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REPULSIVE FIXED-POINTS OF THE LAGUERRE-LIKE ITERATION FUNCTIONS

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Abstract. Let f be an analytic function with a simple zero in the reals or the complex numbers. An extraneous fixed-point of an iteration function is a fixed-point different from a zero of f. We prove that all extraneous fixed-points of Laguerre-like iteration functions and general Laguerre-like functions are repulsive.

1. Introduction

Suppose that $f(z)$ is analytic with a simple zero at α in either the reals or the complex numbers. Let $L_0(z) = 1$ and

$$L_m(z) = \det \begin{pmatrix} f'(z) & f(z) & 0 & \cdots & 0 \\ f''(z) & f'(z) & f(z) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ \frac{f^{(m-1)}(z)}{(m-2)!} & \frac{f^{(m-2)}(z)}{(m-2)!} & \frac{f^{(m-3)}(z)}{(m-3)!} & \cdots & f(z) \\ \frac{f^{(m)}(z)}{(m-1)!} & \frac{f^{(m-1)}(z)}{(m-1)!} & \frac{f^{(m-2)}(z)}{(m-2)!} & \cdots & f'(z) \end{pmatrix}, \tag{1.1}$$

where $\det(\cdot)$ denotes the determinant. $L_m(z)$ is the determinant of a Toeplitz-like matrix (see [1, 6]). For each $m \geq 2$, recursively define

$$K_m(z) = z - f(z)\frac{L_{m-1}(z)}{L_m(z)} \tag{1.2}$$

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and the general form of the iteration (1.2) by

$$U_m(v,z) = z - f(z)\frac{(v-z)L_{m-1}(z) + f(z)L_{m-2}(z)}{(v-z)L_m(z) + f(z)L_{m-1}(z)} \tag{1.3}$$

for a complex constant v in [6]. The Laguerre case (see [2, 7]) can be obtained from (1.3) by taking a polynomial f with $m = 2$. We call $K_m(z)$ a Laguerre-like iteration function.

We first give some relevant properties of the Laguerre-like iteration functions before proving the main theorem.

In [1], the recursion formula for L_m is obtained by

$$L_m(z) = f'(z)\, L_{m-1}(z) - \tfrac{1}{m-1} f(z)\, L'_{m-1}(z)\,,\ m \geq 2. \tag{1.4}$$

THEOREM 1.1. [1] *Let $f(z)$ be an analytic function with a simple zero at α. Suppose L_m satisfies the identity (1.4) for each $m \geq 2$ and define $K_m(z)$ as in (1.2). Then the fixed-point iteration $z_{n+1} = K_m(z_n)$, $n = 1, 2, \ldots$ has mth-order of convergence.*

From (1.3), $\lim_{v\to z} U_m(v,z) = z - f(z)\frac{L_{m-2}(z)}{L_{m-1}(z)}$ and $\lim_{v\to\infty} U_m(v,z) = z - f(z)\frac{L_{m-1}(z)}{L_m(z)}$ which have the order of convergence $m-1$ and m, respectively.

THEOREM 1.2. [1] *Let $f(z)$ be an analytic function with a simple zero at α. Suppose v is a complex constant with $v \neq \alpha$. For each $m \geq 2$, define $U_m(v,z)$ as in (1.3). Then the iterations*

$$z_{n+1} = U_m(v, z_n),\ n = 1, 2, \ldots$$

converge to α and the order of convergence is m.

If α is a zero of f, then it is necessarily to be a fixed-point of K_m, i.e. $f(\alpha) = 0$ implies $K_m(\alpha) = \alpha$. The converse however may not be true.

DEFINITION 1.3. *If $K_m(\alpha) = \alpha$ but $f(\alpha) \neq 0$, then α is said to be an extraneous fixed-point of K_m. An extraneous fixed-point is said to be repulsive if it satisfies the following property:*

$$|K'_m(\alpha)| > 1. \tag{1.5}$$

The definition of extraneous fixed-points and repulsive fixed-point applies to more general iteration functions for root-finding (see [8]) and the extraneous fixed-points of the basic family are repulsive in [5]. The basic family is a family of iteration functions consisting of the determinant of a Toeplitz matrix for the normalized derivative for an analytic function with a simple zero ([3, 4]).

In this paper, we shall examine the property of repulsive for the Laguerre-like iteration functions (1.2) and (1.3).

2. Repulsive fixed points

THEOREM 2.1. *For any $m \geq 2$, the extraneous fixed-points of $K_m(z)$ are repulsive. More specifically, if α is an extraneous fixed-point of K_m, then $|K_m'(\alpha)| > 1$.*

Proof. Suppose that $K_m(\alpha) = \alpha$, but $f(\alpha) \neq 0$. From the equation (1.2), we have $L_{m-1}(\alpha) = 0$ and thus (1.4) implies that

$$L_m(\alpha) = -\frac{1}{m-1} f(\alpha)\, L_{m-1}'(\alpha). \tag{2.1}$$

Direct differentiation of $K_m(z)$ yields

(2.2)

$$K_m'(z) = 1 - f'(z)\,\frac{L_{m-1}(z)}{L_m(z)} - f(z)\frac{L_{m-1}'(z)\,L_m(z) - L_{m-1}(z)\,L_m'(z)}{L_m(z)^2}.$$

Substituting $L_{m-1}(\alpha) = 0$ and $L_m(\alpha)$ from (2.1) into (2.2), and simplifying then we obtain

$$K_m'(\alpha) = 1 + (m-1) = m. \tag{2.3}$$

The above proof assumes that $L_m(\alpha)$ is non-zero which implies that α is a simple root of L_{m-1}. Hence, α is repulsive. □

