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SOME PROPERTIES OF GM MODULES AND MR GROUPS

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Abstract. The purpose of this paper, GM modules are defined as a generalization of AGR rings. Also, from the faithful GM-property, we get a commutativity of rings. Finally, for a nearring R, we will introduce MR groups, and also derive some properties of GM modules and MR groups.

1. Introduction

Throughout this paper, we start with the study of rings in which all the additive endomorphisms or only the left multiplication endomorphisms are generated by ring endomorphisms. This research was motivated by the work on the Sullivan's Research Problem (that is, characterize those rings in which every additive endomorphism is a ring endomorphism, these rings are called *AE rings*) [15], [2], [3], [4], [5], [6] and [8], and the investigation of LSD-generated rings and SD-generated rings [2] and [7].

Let R be an associative ring or nearring not necessarily with unity, G be an additive group (not necessarily abelian) and M a right R-module.

We introduce some notions of nearring in [13]. We consider the following notations: Given a nearring R, $R_0 = \{a \in R \mid 0a = 0\}$, $R_c = \{a \in R \mid 0a = a\}$ and $R_d = \{a \in R \mid a \textit{ is distributive}\}$.

We note that R_0 and R_c are subnearrings of R, but R_d is not a subnearring of R. A nearring R with the extra axiom $0a = 0$ for all $a \in R$, that is, $R = R_0$ is said to be *zero symmetric,* also, in case $R = R_c$, R is called a *constant* nearring, and in case $R = R_d$, R is called a *distributive* near-ring.

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We denote that for a ring R, $End(R,+)$ is the ring of additive endomorphisms of R, $End(R,+,\cdot)$ the monoid of ring endomorphisms of R, $End(M)$ the ring of additive endomorphisms of M and $End_R(M)$ the ring of R-endomorphisms of the right R-module M. For $X \subseteq R$, we use $gp(X)$ for the subgroup of $(R,+)$ generated by X.

Let $(G,+)$ be a group (not necessarily abelian). We will use right operations (that is, operations on the right side of the variables) in nearring case to distinguish from left operations in ring case in this paper. In the set

$$M(G) := \{f \mid f : G \longrightarrow G\}$$

of all self maps of G,

If we define the sum $f+g$ of any two mappings f, g in $M(G)$ by the rule $x(f+g) = xf + xg$ for all $x \in G$ (called the *pointwise addition of maps*) and the product $f \cdot g$ by the rule $x(f \cdot g) = (xf)g$ for all $x \in G$, then $(M(G),+,\cdot)$ becomes a near-ring. It is called the *nearring of self maps* on G. Also, if we define the set

$$M_0(G) := \{f \in M(G) \mid of = o\}$$

for the additive group G with identity o, then $(M_0(G),+,\cdot)$ is a zero symmetric nearring.

Let R and S be two near-rings. Then a mapping θ from R to S is called a *near-ring homomorphism* if for all $a,\ b \in R$, (i) $(a+b)\theta = a\theta + b\theta$ and (ii) $(ab)\theta = a\theta b\theta$.

Let R be any nearring and G an additive group. Then G is called an *R-group* if there exists a nearring homomorphism

$$\theta : (R,+,\cdot) \longrightarrow (M(G),+,\cdot).$$

Such a homomorphism θ is called a *representation* of R on G.

We write xr (right scalar multiplication in R) for $x(\theta_r)$ for all $x \in G$ and $r \in R$. If R is unitary, then R-group G is called *unitary*. Thus an R-group is an additive group G satisfying (i) $x(a+b) = xa + xb$, (ii) $x(ab) = (xa)b$ and (iii) $x1 = x$ (If R has a unity 1), for all $x \in G$ and $a,\ b \in R$.

An R-group G with the property that for each $x,\ y \in G$ and $a \in R$, $(x+y)a = xa + ya$ is called a *distributive R-group*, and also an R-group

G with $(G, +)$ abelian is called an *abelian R-group.* For example, if $(G, +)$ is abelian, then $M(G)$ is an abelian nearring and moreover, G is an abelian $M(G)$-group, on the other hand, every distributive nearring R is a distributive R-group. Also, the existence of distributive abelian R-groups is shown in [13,14].

