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THE NON-EXISTENCE AND EXISTENCE OF POSITIVE SOLUTION TO THE COOPERATION MODEL WITH GENERAL COOPERATION RATES

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Abstract. The non-existence and existence of the positive solution for the generalized cooperation biological model for two species of animals

$$\begin{aligned}&\Delta u + u(a - bu + g(v)) = 0 \text{ in } \Omega\\&\Delta v + v(d + h(u) - cv) = 0 \text{ in } \Omega\\&u = v = 0 \text{ on } \partial\Omega,\end{aligned}$$

are investigated. The techniques used in this paper are elliptic theory, upper-lower solutions, maximum principles and spectrum estimates. The arguments also rely on some detailed properties for the solution of logistic equations.

1. Introduction

A lot of research has been focused on reaction-diffusion equations modeling of the elliptic steady states of cooperative interacting processes with Dirichlet boundary conditions. Our knowledge about the existence of positive solutions is limited to somewhat rather special systems, whose relative growth rates are linear; the results established are only for the following cooperation models(see [1],[2],[3],[4],[5].)

$$\begin{aligned}&\Delta u + u(a - bu + cv) = 0 \text{ in } \Omega\\&\Delta v + v(d + eu - fv) = 0 \text{ in } \Omega\\&u = v = 0 \text{ on } \partial\Omega,\end{aligned}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, $a, d > 0$ are reproduction rates, $b, f > 0$ are self-limitation rates and $c, e > 0$ are

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cooperation rates.
The question in this paper concerns the existence of positive coexistence when the cooperation growth rates are nonlinear, more precisely, the existence of the positive steady

state of

$$\begin{array}{l} \Delta u + u(a - bu + g(v)) = 0 \ \text{ in } \ \Omega \\ \Delta v + v(d + h(u) - cv) = 0 \ \text{ in } \ \Omega \\ u = v = 0 \ \text{ on } \ \partial\Omega, \end{array}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, b, c are positive constants, $g, h \in C^1$ are strictly increasing, and $g(0) = h(0) = 0$. In section 3, we will see when they can not coexist, that is, some sufficient conditions that either one of the species is excluded by the other using a Maximum Principles and spectrum theory. In section 4, we provide the coexistence region of the reproduction rates (a, d) by virtue of Maximum Principles, upper-lower solutions method and the properties of the logistic equation.

2. Preliminaries

In this section, we state some preliminary results which will be useful for our later arguments.

Definition 2.1. *(upper and lower solutions)*

$$\left\{ \begin{array}{l} \Delta u + f(x, u) = 0 \ \ in \ \ \Omega, \\ u|_{\partial\Omega} = 0 \end{array} \right. \tag{1}$$

where $f \in C^\alpha(\bar{\Omega} \times R)$ and Ω is a bounded domain in R^n.
(A) A function $\bar{u} \in C^{2,\alpha}(\bar{\Omega})$ satisfying

$$\left\{ \begin{array}{l} \Delta \bar{u} + f(x, \bar{u}) \leq 0 \ \ in \ \ \Omega, \\ \bar{u}|_{\partial\Omega} \geq 0 \end{array} \right.$$

is called an upper solution to (1).
(B) A function $\underline{u} \in C^{2,\alpha}(\bar{\Omega})$ satisfying

$$\left\{ \begin{array}{l} \Delta \underline{u} + f(x, \underline{u}) \geq 0 \ \ in \ \ \Omega, \\ \underline{u}|_{\partial\Omega} \leq 0 \end{array} \right.$$

is called a lower solution to (1).

Lemma 2.1. *Let $f(x,\xi)\in C^{\alpha}(\bar{\Omega}\times R)$ and let $\bar{u},\underline{u}\in C^{2,\alpha}(\bar{\Omega})$ be respectively, upper and lower solutions to (1) which satisfy $\underline{u}(x)\le\bar{u}(x), x\in\bar{\Omega}$. Then (1) has a solution $u\in C^{2,\alpha}(\bar{\Omega})$ with $\underline{u}(x)\le u(x)\le\bar{u}(x), x\in\bar{\Omega}$.*

