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A NOTE ON THE RETURN TIME OF STURMIAN SEQUENCES

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Abstract. Let R_n be the the first return time to its initial n-word. Then the Ornstein-Weiss first return time theorem implies that $\log R_n$ divided by n converges to entropy. We consider the convergence of $\log R_n$ for Sturmian sequences which has the lowest complexity. In this case, we normalize the logarithm of the first return time by $\log n$. We show that for any numbers $1 \le \alpha, \beta \le \infty$, there is a Sturmian sequence of which limsup is α and liminf is $1/\beta$.

1. Introduction

The asymptotic behavior of the first return time (recurrence time) is one of the main ingredients in studying dynamical systems. Let $\{X_n : n \in \mathbb{N}\}$ be a stationary ergodic process on the space of infinite sequences $(\mathcal{A}^{\mathbb{N}}, \Sigma, \mu)$, where $\mathcal{A}$ is a finite set, Σ is the σ-field generated by finite dimensional cylinders and μ is a shift invariant ergodic probability measure. In this paper, we will consider $\mathcal{A} = \{0, 1\}$ and a binary sequence $x = x_1x_2x_3\dots$. Define $R_n(x)$ to be the first return time of the initial n-word $x_1 \dots x_n$, i.e.,

$$R_n(x) := \min\{j \ge 1 : x_1 \dots x_n = x_{j+1} \dots x_{j+n}\}.$$

The convergence of $\log R_n$ was first studied in relation to data compression algorithm. After Wyner and Ziv's work[12] for the convergence in probability, Ornstein and Weiss[9] showed that for an ergodic process

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with entropy h

$$\lim_{n\to\infty} \frac{1}{n} \log R_n(x) = h \tag{1}$$

almost surely.

A sequence $u = u_1u_2u_3\ldots$ is called Sturmian if its complexity function is $p_u(n) = n+1$, where the complexity function $p_u(n)$ denote the number of different words of length n occurring in u. Morse and Hedlund showed that Sturmian sequence has the least complexity function except for eventually periodic sequences (see e.g. [8]). A famous example of Sturmian sequence is the Fibonacci sequence 1011010110110101101 01 ...

Let $T : (X,\mu) \to (X,\mu)$ be a measure preserving transformation and $\mathcal{P} = \{I_0, I_1\}$ be a partition of X. Let $\{\mathcal{P}_n\}$ be a sequence of partitions of X obtained by $\mathcal{P}_n = \mathcal{P} \vee T^{-1}\mathcal{P} \vee \cdots \vee T^{-(n-1)}\mathcal{P}$, where $\mathcal{P} \vee \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}$. We call $u = u_1u_2\ldots u_n\ldots$ the $\mathcal{P}$-trajectory of x under T if $T^{i-1}x \in I_{u_i}$ for $i = 1,\ldots,n$. Let $I_n(x)$ be the element of $\mathcal{P}_n$ which contains x.

For a measurable subset $E \subset X$ with $\mu(E) > 0$ and a point $x \in E$ which returns to E under iteration of T, we define the first return time R_E on E by

$$R_E(x) = \min\left\{j \geq 1 : T^j x \in E\right\}.$$

For each trajectory u of $x \in X$, we have $R_{I_n(x)}(x) = R_n(u)$.

Let $0 < \theta < 1$ be an irrational number and $T : [0,1) \to [0,1)$ be an irrational rotation by θ, which preserve the Lebesgue measure μ on $X = [0,1)$ i.e.,

$$T(x) = x + \theta \pmod 1.$$

Let $\mathcal{P}$ be a partition of X given by $\mathcal{P} = \{[0, 1-\theta), [1-\theta, 1)\}$. A necessary and sufficient condition for the Sturmian sequence is that $u = u_1u_2u_3\ldots$ is a $\mathcal{P}$-trajectory of an irrational rotation (see e.g. [8]). For an example, the Fibonacci sequence is a $\mathcal{P}$-trajectory of an irrational rotation by the golden mean $\frac{\sqrt{5}-1}{2}$. Since an irrational rotation has zero entropy, if we normalize $\log R_n$ by n as in (1), it should converges to 0 in almost everywhere sense. Considering the Shannon-McMillan-Breiman Theorem, we might expect that $\log R_n$ should be normalized by the logarithm of the size of each element of $\mathcal{P}_n$, which is roughly $1/(n+1)$ when $n = q_k - 1$ (Section 3).