THEOREM 2.2. *Suppose v is a complex constant with $v \neq \alpha$. Then for each $m \geq 2$ the extraneous fixed-points of $U_m(v, z)$ are repulsive. More specifically, if α is an extraneous fixed-point of U_m, then $|U_m'(\alpha)| > 1$.*

Proof. Suppose that α is an extraneous fixed-point of $U_m(v, z)$, i.e., $U_m(\alpha) = \alpha$, but $f(\alpha) \neq 0$. Equation (1.3) is

$$U_m(v, z) = z - f(z)\,\frac{P_{m-1}(z)}{P_m(z)}\,, \tag{2.4}$$

where $P_m(z) = (v - z)L_m(z) + f(z)L_{m-1}(z)$ for $m \geq 2$. Since α is an extraneous fixed-point of $U_m(v, z)$, we have the following two cases in order to satisfy $P_{m-1}(\alpha) = 0$;

(*i*) $L_{m-1}(\alpha) = 0$ and $L_{m-2}(\alpha) = 0$ but $L_m(\alpha) \neq 0$, $m \geq 2$

(*ii*) $v = K_{m-1}(\alpha)$ but $v \neq K_m(\alpha)$, $m \geq 2$

where $K_m(\alpha)$ is defined as in (1.2). We note that $L_{m-1}(\alpha) \neq 0$ and $L_{m-2}(\alpha) \neq 0$ since $v \neq \alpha$ in the case of (*ii*).

Differentiating $U_m(v, z)$ with respect to z, we obtain

$$U_m'(v, z) = 1 - f'(z)\,\frac{P_{m-1}(z)}{P_m(z)} - f(z)\frac{P_{m-1}'(z)P_m(z) - P_{m-1}(z)P_m'(z)}{P_m(z)^2}$$

and then evaluating at α, then

$$U_m'(v, \alpha) = 1 - f(\alpha)\frac{P_{m-1}'(\alpha)}{P_m(\alpha)}. \tag{2.5}$$

In the case of (*i*), we have

$$\begin{aligned} &P_{m-1}(\alpha) = 0,\ P_m(\alpha) = (v-\alpha)L_m(\alpha), \\ &P_{m-1}'(\alpha) = (v-\alpha)L_{m-1}'(\alpha) + f(\alpha)L_{m-2}'(\alpha) = (v-\alpha)L_{m-1}'(\alpha) \end{aligned} \tag{2.6}$$

since $f(\alpha)L_{m-2}'(\alpha) = (m-2)(f'(\alpha)L_{m-2}(\alpha) - L_{m-1}(\alpha)) = 0$ by (1.4). Substituting (2.6) into (2.5), and applying (2.1)

$$U_m'(\alpha) = 1 - f(\alpha)\frac{L_{m-1}'(\alpha)}{L_m(\alpha)} = 1 + (m-1) = m.$$

The above proof assume that $L_m(\alpha)$ is non-zero which implies that α is a simple zero of $L_{m-1}(z)$. Hence in the case of (*i*), the simple zero at α of $L_{m-1}(z)$ is repulsive.

We now consider the case of (*ii*). Differentiating $P_{m-1}(z)$ and then

$$\begin{aligned} P_{m-1}'(\alpha) &= (v-\alpha)L_{m-1}'(\alpha) - L_{m-1}(\alpha) + f'(\alpha)L_{m-2}(\alpha) + f(\alpha)L_{m-2}'(\alpha) \\ &= -f(\alpha)\,L_{m-1}'(\alpha)\,\tfrac{L_{m-2}(\alpha)}{L_{m-1}(\alpha)} - L_{m-1}(\alpha) + f'(\alpha)L_{m-2}(\alpha) + f(\alpha)L_{m-2}'(\alpha) \\ &= -(m-1)(f'(\alpha)L_{m-1}(\alpha) - L_m(\alpha))\,\tfrac{L_{m-2}(\alpha)}{L_{m-1}(\alpha)} - L_{m-1}(\alpha) \\ &\quad + f'(\alpha)L_{m-2}(\alpha) + (m-2)(f'(\alpha)L_{m-2}(\alpha) - L_{m-1}(\alpha)) \end{aligned}$$

by the recursion formula (1.4). Therefore, we have

$$\begin{aligned} &P_{m-1}'(\alpha) = (m-1)\Big(\tfrac{L_{m-2}(\alpha)}{L_{m-1}(\alpha)}\,L_m(\alpha) - L_{m-1}(\alpha)\Big), \\ &P_m(\alpha) = (v-\alpha)L_m(\alpha) + f(\alpha)L_{m-1}(\alpha) = -f(\alpha)\Big(\tfrac{L_{m-2}(\alpha)}{L_{m-1}(\alpha)}L_m(\alpha) - L_{m-1}(\alpha)\Big). \end{aligned} \tag{2.7}$$

Since $v \neq K_m(\alpha)$, we have $\frac{L_{m-2}(\alpha)}{L_{m-1}(\alpha)} \neq \frac{L_{m-1}(\alpha)}{L_m(\alpha)}$. Plugging (2.7) into (2.5), then we obtain

$$U_m'(\alpha) = 1 - f(\alpha)\frac{P_{m-1}'(\alpha)}{P_m(\alpha)} = 1 + (m-1) = m.$$

From both cases of (i) and (ii), the extraneous fixed-point α is always repulsive. □

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