) By (3.5), for $U \in E^+$

$$\begin{aligned} J(U) = &\frac{1}{2}\int_Q \mathcal{L}U \cdot U + \frac{1}{2}\int_Q (A(U+U_0)_+, U)_{R^2} - \int_Q (AU_0, U)_{R^2} \\ &-\frac{2}{\alpha+\beta}\int_Q (u+u_0)_-^\alpha (v+v_0)_-^\beta dxdt \\ = &\frac{1}{2}\int_Q \mathcal{L}U \cdot U + \frac{1}{2}\int_Q (AU,U)_{R^2} + \frac{1}{2}\int_Q (A(U+U_0)_-, U)_{R^2} \\ &-\frac{2}{\alpha+\beta}\int_Q (u+u_0)_-^\alpha (v+v_0)_-^\beta dxdt \end{aligned}$$

Since $a < 0$, $b < 0$ and $(A(U + U_0)_-, U)_{R^2} \geq (AU_-, U_-)_{R^2}$, we have

$$J(U) \geq \frac{1}{2}\int_Q \mathcal{L}U \cdot U + \frac{1}{2}\int_Q (AU, U)_{R^2} - \frac{1}{2}\int_Q (AU_-, U_-)_{R^2}$$

$$-\frac{2}{\alpha+\beta}(C_{(\alpha,\beta)}(Q))^{-\frac{\alpha+\beta}{2}}\|U\|_E^{\alpha+\beta}$$

$$= \frac{1}{2}\int_Q \mathcal{L}U \cdot U + \frac{1}{2}\int_Q (AU_+, U_+)_{R^2} - \frac{2}{\alpha+\beta}(C_{(\alpha,\beta)}(Q))^{-\frac{\alpha+\beta}{2}}\|U\|_E^{\alpha+\beta}$$

$$\geq \frac{1}{2}\frac{1}{\lambda_{00}}\|U\|_E^2 - \sqrt{ab}\|U_+\|^2_{L^2(Q} - \frac{2}{\alpha+\beta}(C_{(\alpha,\beta)}(Q))^{-\frac{\alpha+\beta}{2}}\|U\|_E^{\alpha+\beta}$$

$$\geq \frac{1}{2}\frac{1-\sqrt{ab}}{\lambda_{00}}\|U\|_E^2 - \frac{2}{\alpha+\beta}(C_{(\alpha,\beta)}(Q))^{-\frac{\alpha+\beta}{2}}\|U\|_E^{\alpha+\beta}$$

Since $\sqrt{ab} < 1 = \lambda_{00}$ and $\alpha + \beta > 2$, there exist a small number $\rho > 0$ and a small ball $B_\rho \subset E^+$ with radius ρ such that if $U \in \partial B_\rho \subset E^+$, then $\inf J(U) > 0$.

(ii) Let us choose an element $e \in E^+$. Let us fix $\tilde{U} = (\tilde{u}, \tilde{v}) = P^-\tilde{U} + re(\neq (0,0)) \in E^0 \oplus E^- \oplus \{re|\ 0 < r\}$ such that

$$\int_Q (\tilde{u}+1)_-^\alpha(\tilde{v}+1)_-^\beta > 0. \tag{3.6}$$

For $s > 0$ we have

$$J(s\tilde{U}) = \frac{s^2}{2}\int_Q \mathcal{L}\tilde{U}\cdot\tilde{U}dxdt + \frac{s^2}{2}\int_Q (A(\tilde{U} + \frac{U_0}{s})_+, \tilde{U})_{R^2}$$

$$-s\int_Q (AU_0, \tilde{U})_{R^2}dxdt - \frac{2}{\alpha+\beta}s^{\alpha+\beta}\int_Q (\tilde{u} + \frac{u_0}{s})_-^\alpha(\tilde{v} + \frac{v_0}{s})_-^\beta dxdt$$