Let G and T be two R-groups. Then the mapping $f : G \longrightarrow T$ is called an *R-group homomorphism* if for all $x,\ y \in G$ and $a \in R$, (i) $(x+y)f = xf + yf$ and (ii) $(xa)f = (xf)a$. In this paper, we call that the mapping $f : G \longrightarrow T$ with the condition $(xa)f = (xf)a$ an *R-map* or *R-homogeneous map* [12].

A nearring R is called *distributively generated* (briefly, *D.G.*) by S if

$$(R, +) = gp < S >= gp < R_d >$$

where S is a semigroup of distributive elements in R, in particular, $S = R_d$ (this is motivated by the set of all distributive elements of R is multiplicatively closed and contain the unity of R if it exists), where $gp < S >$ is a group generated by S, we denote this d.g. nearring R generated by S as (R, S).

For the remaining concepts and results on ring case and nearrings case, we refer to J. D. P. Meldrum [13] and G. Pilz [14].

2. Some results of GM modules and MR groups

Hereafter, we can introduce similar notions of AR rings or almost AR rings in right R-modules and R-groups. First, we introduce a new concept of right R-modules and investigate it's properties.

For any ring R, right R-modules M and N, the set of all R-module homomorphisms from M to N is denoted by $Hom_R(M, N)$ and the set of all group homomorphisms from M to N is by $Hom(M, N) := Hom_{\mathbb{Z}}(M, N)$, in particular we denote that $End_R(M) := Hom_R(M, M)$ and $End(M) := End_{\mathbb{Z}}(M) = Hom_{\mathbb{Z}}(M, M)$, In this case, M is called a *GM module* over R if every group homomorphism of M is an module homomorphism, that is,

$$End(M) = End_R(M).$$

In particular, R is called a *GM ring* if R is a GM module as a right R-module, that is, for all $f \in End_{\mathbb{Z}}(R),\ x,\ r \in R$, we have $f(xr) = f(x)r$.

For example, $\mathbb{Q}$ (: rational field) is a GM ring, because of $End_{\mathbb{Z}}\mathbb{Q} = \mathbb{Q} = End_{\mathbb{Q}}\mathbb{Q}$, also, every $\mathbb{Q}$-module is a GM module.

In order to show some characterizations of GM modules and MR groups, we introduce the following useful lemma from [1].

LEMMA 2.1 [1]. *Let $\{M_i | i = 1, 2, \cdots, n\}$ be a finite family of right R-modules and let $M := \bigoplus_{i=1}^n M_i$ a direct sum of the M_i. Then*

$$End_R(M) \cong \begin{bmatrix} End_R(M_1) & Hom_R(M_2, M_1) & \cdots & Hom_R(M_n, M_1) \\ Hom_R(M_1, M_2) & End_R(M_2) & \cdots & Hom_R(M_n, M_2) \\ \vdots & \vdots & \cdots & \vdots \\ Hom_R(M_1, M_n) & Hom_R(M_2, M_n) & \cdots & End_R(M_n) \end{bmatrix}$$

as rings.

PROPOSITION 2.2. *Let $\{M_i | i = 1, 2, \cdots, n\}$ be a finite family of right R-modules and let $M := \bigoplus_{i=1}^n M_i$ a direct sum of the M_i. Then M is a GM module if and only if we have that $Hom(M_i, M_j) = Hom_R(M_i, M_j)$ for all i, j in $\{1, 2, 3, \cdots, n\}$. In particular, each M_i is a GM module for all i in $\{1, 2, 3, \cdots, n\}$.*

