

## APPROXIMATELY ADDITIVE MAPPINGS OVER $p$ -ADIC FIELDS

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ABSTRACT. In this paper, we prove the Hyers–Ulam–Rassias stability of the Cauchy functional equation  $f(x+y) = f(x) + f(y)$  and of the Jensen functional equation  $2f(\frac{x+y}{2}) = f(x) + f(y)$  over the  $p$ -adic field  $\mathbb{Q}_p$ . The concept of Hyers–Ulam–Rassias stability originated from the Th.M. Rassias' stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297–300.

### 1. Introduction and preliminaries

In [10], Hensel introduced the concept of  $p$ -adic numbers as a tool for solving problems in algebra and number theory. His idea was to extend the analogies between the ring of integers  $\mathbb{Z}$  and the field of rational numbers  $\mathbb{Q}$  to the field of rational functions and Laurent series. The way this was accomplished was by expressing any rational number  $x \in \mathbb{Q}$  as the sum

$$x = \sum_{n \geq n_0}^{\infty} a_n p^n,$$

where  $p$  is a prime number and  $n_0, a_n \in \mathbb{Z}$  ( $a_n \leq p-1$ ). For a fixed value of  $p$ , we denote by  $\mathbb{Q}_p$  the complete field of  $p$ -adic numbers (see [9]).

Ulam [43] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems.

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Among these was the following question concerning the stability of homomorphisms.

*We are given a group  $G$  and a metric group  $G'$  with metric  $\rho(\cdot, \cdot)$ . Given  $\epsilon > 0$ , does there exist a  $\delta > 0$  such that if  $f : G \rightarrow G'$  satisfies  $\rho(f(xy), f(x)f(y)) < \delta$  for all  $x, y \in G$ , then a homomorphism  $h : G \rightarrow G'$  exists with  $\rho(f(x), h(x)) < \epsilon$  for all  $x \in G$ ?*

By now an affirmative answer has been given in several cases, and some interesting variations of the problem have also been investigated. We shall call such an  $f : G \rightarrow G'$  an *approximate homomorphism*.

Hyers [11] considered the case of approximately additive mappings  $f : E \rightarrow E'$ , where  $E$  and  $E'$  are Banach spaces and  $f$  satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all  $x, y \in E$ . It was shown that the limit

$$L(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$$

exists for all  $x \in E$  and that  $L : E \rightarrow E'$  is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \leq \epsilon.$$

No continuity conditions are required for this result, but if  $f(tx)$  is continuous in the real variable  $t$  for each fixed  $x \in E$ , then  $L$  is linear, and if  $f$  is continuous at a single point of  $E$  then  $L : E \rightarrow E'$  is also continuous.

Th.M. Rassias [34] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded*.

**THEOREM 1.1.** (Th.M. Rassias). *Let  $f : E \rightarrow E'$  be a mapping from a normed vector space  $E$  into a Banach space  $E'$  subject to the inequality*

$$(1.1) \quad \|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all  $x, y \in E$ , where  $\epsilon$  and  $p$  are constants with  $\epsilon > 0$  and  $p < 1$ . Then the limit

$$L(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$$

exists for all  $x \in E$  and  $L : E \rightarrow E'$  is the unique additive mapping which satisfies

$$(1.2) \quad \|f(x) - L(x)\| \leq \frac{2\epsilon}{2-2^p} \|x\|^p$$

for all  $x \in E$ . If  $p < 0$  then inequality (1.1) holds for  $x, y \neq 0$  and (1.2) for  $x \neq 0$ .

Th.M. Rassias [35] during the 27<sup>th</sup> International Symposium on Functional Equations asked the question whether such a theorem can also be proved for  $p \geq 1$ . In 1991, Z. Gajda [7] following the same approach as in Th.M. Rassias [34], gave an affirmative solution to this question for  $p > 1$ . It was shown by Z. Gajda [7], as well as by Th.M. Rassias and P. Šemrl [40] that one cannot prove a Th.M. Rassias' type Theorem when  $p = 1$ . The counterexamples of Z. Gajda [7], as well as of Th.M. Rassias and P. Šemrl [40] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. P. Găvruta [8], S. Czerwik [4], S. Jung [19], who among others studied the Hyers–Ulam–Rassias stability of functional equations. The inequality (1.1) that was introduced for the first time by Th.M. Rassias [34] provided a lot of influence in the development of a generalization of the Hyers–Ulam stability concept. This new concept is known as *Hyers–Ulam–Rassias stability* of functional equations (cf. the books of P. Czerwik [5], D.H. Hyers, G. Isac and Th.M. Rassias [12], S. Jung [20]).

Beginning around the year 1980 the topic of approximate homomorphisms and their stability theory in the field of functional equations and inequalities was taken up by several mathematicians (cf. D.H. Hyers and Th.M. Rassias [14], Th.M. Rassias [38] and the references therein).