Let θ be an irrational number of type η, which is defined in Section 2. In [6] it is shown that for almost every Sturmian sequence u from the

$\mathcal{P}$-trajectory under the rotation by θ

$$\liminf_{n\to\infty} \frac{\log R_n(u)}{\log n} = \frac{1}{\eta} \quad \text{and} \quad \limsup_{n\to\infty} \frac{\log R_n(u)}{\log n} = 1.$$

That is, there exist many Sturmian sequences with the property that $\log R_n(u)/\log n$ does not converge.

An infinite sequence $u = u_1u_2u_3\dots$ is called k-automatic if it is generated by a k-automaton. An infinite sequence is k-automatic if and only if it is the image under a coding of a fixed point of a k-uniform morphism[3]. A well known example of the automatic sequences is the Thue-Morse sequence, which is a fixed point of the 2-uniform morphism of $\sigma(0) = 01$ and $\sigma(1) = 10$. It is known[2] that for the Thue-Morse sequence

$$\limsup \frac{p_u(n)}{n} = \frac{10}{3}, \quad \liminf \frac{p_u(n)}{n} = 3.$$

Let u be a non-eventually periodic automatic sequence on alphabet $\mathcal{A}$. Then it is shown[6] that

$$\lim_{n\to\infty} \frac{\log R_n(u)}{\log n} = 1.$$

In this paper we consider the first return time of Sturmian sequences. We can construct a Sturmian sequence with arbitrary limsup and liminf of the $\log R_n/\log n$:

Theorem 1.1. *For any $1 \le \alpha, \beta \le \infty$, there is a Sturmian sequence u such that*

$$\limsup_{n\to\infty} \frac{\log R_n(u)}{\log n} = \alpha, \qquad \liminf_{n\to\infty} \frac{\log R_n(u)}{\log n} = \frac{1}{\beta}.$$

2. The diophantine types of irrational numbers

We need some properties on diophantine approximations. For more details, consult [4] and [10]. For an irrational number $0 < \theta < 1$, we have a unique continued fraction expansion;

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

if $a_i \geq 1$ for all $i \geq 1$. Put $p_0 = 0$ and $q_0 = 1$. Choose p_i and q_i for $i \geq 1$ such that $(p_i, q_i) = 1$ and

$$\frac{p_i}{q_i} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + 1/a_i}}}.$$

We call each a_i the i-th partial quotient and p_i/q_i the i-th convergent. Then the denominator q_i and the numerator p_i of the i-th convergent satisfy the following properties: $q_{i+2} = a_{i+2}q_{i+1} + q_i$, $p_{i+2} = a_{i+2}p_{i+1} + p_i$ and

$$\frac{1}{2q_{i+1}} < \frac{1}{q_{i+1} + q_i} < \|q_i\theta\| < \frac{1}{q_{i+1}} \tag{2}$$

for $i \geq 1$.

For $t \in \mathbb{R}$ we denote $\|\cdot\|$ and $\{\cdot\}$ by the distances, respectively, to the nearest integer and the greatest integer less than or equal to t, i.e.,

$$\|t\| = \min_{n \in \mathbb{Z}} |t - n|, \qquad \{t\} = t - \lfloor t \rfloor,$$

An irrational number θ, $0 < \theta < 1$, is said to be of type η if

$$\eta = \sup\{t > 0 : \liminf_{j \to \infty} j^t \|j\theta\| = 0\}.$$

Note that every irrational number is of type $\eta \geq 1$. The set of irrational numbers of type 1 has measure 1 and includes the set of irrational numbers with bounded partial quotients, which is of measure 0. There exist numbers of type ∞, called Liouville numbers. Here we introduce a new definition on type of irrational numbers[5]:

Definition 2.1. An irrational number θ, $0 < \theta < 1$, is said to be of *type* (α, β) if

$$\alpha = \sup\{t > 0 : \liminf_{j \to \infty} j^t \{-j\theta\} = 0\},$$
$$\beta = \sup\{t > 0 : \liminf_{j \to \infty} j^t \{j\theta\} = 0\}.$$

For example, if the partial quotients of an irrational number θ is $a_{2i} = 2^{2^i}$ for $i \geq 1$ and $a_{2i+1} = 1$ for $i \geq 0$, then θ is of type $(2, 1)$. Note that $\alpha, \beta \geq 1$ and $\eta = \max\{\alpha, \beta\}$. For each $\alpha, \beta > 1$ there are uncountably many (but measure zero) θ's which are of type (α, β).