Proof. From the Lemma 2.1, we will prove only the "if" direction as following:

$$End(M) := End_{\mathbb{Z}}(M) = End_{\mathbb{Z}}(\bigoplus_{i=1}^{n} M_i)$$

$$\cong \begin{bmatrix} End_{\mathbb{Z}}(M_1) & Hom_{\mathbb{Z}}(M_2, M_1) & \cdots & Hom_{\mathbb{Z}}(M_n, M_1) \\ Hom_{\mathbb{Z}}(M_1, M_2) & End_{\mathbb{Z}}(M_2) & \cdots & Hom_{\mathbb{Z}}(M_n, M_2) \\ \vdots & \vdots & \cdots & \vdots \\ Hom_{\mathbb{Z}}(M_1, M_n) & Hom_{\mathbb{Z}}(M_2, M_n) & \cdots & End_{\mathbb{Z}}(M_n) \end{bmatrix}$$

$$= \begin{bmatrix} End_R(M_1) & Hom_R(M_2, M_1) & \cdots & Hom_R(M_n, M_1) \\ Hom_R(M_1, M_2) & End_R(M_2) & \cdots & Hom_R(M_n, M_2) \\ \vdots & \vdots & \cdots & \vdots \\ Hom_R(M_1, M_n) & Hom_R(M_2, M_n) & \cdots & End_R(M_n) \end{bmatrix}$$

$$\cong End_R(\bigoplus_{i=1}^{n} M_i) = End_R(M).$$

□

THEOREM 2.3. *Let $\{M_i | i \in \Lambda\}$ be any family of right R-modules. Then each M_i is a GM module if and only if $M := \bigoplus M_i$ is a GM module.*

Proof. Let $f \in End(M)$, and let $x = (x_i)_{i \in \Lambda} =: (x_i) \in M$, where $x_i \in M_i,\ x_i = 0$ except finitely many $i \in \Lambda$ and $r \in R$. For each $i \in \Lambda$, we define $f_i \in End(M_i)$ by $x_i \longmapsto f_i(x_i)$ as following:

$$f(0, \cdots, x_i, \cdots, 0) \ = \ (0, \cdots, f_i(x_i), \cdots, 0)$$

Then f_i is well-defined. Indeed, if $x_i = y_i$ in M_i, then clearly,

$$f(0, \cdots, x_i, \cdots, 0) = f(0, \cdots, y_i, \cdots, 0)$$

this implies that

$$((0, \cdots, f_i(x_i), \cdots, 0) = (0, \cdots, f_i(y_i), \cdots, 0)$$

Also, this implies that $f(x_i) = f(y_i)$, so that f_i is well-defined.

Since $f \in End(M)$, we have the following equalities:

$$(0, \cdots, f_i(x_i + y_i), \cdots, 0) = f(0, \cdots, x_i + y_i, \cdots, 0)$$

$$= f(0, \cdots, x_i, \cdots, 0) + f(0, \cdots, y_i, \cdots, 0)$$

$$= (0, \cdots, f_i(x_i), \cdots, 0) + (0, \cdots, f_i(y_i), \cdots, 0)$$

$$= (0, \cdots, f_i(x_i) \ + \ f_i(y_i), \cdots, 0)$$

Hence $f_i(x_i + y_i) = f_i(x_i) + f_i(y_i)$, that is, $f_i \in End(M_i)$.

On the other hand, since each M_i is a GM module, that is, $f_i \in End(M_i) = End_R(M_i)$, we get

$f((0, \cdots, x_i r, \cdots, 0)r) = (0, \cdots, f_i(x_i r), \cdots, 0) = (0, \cdots, f_i(x_i)r, \cdots, 0)$.

With this and $x = (x_i)$ is a finite sum of the form $\sum(0, \cdots, x_i, \cdots, 0)$, where $x_i \neq 0$ for $i \in \Lambda$, We see that

$$f(xr) = f(\sum(0, \cdots, x_i, \cdots, 0)r) = f(\sum(0, \cdots, x_i r, \cdots, 0)$$
$$= (\sum f(0, \cdots, x_i r, \cdots, 0)) = (\sum(0, \cdots, f_i(x_i)r, \cdots, 0))$$
$$= (\sum(0, \cdots, f(x_i), \cdots, 0)r) = f(x)r.$$

Hence f is an R-module homomorphism. Consequently, $M = \bigoplus M_i$ is a GM module.

