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ON PROPERTIES OF QUASI-CLASS A OPERATORS

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Abstract. Let the set of all quasi-class A operators for which $\ker(A) \subseteq \ker(A^*)$ be denoted by $A \in \mathcal{QA}^*$. In this paper it is proved that an operator $T \in \mathcal{QA}^*$ is normal if and only if the Duggal transform of T is normal.

1. Introduction

Let $\mathscr{L}(\mathscr{H})$ denote the algebra of bounded linear operators on a complex infinite dimensional Hilbert space $\mathscr{H}$. Recall ([1], [3], [7]) that $T \in \mathscr{L}(\mathscr{H})$ is called *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$ for $p \in (0,1]$, and T is called *paranormal* if $||T^2x|| \geq ||Tx||^2$ for all unit vector $x \in \mathscr{H}$. Following [6] and [5] we say that $T \in \mathscr{L}(\mathscr{H})$ belongs to *class A* if $|T^2| \geq |T|^2$. Recall ([9], [11]) that T is called *p-quasihyponormal* if $T^*(T^*T)^pT \geq T^*(TT^*)^pT$ for $p \in (0,1]$. For brevity, we shall denote classes of p-hyponormal operators, p-quasihyponormal operators, paranormal operators, and class A operators by $\mathcal{H}(p)$, $\mathcal{QH}(p)$, $\mathcal{PN}$, and $\mathcal{A}$, respectively. It is well known that

$$\mathcal{H}(p) \subset \mathcal{A} \subset \mathcal{PN} \text{ and } \mathcal{H}(p) \subset \mathcal{QH}(p) \subset \mathcal{PN}. \tag{1.1}$$

In [8] Jeon and Kim considered an extension of the notion of class A operators, similar in spirit to the extension of the notion of p-hyponormality to p-quasihyponormality; we say that $T \in \mathscr{L}(\mathscr{H})$ is *quasi-class A* if

$$T^*|T^2|T \geq T^*|T|^2T.$$

For brevity, we shall denote the set of quasi-class A operators by $\mathcal{QA}$. As shown in [8], the class of quasi-class A operators properly contains

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classes of class A operators and p-quasihyponormal operators, i.e., the following inclusions holds;

$$\mathcal{H}(p) \subset \mathcal{QH}(p) \subset \mathcal{QA} \text{ and } \mathcal{H}(p) \subset \mathcal{A} \subset \mathcal{QA} \tag{1.2}$$

In view of inclusions (1.1) and (1.2), it seems reasonable to expect that the operators in class $\mathcal{QA}$ are paranormal. But there exists an example [8] that one would be wrong in such an expectation.

Throughout this paper, restricting ourselves to those $A \in \mathcal{QA}$ for which $\ker(A) \subseteq \ker(A^*)$, denoted $A \in \mathcal{QA}^*$, we prove that an operator $T \in \mathcal{QA}^*$ is normal if and only if the Duggal transform of T is normal. From this result it is also proved that every operator $A \in \mathcal{QA}^*$ satisfies property (β), and for densely intertwined operators $\mathcal{QA}^*$, their spectra, essential spectra, and indices are preserved.

2. Main results

If $T \in \mathscr{L}(\mathscr{H})$ has the polar decomposition $T = U|T|$, then $T^{\backsim} = |T|U$ is called to be *Duggal transform* of T. It is well known that Duggal transform is one of very useful tools to study properties of operators ([4]). As an essential tool, to prove our main results below, we will use Duggal transforms of $\mathcal{QA}^*$ operators.

THEOREM 2.1. *The operator $A \in \mathcal{QA}^*$ is normal if and only if $A^{\backsim}$ is normal, and then $A = A^{\backsim}$.*

Proof. First, we claim that

$$A \in \mathcal{QA}^* \text{ if and only if } A^{\backsim} \in \mathcal{A}.$$

Let A have a decomposition $A = A_1 \oplus A_2$ with respect to some decomposition $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ of $\mathscr{H}$, such that $A_1 = A|_{\mathscr{H}_1}$ is normal and $A_2 = A|_{\mathscr{H}_2}$ is pure. Let A_i have the polar decomposition $A_i = U_i|A_i|$. Then the partial isometry U_2 is an isometry, and we may choose the partial isometry U_1 to be a unitary such that the commutator

$$[|A_1|, U_1] = |A_1|U_1 - U_1|A_1| = 0.$$

Define the Duggal transform $A^{\backsim}$ of A by

$$A^{\backsim} = A_1 \oplus A_2^{\backsim} = (|A_1| \oplus |A_2|)(U_1 \oplus U_2).$$

Since $A_2 \in \mathcal{QA}^*$, A_2 is injective and $|A_2|$ is a quasiaffinity, it follows from the equivalence

$$A_2^*(|A_2{}^2| - |A_2|^2)A_2 \geq 0 \Longleftrightarrow U_2^*(|A_2{}^2| - |A_2|^2)U_2 \geq 0$$

that

$$|\widetilde{A_2}|^2 = U_2^*|A_2|^2U_2 \leq U_2^*|A_2^2|U_2 \leq (U_2^*A_2^{*2}A_2^2U_2)^{\frac{1}{2}} = |\widetilde{A_2}^2|,$$

i.e., $\widetilde{A_2} = |A_2|U_2 \in \mathcal{A}$. Hence $\widetilde{A}$ is a class A operator.