Lemma 2.2. *(The first eigenvalue)*

$$\left\{\begin{array}{l} -\Delta u+q(x)u=\lambda u \ \text{ in } \ \Omega,\\ u|_{\partial\Omega}=0, \end{array}\right. \tag{2}$$

where $q(x)$ is a smooth function from Ω to R and Ω is a bounded domain in R^n.
(A) The first eigenvalue $\lambda_1(q)$, denoted by simply λ_1 when $q\equiv 0$, is simple with a positive eigenfunction.
(B) If $q_1(x)<q_2(x)$ for all $x\in\Omega$, then $\lambda_1(q_1)<\lambda_1(q_2)$.
(C)(Variational Characterization of the first eigenvalue)

$$\lambda_1(q)=\min_{\phi\in W_0^1(\Omega),\phi\neq 0}\frac{\int_\Omega(|\nabla\phi|^2+q\phi^2)dx}{\int_\Omega\phi^2dx}.$$

(D) If $a(x)\in C(\bar{\Omega}), a(x)>0(x\in\Omega)$ and $\alpha\in R$, then

$$\lim_{\alpha\to+\infty}\lambda_1(\alpha a(x))=+\infty.$$

Lemma 2.3. *If $a(x)\in C^{\alpha}(\bar{\Omega})$ for some $\alpha\in(0,1)$ and $\lambda_1(-\Delta+a(x))>0, w\in C^2(\bar{\Omega})$ and satisfies*

$$\left\{\begin{array}{l} -\Delta w+a(x)w\ge 0 \ \text{ in } \ \Omega\\ w=0 \ \text{ on } \ \partial\Omega. \end{array}\right.$$

Then $w\ge 0$ in Ω.

Lemma 2.4. *(Maximum Principles)*

$$Lu=\sum_{i,j=1}^n a_{ij}(x)D_{ij}u+\sum_{i=1}^n a_i(x)D_iu+a(x)u=f(x) \ \text{ in } \ \Omega,$$

where Ω is a bounded domain in R^n.
(M1) $\partial\Omega\in C^{2,\alpha}(0<\alpha<1)$
(M2) $|a_{ij}(x)|_\alpha, |a_i(x)|_\alpha, |a(x)|_\alpha\le M(i,j=1,...,n)$
(M3) L is uniformly elliptic in $\bar{\Omega}$, with ellipticity constant γ, i.e., for every $x\in\bar{\Omega}$ and every real vector $\xi=(\xi_1,...,\xi_n)$

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j\ge\gamma\sum_{i=1}^n|\xi_i|^2.$$

Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution of $Lu \geq 0 (Lu \leq 0)$ in Ω.
(A) If $a(x) \equiv 0$, then $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u (\min_{\bar{\Omega}} u = \min_{\partial\Omega} u)$.
(B) If $a(x) \leq 0$, then $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+ (\min_{\bar{\Omega}} u \geq -\max_{\partial\Omega} u^-)$, where $u^+ = \max(u, 0), u^- = -\min(u, 0)$.
(C) If $a(x) \equiv 0$ and u attains its maximum (minimum) at an interior point of Ω, then u is identically a constant in Ω.
(D) If $a(x) \leq 0$ and u attains a nonnegative maximum (nonpositive minimum) at an interior point of Ω, then u is identically a constant in Ω.

We also need some information on the solutions of the following logistic equations.

Lemma 2.5.
$$\begin{cases} \Delta u + u f(u) = 0 \ \text{ in } \ \Omega, \\ u|_{\partial\Omega} = 0, u > 0, \end{cases}$$
where f is a decreasing C^1 function such that there exists $c_0 > 0$ such that $f(u) \leq 0$ for $u \geq c_0$ and Ω is a bounded domain in R^n.
If $f(0) > \lambda_1$, then the above equation has a unique positive solution, where λ_1 is the first eigenvalue of $-\Delta$ with homogeneous boundary condition. We denote this unique positive solution as θ_f.

The main property about this positive solution is that θ_f is increasing as f is increasing.

Especially, for $a > \lambda_1$, we denote θ_a as the unique positive solution of
$$\begin{cases} \Delta u + u(a - u) = 0 \ \text{ in } \ \Omega, \\ u|_{\partial\Omega} = 0, u > 0. \end{cases}$$
Hence, θ_a is increasing as $a > 0$ is increasing.

3. Nonexistence of steady state

We consider
$$\begin{aligned} &\Delta u + u(a - bu + g(v)) = 0 \ \text{ in } \ \Omega \\ &\Delta v + v(d + h(u) - cv) = 0 \ \text{ in } \ \Omega \\ &u = v = 0 \ \text{ on } \ \partial\Omega, \end{aligned} \tag{3}$$
where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, b, c are positive constants, $g, h \in C^1$ are strictly increasing, and $g(0) = h(0) = 0$.