Since $a < 0$, $b < 0$ and $(A(\tilde{U} + \frac{U_0}{s})_+, \tilde{U})_{R^2} \leq (A\tilde{U}_+, \tilde{U}_+)_{R^2}$, we have that

$$J(s\tilde{U}) \leq \frac{s^2}{2}(-3)\|P^-\tilde{U}\|^2_{L^2(Q)} + \frac{s^2}{2}\int_Q (A(P^-\tilde{U}+, P^-\tilde{U}_+)_{R^2} + \frac{s^2}{2}\lambda_{mn}\|re\|^2_{L^2(Q)}$$

$$+\frac{s^2}{2}\int_Q (A(re)_+, (re)_+)_{R^2} - s\int_Q (AU_0, \tilde{U})_{R^2}dxdt$$

$$-\frac{2}{\alpha+\beta}s^{\alpha+\beta}\int_Q (\tilde{u} + \frac{u_0}{s})_-^\alpha(\tilde{v} + \frac{v_0}{s})_-^\beta dxdt$$

$$\leq \frac{s^2}{2}(-3 + \sqrt{ab})\|P^-\tilde{U}\|^2_{L^2(Q)} + \frac{s^2}{2}(\lambda_{mn} + \sqrt{ab})\|re\|^2_{L^2(Q)}$$

$$-s\int_Q (AU_0,\tilde{U})_{R^2}dxdt - \frac{2}{\alpha+\beta}s^{\alpha+\beta}\int_Q (\tilde{u}+\frac{u_0}{s})_-^{\alpha}(\tilde{v}+\frac{v_0}{s})_-^{\beta}dxdt$$

for some $\lambda_{mn} > 0$ Choosing $s_1 > 0$ such that $\frac{u_0(x,t)}{s_1}, \frac{v_0(x,t)}{s_1} \geq 1, \forall (x,t) \in Q$, we get, by (3.6) that the last integral in the inequality above is positive for $s \leq s_1$ since $\alpha+\beta > 2$. Since $\alpha+\beta > 2$, $J(s\tilde{U}) \to -\infty$ as $s \to \infty$. Therefore we can choose a large number $R > 0$ and a large ball $(D_R \subset E^0 \oplus E^-) \oplus \{re|\ 0 < r < R\}$ with radius $R > 0$ such that if $U \in \partial(\bar{D}_R \cap E^0 \oplus E^-) \oplus \{re|\ 0 < r < R\}$, then $\sup J(U) \leq 0$. So the assertion (ii) hold. Thus the lemma is proved. □

We shall prove that the functional J satisfies the $(P.S.)_c^*$ condition with respect to $(E_n)_n$ for any $c \in R$.

LEMMA 8. *Assume that the conditions (1.3), (1.4) and (1.5) hold. Then the functional J satisfies the $(P.S.)_c^*$ condition with respect to $(E_n)_n$ for any real number c.*

Proof. Let $c \in R$ and (h_n) be a sequence in N such that $h_n \to +\infty$, $(U_n)_n$ be a sequence such that

$$U_n = (u_n, v_n) \in E_{h_n}, \forall n, J(U_n) \to c, P_{E_{h_n}}\nabla J(U_n) \to 0.$$

We claim that $(U_n)_n$ is bounded. By contradiction we suppose that $\|U_n\|_E \to +\infty$ and set $\hat{U}_n = \frac{U_n}{\|U_n\|_E}$. Then

$$\langle P_{E_{h_n}}\nabla J(U_n), \hat{U}_n\rangle = 2\frac{J(U_n)}{\|U_n\|_E} -$$

$$\frac{\int_Q (\frac{2\alpha}{\alpha+\beta}(u_n+u_0)_-^{\alpha-1}(v_n+v_0)_-^{\beta}u_n + \frac{2\beta}{\alpha+\beta}(u_n+u_0)_-^{\alpha}(v_n+v_0)_-^{\beta-1}v_n}{\|U_n\|_E}$$

$$\frac{-\frac{4}{\alpha+\beta}(u_n+u_0)_-^{\alpha}(v_n+v_0)_-^{\beta})dxdt}{\|U_n\|_E} \longrightarrow 0.$$