Now, if A is normal, then A_2 acts on the trivial space $\{0\}$, and so $A = A_1 = \widetilde{A}$. If, instead, $\widetilde{A}$ is normal, then the normality of $\widetilde{A_2}$ implies that

$$U_2^*|A_2|^2U_2 = |A_2|^2 = |A_2|U_2U_2^*|A_2|.$$

Since $|A_2|$ is a quasiaffinity, $U_2U_2^* = I|_{H_2}$, so that U_2 is a unitary. But then $U_2^*|A_2|^2U_2 = |A_2|^2$, which implies that $[|A_2|, U_2] = 0$, i.e., A_2 is normal. Since A_2 is pure, A_2 acts on the trivial space $\{0\}$. Hence $A = \widetilde{A} = A_1$. □

Recall that a $T \in \mathscr{L}(\mathscr{H})$ has (Bishop's) property (β) if, for every open subset U of $\mathbb{C}$ and every sequence of analytic functions $f_n : U \longrightarrow \mathscr{H}$ with the property that

$$(T - \lambda)f_n(\lambda) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

uniformly on all compact subsets of U, it follows that $f_n(\lambda) \longrightarrow 0$ as $n \longrightarrow \infty$ locally uniformly on U [10, Definition 1.2.5].

Corollary 2.2. *Every operator $A \in \mathcal{QA}^*$ satisfies property (β).*

Proof. Let A and $\widetilde{A}$ be defined as in the proof of Theorem 2.1. Then observe that A has property (β) if and only if A_2 has property (β). Recall that if $R, S \in \mathscr{L}(\mathscr{H})$ are injective, then RS satisfies property (β) if and only if SR satisfies property (β) [3, Theorem 5]. Since the operators $|A_2|$ and U_2 are injective, and the operator $\widetilde{A_2} = |A_2|U_2 \in \mathcal{A}$ satisfies property (β) [2], it follows that $A_2 = U_2|A_2|$ satisfies property (β). □

Recall that an operator $T \in \mathscr{L}(\mathscr{H})$ is *Fredholm* if $T(\mathscr{H})$ is closed, $\alpha(T) = \dim T^{-1}(0) < \infty$ and $\beta(T) = \dim T^{*-1}(0) < \infty$). If $T \in \mathscr{L}(\mathscr{H})$ is Fredholm, then the *index* of T, denoted $i(T)$, is defined by $i(T) = \alpha(T) - \beta(T)$. The essential spectrum of T, denoted $\sigma_e(T)$, is the set of all $\lambda \in \mathbb{C}$ such that $T-\lambda I$ is not Fredholm. The operators $S, T \in \mathscr{L}(\mathscr{H})$ are said to be *densely intertwined* if there exist operators $X, Y \in \mathscr{L}(\mathscr{H})$ with dense range such that $SX = XT$ and $TY = YS$.

Corollary 2.3. *If $A, B \in \mathcal{QA}^*$ are densely intertwined, then $\sigma(A) = \sigma(B)$, $\sigma_e(A) = \sigma_e(B)$ and $i(A-\lambda) = i(B-\lambda)$ for all $\lambda \in \mathbb{C} \setminus \sigma_e(A)$.*

Proof. Both A and B have property (β), and [10, Theorem 3.7.15] applies. $\square$

We conclude with comments on Putnam-Fuglede type commutativity theorem. In facts, we have tried to get a Putnam-Fuglede type commutativity theorem for operators in $\mathcal{QA}^*$. But, first of all, we must find the proof of the following conjecture on Putnam-Fuglede type commutativity theorem for operators in $\mathcal{A}$.

Conjecture 2.4. *If $A, C^* \in \mathcal{A}$ are such that $AX = XC$ for some non-trivial $X \in \mathscr{L}(\mathscr{H})$, then $A^*X = XC^*$, $\overline{ranX}$ reduces A, $ker^{\perp}(X)$ reduces C, and $A|_{\overline{ranX}}$ and $C|_{ker^{\perp}(X)}$ are unitarily equivalent normal operators.*

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