Korean J. Math. **16** (2008), No. 3, pp. 363–367

S-DISTAL EXTENSIONS OF FLOWS

Young key Kim* and Woo hwan Park

Abstract. In this paper, we define the S-distal flow and S-distal homomorphism which are motivated by the distal flow and the distal homomorphism respectively and obtain some results and that an S-distal extension of an S-distal flow is S-distal.

1. Definitions and Preliminaries

Definition 1.1. Let (X, T) be a flow. A subset M of X is *invariant* if $MT \subset M$. A closed nonempty subset A of X is called a *minimal set* if for every $x \in A$, the orbit xT is a dense subset of A. If X is itself minimal, we say that it is a minimal flow.

Let (X, T) be any flow with compact Hausdorff phase space X. As is customary, let X^X denote the set of all functions from X to X provided with the topology of pointwise convergence and consider T as a subset of X^X. The *enveloping semigroup* $E(X)$ of the flow (X, T) is the closure of T in X^X.Then $E(X)$ is a compact Hausdorff space, and we may consider $(E(X), T)$ as a flow, whose phase space $E(X)$ admits a semigroup structure. The *minimal right ideals* I of $E(X)$ is the nonempty subsets I of $E(X)$ such that $IE(X) \subset I$, which contains no proper nonempty subsets with the same property.

Definition 1.2. Let (X, T) and (Y, T) be flows and $\Psi : X \to Y$ be a function. Then Ψ is called a *homomorphism* if Ψ is continuous and $\Psi(xt) = \Psi(x)t$, $(x \in X, t \in T)$. If Y is minimal, Ψ is always onto. If $\Psi : X \to Y$ is a homomorphism and onto, we say that Ψ is an *epimorphism.* If there is a homomorphism Ψ from X onto Y, we say that Y is a *factor* of X, and that X is an *extension* of Y . Especially, a homomorphism Ψ from (X, T) into itself (not necessarily onto) is called

Received July 22, 2008. Revised July 31, 2008.
2000 Mathematics Subject Classification: 54H20.
Key words and phrases: distal, S-distal, flow.

an *endomorphism* of (X,T) and if Ψ is bijective, then Ψ is called an *automorphism* of (X,T). The set of endomorphisms of (X,T) is denoted by $H(X)$, and the set of automorphisms of (X,T) is denoted by $A(X)$.

THEOREM 1.3. *([2], Prorosition 3.3 and Proposition 3.8)*
Suppose that $E(X)$ is the enveloping semigroup of X.
Then
(1) The maps $\Theta_x : E(X) \to X$ defined by $\Theta_x(p) = xp$ are homomorphisms with
range $\overline{xT}$.
(2) Given an epimorphism $\Psi : X \to Y$, there exists a unique epimorphism
$\Theta : (E(X),T) \to (E(Y),T)$ such that the diagram

$$\begin{array}{ccc} E(X) & \xrightarrow{\Theta} & E(Y) \\ {\scriptstyle \Theta_x}\downarrow & & \downarrow{\scriptstyle \Theta_{\Psi(x)}} \\ X & \xrightarrow{\Psi} & Y \end{array} \tag{1.1}$$

commutes, $^{\forall}x \in X$.

DEFINITION 1.4. Let T be a topological group and A be a subset of T. Then A is called *syndetic* if there exists a compact subset K of T such that $T = AK$.

DEFINITION 1.5. A point x in a flow (X,T) is said to be *almost periodic* if given any neighborhood U of x, the set $A = \{t \in T \mid xt \in U\}$ is syndetic.

THEOREM 1.6. *([2], Proposition 3.7) Let (X,T) be a flow and $x \in X$. Let E be its enveloping semigroup and I be a minimal ideal in E. Then the the following statements are equivalent :*
(1) x is an almost peridic point of (X,T).
(2) $\overline{xT} = xI = \{xp \mid p \in I\}$.
(3) there exists an idempotent $u \in I$ such that $xu = x$.

THEOREM 1.7. *([2],Proposition 6.1) Let $\Psi : (X,T) \to (Y,T)$ be an epimorphism and let y be an almost periodic point of (Y,T). Then there exists an almost periodic point x of (X,T) such that $\Psi(x) = y$.*

DEFINITION 1.8. Let (X,T) be a flow and $x, y \in X$.
(1) x and y are *weakly proximal* if there exists a net (t_α) in T such that

$\lim_\alpha xh(t_\alpha) = \lim_\alpha yt_\alpha$, for some $h \in H(T)$.

(2) The set of all weakly proximal pairs in X is called the *weakly proximal relation* and is denoted by $WP(X,T)$ or WP.

(3) A flow (X,T) is said to be *weakly proximal* if $WP(X,T) = X \times X$

2. S-distal homomorphisms

DEFINITION 2.1. Let (X,T) be a flow and $x, y \in X$. Then (X,T) is S-*distal* provided that if there exists a net (t_α) in T such that $\lim_\alpha xh(t_\alpha) = \lim_\alpha yt_\alpha$, for some $h \in H(T)$, then $x = y$.

DEFINITION 2.2. A homomorphism $\Psi : (X,T) \to (Y,T)$ is S-*distal* if $\Psi(x) = \Psi(x^{'})$ and $(x, x^{'}) \in WP(X,T)$ imply $x = x^{'}$.