Hence

$$\frac{\int_Q (\frac{2\alpha}{\alpha+\beta}(u_n+u_0)_-^{\alpha-1}(v_n+v_0)_-^{\beta}u_n + \frac{2\beta}{\alpha+\beta}(u_n+u_0)_-^{\alpha}(v_n+v_0)_-^{\beta-1}v_n)dxdt}{\|U_n\|_E}$$

$$-\frac{\frac{4}{\alpha+\beta}\int_Q (u_n+u_0)_-^{\alpha}(v_n+v_0)_-^{\beta}dxdt}{\|U_n\|_E} \longrightarrow 0.$$

Since $\|U_n\|_E \to \infty$ and $\frac{\frac{4}{\alpha+\beta}\int_Q (u_n+u_0)_-^{\alpha}(v_n+v_0)_-^{\beta}dxdt}{\|U_n\|_E}$ is bounded in Q,

$$\frac{\text{grad}(\int_Q \frac{2}{\alpha+\beta}(u_n+u_0)_-^{\alpha}(v_n+v_0)_-^{\beta}dxdt)\cdot U_n}{\|U_n\|_E} \quad \text{converges to } 0$$

and $\hat{U_n} \rightharpoonup 0$. We get

$$\frac{P_{E_{h_n}}\nabla J(U_n)}{\|U_n\|_E} = P_{E_{h_n}}\int_Q [\mathcal{L}\hat{U_n} + A(\hat{U}n + \frac{U_0}{\|U_n\|_E})_+ - A\frac{U_0}{\|U_n\|_E}]$$
$$-\frac{P_{E_{h_n}}\text{grad}(\int_Q \frac{2}{\alpha+\beta}(u_n+u_0)_-^{\alpha}(v_n+v_0)_-^{\beta}dxdt)}{\|U_n\|_E} \longrightarrow 0,$$

so $P_{E_{h_n}}\mathcal{L}\hat{U_n} + A(\hat{U}n + \frac{U_0}{\|U_n\|_E})_+ - A\frac{U_0}{\|U_n\|_E}$ converges. Since $(\hat{U_n})_n$ and $\frac{U_0}{\|U_n\|_E}$ are bounded and $\mathcal{L}$ and A are compact mappings, up to subsequence, $(\hat{U_n})_n$ has a limit. Since $\hat{U_n} \rightharpoonup (0,0)$, we get $\hat{U_n} \to (0,0)$, which is a contradiction to the fact that $\|\hat{U_n}\|_E = 1$. Thus $(U_n)_n$ is bounded. We can now suppose that $U_n \rightharpoonup U$ for some $U \in E$. Since the mapping $U \mapsto \text{grad}(\int_Q \frac{2}{\alpha+\beta}(u_n+u_0)_-^{\alpha}(v_n+v_0)_-^{\beta}dxdt)$ is a compact mapping, $\text{grad}(\int_Q \frac{2}{\alpha+\beta}(u_n+u_0)_-^{\alpha}(v_n+v_0)_-^{\beta}dxdt) \longrightarrow \text{grad}(\int_Q \frac{2}{\alpha+\beta}(u+u_0)_-^{\alpha}(v+v_0)_-^{\beta}dxdt)$. Thus $(P_{E_{h_n}}(\mathcal{L}U_n + A(U_n+U_0)_+)_n$ converges. Since $\mathcal{L}$ and A are compact operators and $(U_n)_n$ is bounded, we deduce that, up to a subsequence, $(U_n)_n$ converges to some U strongly with $\nabla J(U) = \lim \nabla J(U_n) = 0$. Thus we prove the lemma. □