By virtue of Lemma 2.3, we have the following estimates of solutions to (3).

LEMMA 3.1. *Let (u, v) be a solution of (3).*

(1) If $a \geq d, 0 < b \leq 1, c \geq 1$, then $[\sup(h') + 1]u \geq [\inf(g') + 1]v$.
(2) If $a \leq d, 0 < c \leq 1, b \geq 1$, then $[\inf(h') + 1]u \leq [\sup(g') + 1]v$.

PROOF. (1) Let $w = [\sup(h') + 1]u - [\inf(g') + 1]v$. By the mean value theorem and the fact $g(0) = h(0) = 0$, we have

$$\sup(h')u \geq h(u), \inf(g')v \leq g(v).$$

Hence,

$$\begin{aligned}
& -\Delta w + (-d + bu + cv)w \\
= \; & -[\sup(h') + 1]\Delta u + [\inf(g') + 1]\Delta v + (-d + bu + cv)[\sup(h') + 1]u \\
& -(-d + bu + cv)[\inf(g') + 1]v \\
= \; & [\sup(h') + 1]au + [\sup(h') + 1]ug(v) - [\inf(g') + 1]dv - [\inf(g') + 1]vh(u) \\
& -[\sup(h') + 1]du + [\sup(h') + 1]cuv + [\inf(g') + 1]dv - [\inf(g') + 1]buv \\
\geq \; & h(u)g(v) + ug(v) - g(v)h(u) - vh(u) + cvh(u) - bug(v) + cuv - buv \\
= \; & ug(v)(1 - b) - vh(u)(1 - c) + (c - b)uv \\
\geq \; & 0.
\end{aligned}$$

Since (u, v) is a positive solution of (3), from the monotonicity of the first eigenvalue,

$$\lambda_1(-\Delta - d + bu + cv) > \lambda_1(-\Delta - d - h(u) + cv) = 0.$$

Hence, 2.3 implies $w \geq 0$ and we get the desired result.

(2) Let $w = [\sup(g') + 1]v - [\inf(h') + 1]u$. By the mean value theorem and the fact $g(0) = h(0) = 0$, we have

$$\inf(h')u \leq h(u), \sup(g')v \geq g(v).$$

Hence,

$$\begin{aligned}
& -\Delta w + (-a+bu+cv)w \\
= & -[\sup(g')+1]\Delta v + [\inf(h')+1]\Delta u + (-a+bu+cv)[\sup(g')+1]v \\
& -(-a+bu+cv)[\inf(h')+1]u \\
= & [\sup(g')+1]dv + [\sup(g')+1]vh(u) - [\inf(h')+1]au - [\inf(h')+1]ug(v) \\
& -[\sup(g')+1]av + [\sup(g')+1]buv + [\inf(h')+1]au - [\inf(h')+1]cuv \\
\geq & \; g(v)h(u) + vh(u) - h(u)g(v) - ug(v) + g(v)bu - h(u)cv + buv - cuv \\
= & \; vh(u)(1-c) - ug(v)(1-b) + (b-c)uv \\
\geq & \; 0.
\end{aligned}$$

Since (u,v) is a positive solution of (3), from the monotonicity of the first eigenvalue,

$$\lambda_1(-\Delta - a + bu + cv) > \lambda_1(-\Delta - a + bu - g(v)) = 0.$$

Hence, the Lemma 2.3 implies $w \geq 0$ and we get the desired result.

Now, we have the following nonexistence results.

THEOREM 3.2. *Let $a, d > \lambda_1$.*
(i) If $a \geq d, 0 < b \leq 1, c \geq 1$ and $\inf(h')\inf(g') + \inf(h') - c\sup(h') \geq c$, then (3) has no positive solution.
(ii) If $a \leq d, 0 < c \leq 1, b \geq 1$ and $\inf(g')\inf(h') + \inf(g') - b\sup(g') \geq b$, then (3) has no positive solution.