Conversely, let $g_i \in End(M_i)$ and let $x_i \in M_i,\ r \in R$, Consider, $x = (x_i)_{i \in \Lambda} =: (x_i) \in \bigoplus M_i$, where $x_i \in M_i,\ x_i = 0$ except finitely many $i \in \Lambda$ in particular, $x = (0, \cdots, x_i, \cdots, 0)$ in M. Define

$$f : \bigoplus M_i \longrightarrow \bigoplus M_i$$

$f(x) = (g_i(x_i))_{i \in \Lambda} =: (g_i(x_i))$. Then f is well defined, for, if $x =: (x_i) = (y_i) := y$ in $\bigoplus M_i$, then $g_i(x_i) = g_i(y_i)$, for each $i \in \Lambda$, so that $f(x) = f(y)$. Thus f is well-defined.

Let $x,\ y \in M$, by the above notation. Then

$f(x+y) = f((x_i+y_i)) = (g_i(x_i+y_i)) = (g_i(x_i))+(g_i(y_i)) = f(x)+f(y)$

Since $M := \bigoplus M_i$ is a GM module, above equalities implies that $f \in End(M) = End_R(M)$.

¿From this fact, since

$$f(xr) = f((x_i)r) = f((x_i r)) = (g_i(x_i r))$$

and

$$f(x)r = f((x_i))r = (g_i(x_i))r = (g_i(x_i)r)$$

we derive that

$$g_i(x_i r) = (g_i(x_i)r)$$

for all $i \in \Lambda$; that is, $g_i \in End_R(M_i)$. Therefore each M_i is a GM module. □

From the faithful GM-property, we get a commutativity of rings.

Theorem 2.4. *Let M be a right R-module. If M is a faithful GM module, then R is a commutative ring.*

Proof. Let $f \in End(M)$ and let $a, b \in R$, where $f(x) = xa$, for all $x \in M$. Then

$$f(xb) = (xb)a.$$

On the other hand, since $f \in End(M) = End_R(M)$, we have that

$$f(xb) = f(x)b = (xa)b.$$

Hence $(xb)a = (xa)b$ for all $x \in M$. Since M is faithful, so we see that $ab = ba$. □

Next, we shall treat a d.g. nearring R generated by S, and a faithful R-group G, furthermore, there is a module like concept as follows: Let (R, S) be a d.g. nearring. Then an additive group G is called a *d.g. (R, S)-group* if there exists a d.g. nearring homomorphism

$$\theta : (R, S) \longrightarrow (M(G), End(G)) = E(G)$$

such that $S\theta \subseteq End(G)$. If we write that xr instead of $x(\theta_r)$ for all $x \in G$ and $r \in R$, then an d.g. (R, S)-group is an additive group G satisfying the following conditions:

$$x(rs) = (xr)s,\ x(r + s) = xr + xs,\ (x + y)s = xs + ys,$$

for all $x,\ y \in G$ and all $r,\ s \in S$.

Such a homomorphism θ is called a *d.g. representation* of (R, S) on G. This d.g. representation is said to be *faithful* if $Ker\theta = \{0\}$. In this case, we say that G is called a *faithful d.g. (R, S)-group* [9, 10, 12, 13, 14].

For any near-ring R and R-group G, we write the set

$$M_R(G) := \{f \in M(G) \mid (xr)f = (xf)r,\ for\ all\ x \in G,\ r \in R\}$$

of all R-maps on G as defined previously.

The following two statements are motivation of MR-property of R-groups defined at the next page.