LEMMA 2.3. *The flow (X,T) is S-distal if and only if $WP(X,T) = \Delta$, where Δ is the diagonal of $X \times X$.*

THEOREM 2.4. *Let $\Psi : (X,T) \to (Y,T)$ be* S-distal *and (Y,T) be pointwise almost periodic. Then (X,T) is pointwise almost periodic.*

proof. *Let x be any point of (X,T). Then $y = \Psi(x)$ is an almost periodic point since (Y,T) is pointwise almost periodic. Thus there exists an almost periodic point z of (X,T) with $\Psi(z) = y$ by Theorem 1.7. Let I be a minimal right ideal of $E(X)$ of the flow (X,T) and $\Theta : (E(X),T) \to (E(Y),T)$ be a homomorphism. Since z is an almost periodic point, there exists an idempotent $u \in I$ such that $zu = z$. Since $y = \Psi(z) = \Psi(zu) = \Psi(z)\Theta(u) = y\Theta(u)$, we have $\Psi(xu) = \Psi(x)\Theta(u) = y\Theta(u) = y$.Thus x and xu belong to $\Psi^{-1}(y)$ and $(x, xu) \in WP(X,T)$. Since Ψ is* S-distal, *we have $x = xu$. Therefore (X,T) is pointwise almost periodic.*

COROLLARY 2.5. *The product of an* S-distal *flow and a pointwise almost periodic flow is pointwise almost periodic.*

proof. *Let (X,T) be an S-distal flow and (Y,T) be pointwise almost periodic. Let $\Pi_y : (X \times Y, T) \to (Y,T)$ be the projection. Then Π_y is an epimorphism and Π_y is an* S-distal *homomorphism. Therefore $(X \times Y, T)$ is pointwise almost periodic by Theorem 2.4.*

THEOREM 2.6. *An S-distal extension of an S-distal flow is S-distal.*

proof. *Let $\Psi : X \to Y$ be an S-distal homomorphism with Y S-distal. Suppose that if $(x_1, x_2) \in WP$, then $(\Psi(x_1), \Psi(x_2)) \in WP$. Since*

Y is S-distal, $\Psi(x_1) = \Psi(x_2)$. Since Ψ is S-distal, $x_1 = x_2$. Therefore X is an S-distal flow.

THEOREM 2.7. *Let $\Psi : (X,T) \to (Y,T)$ be an S-*distal *homomorphism. If (Y,T) is minimal and $y \in Y$, then $\{\overline{xT} \mid x \in \Psi^{-1}(y)\}$ is a partition of X.*

proof. *Let z be any point of (X,T). Then $\Psi(z) \in (Y,T)$. Since (Y,T) is minimal, there exists a net (t_α) in T such that $y = \lim_\alpha \Psi(z)t_\alpha$. We may assume that $\lim_\alpha zt_\alpha$ exists. Since $\Psi(\lim_\alpha zt_\alpha) = \lim_\alpha \Psi(z)t_\alpha = y$, we have $\lim_\alpha zt_\alpha \in \Psi^{-1}(y)$. Now let $\lim_\alpha zt_\alpha = x$. Then $z \in \overline{xT}$ by Theorem 2.4. Therefore $\{\overline{xT} \mid x \in \Psi^{-1}(y)\}$ is a partition of X.*

By Theorem 1.3, we have the following theorem.

THEOREM 2.8. *Let $\Psi : (X,T) \to (Y,T)$ be an S-distal epimorphism. Then $\Theta : (E(X),T) \to (E(Y),T)$ is also an S-distal epimorphism.*

proof. *Let $p,q \in E(X)$ such that $\Theta(p) = \Theta(q)$ and $(p,q) \in WP(E(X),T)$. Then there exists a net (t_α) in T such that $\lim_\alpha ph(t_\alpha) = \lim_\alpha qt_\alpha$ for some $h \in H(T)$. Let x be any element of (X,T).*

Then Θ_x is a homomorphism from $(E(X),T)$ into (X,T). Thus

$$\Theta_x(\lim_\alpha ph(t_\alpha)) = \lim_\alpha \Theta_x(p)h(t_\alpha) = \lim_\alpha (xp)h(t_\alpha),$$

$$\Theta_x(\lim_\alpha qt_\alpha) = \lim_\alpha \Theta_x(q)t_\alpha = \lim_\alpha (xq)t_\alpha.$$

Therefore we get $(xp, xq) \in WP(X,T)$.

Since $\Psi(xp) = \Psi(x)\Theta(p) = \Psi(x)\Theta(q) = \Psi(xq)$ and Ψ is S-distal, we obtain $xp = xq$. Since x is any point of (X,T), we have $p = q$. Therefore Θ is S-distal.

References

[1] J.Auslander, *Minimal Flows and Their Extensions*,North-Holland, 1988.

[2] R.Ellis, *Lectures on topological dynamics*, New York, 1969.

[3] S. Glasner, *Compressibility Properties in Topogical Dynamics*,Amer. J. Math. **97**, 148-171(1975).

[4] Y.K.Kim and H.Y.Byun , *F-proximal Flows*, Comm. Korean Math. Soc. **13** (1) (1998), 131-136.

Department of Mathematics,
MyongJi University,
Seoul 449-728, Korea
E-mail: ykkim@mju.ac.kr

Department of Mathematics,
MyongJi University,
Seoul 449-728, Korea