PROOF. (i) From (1) of Lemma 3.1, we have $[\sup(h')+1]u \geq [\inf(g')+1]v$. Hence, by the mean value theorem and the assumption,

$$\begin{aligned}
0 &= \Delta v + v(d + h(u) - cv) \\
&\geq \Delta v + v\left(d + \inf(h')u - \tfrac{c[\sup(h')+1]u}{\inf(g')+1}\right) \\
&= \Delta v + v\left(d + \tfrac{[\inf(h')\inf(g')+\inf(h')-c\sup(h')-c]u}{\inf(g')+1}\right) \\
&\geq \Delta v + dv.
\end{aligned}$$

By multiplying ϕ_1 to the both sides, we have

$$(d - \lambda_1)\int_\Omega v\phi_1 = \int_\Omega \phi_1(\Delta v + dv) \leq 0.$$

Hence $d \leq \lambda_1$, which is a contradiction to our assumption.

(ii) From (2) of Lemma 3.1, we have $[\inf(h')+1]u \leq [\sup(g')+1]v$.

Hence, by the mean value theorem and the assumption,

$$\begin{aligned} 0 &= \Delta u + u(a + g(v) - bu) \\ &\geq \Delta u + u(a + \inf(g')v - \tfrac{b[\sup(g')+1]v}{\inf(h')+1}) \\ &= \Delta u + u(a + \tfrac{[\inf(g')\inf(h')+\inf(g')-b\sup(g')-b]v}{\inf(h')+1}) \\ &\geq \Delta u + au. \end{aligned}$$

By multiplying ϕ_1 to the both sides, we have

$$(a - \lambda_1)\int_\Omega u\phi_1 = \int_\Omega \phi_1(\Delta u + au) \leq 0.$$

Hence $a \leq \lambda_1$, which is a contradiction to our assumption.

4. Existence region for steady state

We consider

$$\begin{aligned} &\Delta u + u(a - bu + g(v)) = 0 \ \text{ in } \ \Omega \\ &\Delta v + v(d + h(u) - cv) = 0 \ \text{ in } \ \Omega \\ &u = v = 0 \ \text{ on } \ \partial\Omega, \end{aligned} \tag{4}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, b, c are positive constants, $g, h \in C^1$ are strictly increasing, $g(0) = h(0) = 0$, and $bc > \sup(g')\sup(h')$.

First, we see that the two species can not coexist when the reproduction capacities are not strong enough.

Theorem 4.1. *Suppose $a \leq \lambda_1, d \leq \lambda_1$.*
Then $u = v \equiv 0$ is the only nonnegative solution to (4).

Proof. Let (u, v) be a nonnegative solution to (4). By the Mean Value Theorem, there are $\tilde{u}, \tilde{v}$ such that

$$\begin{aligned} g(v) &= g(v) - g(0) = g'(\tilde{v})v \\ h(u) &= h(u) - h(0) = h'(\tilde{u})u. \end{aligned}$$

Hence, (4) implies that

$$\begin{aligned} &\Delta u + u(a - bu + g'(\tilde{v})v) = 0 \ \text{ in } \ \Omega, \\ &\Delta v + v(d - cv + h'(\tilde{u})u) = 0 \ \text{ in } \ \Omega. \end{aligned}$$

Hence,

$$\begin{aligned}&\Delta u + u(a - bu + \sup(g')v) \geq 0 \text{ in } \Omega,\\&\Delta v + v(d - cv + \sup(h')u) \geq 0 \text{ in } \Omega.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sup(h')\phi_1\Delta u + \sup(h')\phi_1 u(a - bu + \sup(g')v) \geq 0 \text{ in } \Omega,\\&\sup(g')\phi_1\Delta v + \sup(g')\phi_1 v(d - cv + \sup(h')u) \geq 0 \text{ in } \Omega.\end{aligned}$$

So,

$$\begin{aligned}&\int_\Omega -\sup(h')\phi_1\Delta u dx \leq \int_\Omega[-b\sup(h')u^2 + \sup(g')\sup(h')uv + a\sup(h')u]\phi_1 dx,\\&\int_\Omega -\sup(g')\phi_1\Delta v dx \leq \int_\Omega[-c\sup(g')v^2 + \sup(g')\sup(h')uv + d\sup(g')v]\phi_1 dx.\end{aligned}$$

Hence, by the Green's Identity, we have

$$\begin{aligned}&\int_\Omega \sup(h')\lambda_1\phi_1 u dx \leq \int_\Omega[-b\sup(h')u^2 + \sup(g')\sup(h')uv + a\sup(h')u]\phi_1 dx,\\&\int_\Omega \sup(g')\lambda_1\phi_1 v dx \leq \int_\Omega[-c\sup(g')v^2 + \sup(g')\sup(h')uv + d\sup(g')v]\phi_1 dx.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_\Omega \sup(h')(\lambda_1 - a)u\phi_1 + \sup(g')(\lambda_1 - d)v\phi_1 dx\\\leq\ &\int_\Omega[-b\sup(h')u^2 + 2\sup(g')\sup(h')uv - c\sup(g')v^2]\phi_1 dx.\end{aligned}$$

Since the left hand side is nonnegative and the integrand of the right hand side is negative definite by the assumptions, we conclude that $u = v \equiv 0$.