3. Proof of the main theorem

It is well known[4] that $\|j\theta\| \geq \|q_i\theta\|$ for $0 < j < q_{i+1}$ and $\theta - p_i/q_i$ is positive if and only if i is even. Thus, by the definition of type (α, β) in Definition 2.1, we have

$$\begin{aligned}\eta &= \sup\{t > 0 : \liminf_{i\to\infty} q_i^t \|q_i\theta\| = 0\},\\ \alpha &= \sup\{t > 0 : \liminf_{i\to\infty} q_{2i+1}^t \|q_{2i+1}\theta\| = 0\},\\ \beta &= \sup\{t > 0 : \liminf_{i\to\infty} q_{2i}^t \|q_{2i}\theta\| = 0\}.\end{aligned}$$

And we have the following lemma[5]:

Lemma 3.1. *For any $\epsilon > 0$ and $C > 0$, we have (i)*

$$q_{2i+1}^{\alpha+\epsilon}\|q_{2i+1}\theta\| > C \text{ and } q_{2i}^{\beta+\epsilon}\|q_{2i}\theta\| > C.$$

for sufficiently large integer i, and (ii) there are infinitely many odd i's such that $q_i^{\alpha-\epsilon}\|q_i\theta\| < C$ and even i's such that $q_i^{\beta-\epsilon}\|q_i\theta\| < C$.

For the irrational rotation, generally there are three values for the recurrence time R_E of an interval E[11], but for some specific length of interval the recurrence time has only two values. For the proof consult [7].

Theorem 3.2. *Let $b = \|q_{i-1}\theta\| - c\|q_i\theta\|$, $0 \leq c < a_{i+1}$. If i is even, then*

$$R_{[0,b)}(x) = \begin{cases} q_i, & 0 \leq x < b - \|q_i\theta\|,\\ (c+1)q_i + q_{i-1}, & b - \|q_i\theta\| \leq x < b.\end{cases}$$

If i is odd, then

$$R_{[0,b)}(x) = \begin{cases} (c+1)q_i + q_{i-1}, & 0 \leq x < \|q_i\theta\|,\\ q_i, & \|q_i\theta\| \leq x < b.\end{cases}$$

Let $\mathcal{P} = \{[0, 1-\theta), [1-\theta, 1)\}$ be a partition of $X = [0, 1)$. Note that $\mathcal{P}_n = \vee_{k=0}^{n-1} T^{-k}\mathcal{P}$ is the partition of $[0, 1)$ obtained by the orbit $\{-k\theta\}$, $0 \leq k \leq n$. The followings are well known, which is concerned with the lengths of the elements of $\mathcal{P}_n$ and the number of elements of each length([1], [11]). Let $n = cq_i + q_{i-1} + \ell$, $1 \leq c \leq a_{i+1}$ and $0 \leq \ell < q_i$. Then each length of element of $\mathcal{P}_n$ has only three values: $\|q_i\theta\|$, $\|q_{i-1}\theta\| - c\|q_i\theta\|$ and $\|q_{i-1}\theta\| - (c-1)\|q_i\theta\|$. Moreover, the number of elements of $\mathcal{P}_n$ with length $\|q_i\theta\|$, $\|q_{i-1}\theta\| - c\|q_i\theta\|$ and $\|q_{i-1}\theta\| - (c-1)\|q_i\theta\|$ is $n - q_i + 1$,

$\ell+1$ and $q_i-\ell-1$ respectively. Note that since $q_{i+1}=a_{i+1}q_i+q_{i-1}$ for a given n we can choose i, c and ℓ such that $n=cq_i+q_{i-1}+\ell$ where $1\le c\le a_{i+1}$ and $0\le \ell<q_i$.