Lemma 2.5. *Let G be an abelian d.g. (R,S)-group. Then the set $M_R(G) := \{f \in M(G) \mid (xr)f = (xf)r,\ for\ all\ x \in G,\ r \in R\}$ is a subnearring of $M(G)$.*

Proof. Let $f,\ g \in M_R(G)$. For any $x \in G$ and $r \in R$, since R is a d.g. nearring generated by S, consider that

$$r = \delta_1 s_1 + \delta_2 s_2 + \delta_3 s_3 + \cdots + \delta_n s_n,$$

where $\delta_i = 1$, or -1 and $s_i \in S$ for $i = 1, \cdots, n$. We have that

$$(xr)(f+g) = (xr)f + (xr)g = (xf)r + (xg)r$$

$$= xf(\delta_1 s_1 + \delta_2 s_2 + \cdots + \delta_n s_n) + xg(\delta_1 s_1 + \delta_2 s_2 + \cdots \delta_n s_n)$$

$$= xf\delta_1 s_1 + xg\delta_1 s_1 + xf\delta_2 s_2 + xg\delta_2 s_2 + \cdots + xf\delta_n s_n + xg\delta_n s_n$$

$$= \delta_1 xfs_1 + \delta_1 xgs_1 + \delta_2 xfs_2 + \delta_2 xgs_2 + \cdots + \delta_n xfs_n + \delta_n xgs_n$$

$$= \delta_1(xfs_1 + xgs_1) + \delta_2(xfs_2 + xgs_2) + \cdots + \delta_n(xfs_n + xgs_n)$$

$$= \delta_1(xf + xg)s_1 + \delta_2(xf + xg)s_2 + \cdots + \delta_n(xf + xg)s_n$$

$$= (xf + xg)\delta_1 s_1 + (xf + xg)\delta_2 s_2 + \cdots + (xf + xg)\delta_n s_n$$

$$= (xf + xg)(\delta_1 s_1 + \delta_2 s_2 + \cdots + \delta_n s_n) = (xf + xg)r = x(f+g)r$$

Similary, we have the following equalities:

$$(xr)(-f) = -(xr)f = -(xf)r = x(-f)r$$

and

$$(xr)f \cdot g = ((xr)f)g = ((xf)r)g = (xf)gr = x(f \cdot g)r.$$

Thus $M_R(G)$ is a subnearring of $M(G)$. □

In ring and module theory, we obtain the following important structure for nearring and R-group theory:

Corollary 2.6 (C.J. Maxson) [11]. *Let R be a ring and V a right R-module. Then $M_R(V) := \{f \in M(V) \mid (xr)f = (xf)r,$ for all $x \in V$, $r \in R\}$ is a subnearring of $M(V)$.*

Lemma 2.7 [14]. *Let G be a faithful R-group. Then we have the following conditions:*

(1) *If $(G, +)$ is abelian, then $(R, +)$ is abelian.*
(2) *If G is distributive, then R is distributive.*

Applying this Lemma, we get the following Propositions:

Proposition 2.8. *If G is a distributive abelian faithful R-group, then R is a ring.*

Proposition 2.9. *Let (R, S) be a d.g.. nearring. If G is an abelian faithful d.g. (R, S)-group, then R is a ring.*

Finally, we also introduce the MR-property of R-group, which is motivated by the Lemma 2.5. An R-group G is called an *MR group* over nearring R, provided that every mapping on G is an R-map of G, that is,

$$M(G) = M_R(G)$$

Example 2.10.

(1) *If $R = \mathbb{Z}$ is the nearring of integers, then every regular R-group is an MR group.*
(2) *If $R = M_S(G)$ is a centralizer nearring as in [13, 14], then R-group G is an MR group.*
(3) *Every $\mathbb{Q}$-group is an MR group.*

We also apply Lemma 2.1, Proposition 2.2 and Theorem 2.3 for GM modules to MR groups. Thus we only introduce a characterization of MR groups for direct sum without proof as following.

Theorem 2.11. *Let $\{G_i | i \in \Lambda\}$ be any family of R-groups. Then each G_i is an MR group if and only if $G := \bigoplus G_i$ is an MR group.*

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