In order to prove the main existence results, we will need the following Lemmas.

LEMMA 4.2. *If $u > 0, v > 0$ is a solution to (4), then the system of equations*

$$\begin{aligned}&-bu + \sup(g')v + a = 0\\&\sup(h')u - cv + d = 0\end{aligned} \tag{5}$$

has a unique positive solution (u^*, v^*) and $u \leq u^*, v \leq v^*$ in $\bar{\Omega}$.

PROOF. Let $u > 0, v > 0$ in Ω be a solution to (4) and $K_1 = max_{\bar{\Omega}}u(x) > 0, K_2 = max_{\bar{\Omega}}v(x)$ are occurred at $x_1 \in \Omega, x_2 \in \Omega$, respectively.
We claim
(6)

$$\begin{aligned}&0 \leq K_1(-bu(x_1) + \sup(g')v(x_1) + a) \leq K_1(-bK_1 + \sup(g')K_2 + a)\\&0 \leq K_2(\sup(h')u(x_2) - cv(x_2) + d) \leq K_2(\sup(h')K_1 - cK_2 + d)\end{aligned}$$

In fact, suppose $-bu(x_1)+\sup(g')v(x_1)+a<0$. Then since $\Delta u(x_1)+u(x_1)(-bu(x_1)+\sup(g')v(x_1)+a)\geq 0$ by (4) and the Mean Value Theorem, $\Delta u(x_1)>0$, which contradicts to the Maximum Principles.
Since $bc>\sup(g')\sup(h')$, the system (5) has a unique positive solution

$$\begin{aligned} u^* &= \tfrac{1}{bc-\sup(g')\sup(h')}(ac+d\sup(g')) \\ v^* &= \tfrac{1}{bc-\sup(g')\sup(h')}(a\sup(h')+bd), \end{aligned}$$

and by (6)

$$u\leq K_1\leq u^*, v\leq K_2\leq v^*.$$

Lemma 4.3. *For any $M_0>0$, there are constants $M_1,M_2>M_0$ such that $\bar{u}=M_1,\bar{v}=M_2$ is an upper solution to (4).*

Proof. Since $bc>\sup(g')\sup(h')$, there are $M_1,M_2>M_0$ such that

$$\begin{aligned} \Delta M_1+M_1(a-bM_1+g(M_2)) &\leq M_1(a-bM_1+\sup(g')M_2)\leq 0 \\ \Delta M_2+M_2(d+h(M_1)-cM_2) &\leq M_2(d+\sup(h')M_1-cM_2)\leq 0. \end{aligned}$$

Thus $\bar{u}=M_1,\bar{v}=M_2$ is an upper solution to (4).

Then we prove the main existence results.

Theorem 4.4. *Let $a>\lambda_1[d>\lambda_1]$. Then there is a number $M(a)<\lambda_1[N(d)<\lambda_1]$ such that for any $d>M(a)[a>N(d)]$, (4) has a positive solution in Ω.*

Proof. Suppose $a>\lambda_1$. Let $\underline{u}=\omega_{\frac{a}{b}}$ be the unique positive solution to

$$\begin{aligned} \Delta u+u(a-bu) &= 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Let $M(a)=\lambda_1(-h(\omega_{\frac{a}{b}}))$ be the smallest eigenvalue of

$$\begin{aligned} -\Delta Z-h(\omega_{\frac{a}{b}})Z &= \mu Z \text{ in } \Omega \\ Z &= 0 \text{ on } \partial\Omega. \end{aligned}$$

and $\omega_0(x)$ be the corresponding normalized positive eigenfunction.
By the monotonicity, $M(a)=\lambda_1(-h(\omega_{\frac{a}{b}}))<\lambda_1$.
Let $\underline{v}=\epsilon\omega_0(x)$. Let $d>M(a)$. Then, for sufficiently small $\epsilon>0$,