Proof of Theorem 1.1. Let $\mathcal{P}_n$ be the a partition of $[0,1)$ given by orbit $\{-k\theta\}$, $0\le k\le n$ and $I_n(x)$ be the element of $\mathcal{P}_n$ which contains x.

Since $q_{i+2}=q_i+a_{i+2}q_{i+1}$, for a given n there is odd i and integer c, $0\le c<a_{i+2}$ satisfying that $q_i+cq_{i+1}\le n<q_i+(c+1)q_{i+1}$. Since $\|j\theta\|\ge\|q_i\theta\|$ for $0<j<q_{i+1}$ and $\theta q_i-p_i=(-1)^i\|q_i\theta\|$, we have

$$I_n(0)=[0,\|q_i\theta\|-c\|q_{i+1}\theta\|)$$

for each $q_i+cq_{i+1}\le n<q_i+(c+1)q_{i+1}$, $0\le c<a_{i+2}$. Let u be the trajectory of 0 under the rotation T. Then for each $q_i\le n<q_{i+2}$, i odd, we have by Theorem 3.2

$$R_n(u)=R_{I_n(x)}(0)=q_{i+1}.$$

By (2) we have for $q_i\le n<q_{i+2}$

$$\frac{\log q_{i+1}}{-\log\|q_{i+1}\theta\|}<\frac{\log q_{i+1}}{\log q_{i+2}}<\frac{\log R_n(u)}{\log n}\le\frac{\log q_{i+1}}{\log q_i}<\frac{-\log\|q_i\theta\|}{\log q_i}.$$

For any C and $\epsilon>0$ by Lemma 3.1 (i) we have for large odd i

$$\begin{aligned}\frac{\log C-\log\|q_{i+1}\theta\|}{-(\beta+\epsilon)\log\|q_{i+1}\theta\|}<\frac{\log q_{i+1}}{-\log\|q_{i+1}\theta\|}<\frac{\log R_n(u)}{\log n}&\\ \le\frac{-\log\|q_i\theta\|}{\log q_i}<\frac{-(\alpha+\epsilon)\log\|q_i\theta\|}{\log C-\log\|q_i\theta\|}&.\end{aligned}$$

Therefore, we have

$$\limsup_{n\to\infty}\frac{\log R_n(u)}{\log n}\le\alpha,\quad \liminf_{n\to\infty}\frac{\log R_n(u)}{\log n}\ge\frac{1}{\beta}.$$

By Lemma 3.1(ii) there are infinitely many odd i_k's such that $q_{i_k}^{\alpha-\epsilon}\|q_{i_k}\theta\|<C$. Let $n_k=q_{i_k}$. Then by (2) we have

$$\frac{\log R_{n_k}(u)}{\log n_k}=\frac{\log q_{i_k+1}}{\log q_{i_k}}>\frac{-\log\|q_{i_k}\theta\|-\log 2}{\log q_{i_k}}>\frac{-(\alpha-\epsilon)(\log\|q_{i_k}\theta\|+\log 2)}{\log C-\log\|q_{i_k}\theta\|}.$$

Hence we have

$$\limsup_{n\to\infty}\frac{\log R_n(u)}{\log n}\ge\alpha.$$

Similarly, Lemma 3.1 (ii) states that there are infinitely many even i_k's such that $q_i^{\beta-\epsilon}\|q_i\theta\| < C$. Choose $n_k = q_{i_k+1} - 1$. Then by (2) we have

$$\frac{\log R_{n_k}(u)}{\log n_k} = \frac{\log q_{i_k}}{\log(q_{i_k+1}-1)} = \frac{\log q_{i_k}}{\log q_{i_k+1} + \log(1-1/q_{i_k+1})}$$
$$< \frac{\log C - \log\|q_{i_k}\theta\|}{-(\beta-\epsilon)(\log\|q_{i_k}\theta\| + \log 2 - \log(1-1/q_{i_k+1}))},$$

which yields

$$\liminf_{n\to\infty} \frac{\log R_n(u)}{\log n} \le \frac{1}{\beta}.$$

□

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