4. Existence of the second solution

We note that $J(0,0) = 0$ and $(u,v) \mapsto \text{grad}(\frac{2}{\alpha+\beta}\int_Q (u+u_0)_-^{\alpha}(v+v_0)_-^{\beta}dxdt)$ is a compact mapping. By Lemma 3.2, there exist a small number $\rho > 0$ and a small ball $B_\rho \subset E^0 \oplus E^+$ with radius ρ such that if $U \in \partial B_\rho$, then $\gamma = \inf J(U) > 0$, and there is an $e \in E^+$, $R > \rho > 0$ and a large ball D_R with radius $R > 0$ such that if $W = (\bar{D}_R \cap (E^0 \oplus E^-)) \oplus \{re|\ 0 < r < R\}$, then $\sup_{U \in \partial W} J(U) \le 0$. Let us set $\tau = \sup_W J$. We note that $\tau < +\infty$. Let $(E_n)_n$ be a sequence of subspaces of E satisfying (3.4). Clearly $E^0 \subset E_n$ for all n, and ∂B_ρ and ∂W link. We have, for all $n \in N$,

$$\gamma \le \sup_{\partial W \cap E_n} J < \inf_{\partial B_\rho \cap E_n} J.$$

Moreover, by Lemma 3.3, $J_n = J|_{E_n}$ satisfies the $(P.S.)_c^*$ condition for any $c \in R$. Thus by Lemma 3.1 (Linking Theorem), there exists a critical point (u_n, v_n) for J_n with

$$\gamma \leq \inf_{\partial B_\rho \cap E_n} J \leq J(u_n, v_n) \leq \sup_{W \cap E_n} J \leq \tau.$$

Since J_n satisfies the $(P.S.)_c^*$ condition, we obtain that, up to a subsequence, $(u_n, v_n) \to (\bar{u}, \bar{v})$, with $(\bar{u}, \bar{v})$ a critical point for J such that $\gamma \leq J(\bar{u}, \bar{v}) \leq \tau$. Hence $(\bar{u}, \bar{v}) \neq (0, 0)$. Thus the functional I has two nontrivial solutions, one of which is a positive solution (u_0, v_0) and the second solution of which is $(\bar{u} + u_0, \bar{v} + v_0)$. Thus system (1.1) has at least two nontrivial solutions. Thus Theorem 1.1 is proved.

References

[1] K. C. Chang, *Solutions of asymptotically linear operator equations via Morse theory,* Comm. Pure Appl. Math. **34**, 693-712 (1981).

[2] K. C. Chang, *Infinite dimensional Morse theory and multiple solution problems,* Birkhäuser, (1993).

[3] Q. H. Choi and T. JungT. Jung, *A nonlinear suspension bridge equation with nonconstant load,* Nonlinear Analysis TMA. **35**, 649-668 (1999).

[4] Q. H. Choi and T. JungT. Jung, *Multiplicity results for the nonlinear suspension bridge equation,* Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis, **Vol.9**, 29-38 (2002).

[5] Q. H. Choi, T. JungT. Jung and P. J. McKenna, *The study of a nonlinear suspension bridge equation by a variational reduction method,* Applicable Analysis, **50**, 73-92 (1993).

[6] P. J. McKenna and W. WalterW. Walter, *Nonlinear oscillations in a suspension bridge,* Archive for Rational Mechanics and Analysis **Vol. 98**, No. 2, 167-177 (1987).

[7] J. T. Schwartz, *Nonlinear functional analysis,* Gordon and Breach, New York, (1969).

[8] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, C.B.M.S. Reg. Conf. Ser. in Math. 6, American Mathematical Society, Providence, R1,(1986).

Department of Mathematics
Kunsan National University
Kunsan 573-701 Korea
E-mail: tsjung@kunsan.ac.kr

Department of Mathematics Education
Inha University
Incheon 402-751 Korea
E-mail: qheung@inha.ac.kr