$$\begin{aligned} &\Delta\underline{u}+\underline{u}(a-b\underline{u}+g(\underline{v})) \\ =\ &\Delta(\omega_{\frac{a}{b}})+\omega_{\frac{a}{b}}(a-b\omega_{\frac{a}{b}}+g(\epsilon\omega_0(x))) \\ =\ &\omega_{\frac{a}{b}}g(\epsilon\omega_0(x))>0 \text{ in } \Omega \end{aligned}$$

and

$$\begin{aligned}
& \Delta\underline{v}+\underline{v}(d+h(\underline{u})-c\underline{v})) \\
= \ & \Delta(\epsilon\omega_0)+\epsilon\omega_0(d+h(\omega_{\frac{a}{b}})-c\epsilon\omega_0(x)) \\
= \ & -\epsilon M(a)\omega_0+d\epsilon\omega_0(x)-c\epsilon^2\omega_0^2 \\
= \ & \epsilon\omega_0(d-M(a))-c\epsilon^2\omega_0^2 \\
> \ & 0 \text{ in } \Omega.
\end{aligned}$$

So, $\underline{u}>0, \underline{v}>0$ is a lower solution to (4). But, by the Lemma 4.3, there is an upper solution $M_1>\underline{u}, M_2>\underline{v}$ of (4). Therefore, there is a positive solution of (4).

THEOREM 4.5. *Let $a \le \lambda_1[d \le \lambda_1]$. Then there is a number $M(a) > \lambda_1[N(d)>\lambda_1]$ such that for any $d>M(a)[a>N(d)]$, (4) has a positive solution in Ω.*

PROOF. Suppose $a \le \lambda_1$. Let $d>\lambda_1$ and $\omega_{\frac{d}{c}}$ be the unique positive solution to

$$\begin{aligned}
\Delta v+v(d-cv) &= 0 \text{ in } \Omega \\
v &= 0 \text{ on } \partial\Omega.
\end{aligned}$$

Since

$$\lim_{d\to\infty}\lambda_1(-g(\omega_{\frac{d}{c}})) \le \lim_{d\to\infty}\lambda_1(-\inf(g')\omega_{\frac{d}{c}}) \le \lim_{d\to\infty}\lambda_1(-\inf(g')\frac{d-\lambda_1}{c}\phi_0) = -\infty,$$

there is a number $M(a)\ge\lambda_1$ such that $\lambda_1(-g(\omega_{\frac{d}{c}}))<a$ if $d>M(a)$.
Let $\underline{u}=\epsilon\omega_0$ and $\underline{v}=\omega_{\frac{d}{c}}$, where ω_0 is the normalized positive eigenfunction corresponding to $\lambda_1(-g(\omega_{\frac{d}{c}}))$.
Then if $d>M(a)$, for sufficiently small $\epsilon>0$,

$$\begin{aligned}
& \Delta\underline{u}+\underline{u}(a-b\underline{u}+g(\underline{v})) \\
= \ & \Delta(\epsilon\omega_0)+\epsilon\omega_0(a-b\epsilon\omega_0+g(\omega_{\frac{d}{c}})) \\
= \ & -\lambda_1(-g(\omega_{\frac{d}{c}}))\epsilon\omega_0+a\epsilon\omega_0-b\epsilon^2\omega_0^2 \\
> \ & 0 \text{ in } \Omega
\end{aligned}$$

and

$$\begin{aligned}
& \Delta\underline{v}+\underline{v}(d+h(\underline{u})-c\underline{v}) \\
= \ & \underline{v}h(\underline{u})>0 \text{ in } \Omega.
\end{aligned}$$

So, $\underline{u},\underline{v}$ is a lower solution to (4). Hence, by the Lemma 4.3, if $d>M(a)$, there is a positive solution to (4).

THEOREM 4.6. *If $a<\frac{-d\sup(g')}{c}$, then (4) does not have any positive solution.*

Proof. Since $a < \frac{-d\sup(g')}{c}$, (5) does not have any positive solution, and so by the Lemma 4.2, (4) does not have any positive solution.

The main assumption in this section is $bc > \sup(g')\sup(h')$ which indicates that the two species have stronger self-limitation abilities than cooperation ones. The above results imply that they must have strong enough reproduction capacities in order to survive peacefully under this weak cooperation abilities.

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