J.M. Rassias [30] following the spirit of the innovative approach of Th.M. Rassias [34] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor  $\|x\|^p + \|y\|^p$  by  $\|x\|^p \cdot \|y\|^q$  for  $p, q \in \mathbb{R}$  with  $p + q \neq 1$  (see also [31] for a number of other new results).

P. Găvruta [8] provided a further generalization of Th.M. Rassias' Theorem. In 1996, G. Isac and Th.M. Rassias [15] applied the Hyers–Ulam–Rassias stability theory to prove fixed point theorems and study some new applications in Nonlinear Analysis. In [13], D.H. Hyers, G. Isac and Th.M. Rassias studied the asymptoticity aspect of Hyers–Ulam stability of mappings. In [18], Jun and Lee proved the Hyers–Ulam–Rassias stability of the Jensen functional equation in Banach spaces. In [28], the author introduced the Cauchy–Jensen functional equation and proved the Hyers–Ulam–Rassias stability of the Cauchy–Jensen functional equation in Banach spaces. Several papers have been

published on various generalizations and applications of Hyers–Ulam stability and Hyers–Ulam–Rassias stability to a number of functional equations and mappings, for example: quadratic functional equation, invariant means, multiplicative mappings - superstability, bounded  $n$ th differences, convex functions, generalized orthogonality functional equation, Euler–Lagrange functional equation, Navier–Stokes equations. Several mathematician have contributed works on these subjects; we mention a few: M. Amyari and M.S. Moslehian [1], L.M. Arriola and W.A. Beyer [2], D. Boo, S. Oh, C. Park and J. Park [3], K. Jun and H. Kim [16, 17], C. Park [24, 25], C. Park, J. Park and J. Shin [29], F. Skof [42].

Everett and Ulam [6] presented results on generalizing Lorentz groups over  $p$ -adic fields.  $p$ -adic fields have become of considerable interest to physicists. A key property of  $p$ -adic fields is that they do not satisfy the Archimedean axiom; for all  $a, b > 0$ , there exists an integer  $n$  such that  $a < nb$ . This property has been found to be useful in theoretical physics. In quantum mechanics [22, 23], it has been recognized that fundamental limitations on measuring conjugate quantities such as position-momentum or energy-time exist because of the Heisenberg uncertainty principle. For example, any attempt at taking gravitational measurements at sub-Planck domains, say, of the order of  $l = 10^{-35}m$ , would change the underlying geometry and introduce distortions to  $l$ . Introducing a  $p$ -adic space-time could provide a means of quantifying the non-localization affects.

We recall some definitions and results that will be needed later.

**DEFINITION 1.2.** (Non-Archimedean Valuation) Let  $\mathbb{K}$  denote a scalar field, and  $|\cdot|$  denote the usual absolute value, where  $|\cdot| : \mathbb{K} \rightarrow \mathbb{R}$ . A non-Archimedean valuation is a function  $|\cdot|_p$  that satisfies the strong triangle inequality; namely,

$$|x + y|_p \leq \max\{|x|_p, |y|_p\} \leq |x|_p + |y|_p$$

for all  $x, y \in \mathbb{K}$ . The associated field  $\mathbb{K}$  is referred to as a non-Archimedean field.

**LEMMA 1.3.** [9] *For any nonzero rational number  $x$ , there exists a unique integer  $n \in \mathbb{Z}$  such that  $x = \frac{a}{b}p^n$ , where  $a$  and  $b$  are integers not divisible by  $p$ . The  $p$ -adic valuation is defined by  $|x|_p := p^{-n}$ .*

**DEFINITION 1.4.** ( $p$ -adic Field) For each prime  $p$ , define the  $p$ -adic field  $\mathbb{Q}_p$  to be the set of all  $p$ -adic expansions  $\mathbb{Q}_p := \{x \mid x = \sum_{k \geq n_0}^{\infty} a_k p^k\}$ , where  $a_k \leq p - 1$  are integers.

Throughout this paper, assume that  $B$  is a Banach space with norm  $\|\cdot\|$ .

In this paper, we prove the Hyers–Ulam–Rassias stability of the Cauchy functional equation and of the Jensen functional equation over the  $p$ -adic fields  $\mathbb{Q}_p$ .

## 2. Stability of the Cauchy mapping over the $p$ -adic field $\mathbb{Q}_p$

In this section, we prove the Hyers–Ulam–Rassias stability of the Cauchy functional equation over the  $p$ -adic field  $\mathbb{Q}_p$ .

**THEOREM 2.1.** *Let  $r < 1$  be a nonnegative real number and  $f : \mathbb{Q}_p \rightarrow B$  a mapping such that*

$$(2.1) \quad \|f(x+y) - f(x) - f(y)\| \leq \theta(|x|_p^r + |y|_p^r)$$

for all  $x, y \in \mathbb{Q}_p$ . Then there exists a unique Cauchy additive mapping  $L : \mathbb{Q}_p \rightarrow B$  such that

$$(2.2) \quad \|f(x) - L(x)\| \leq \frac{2\theta}{2-2^r}|x|_p^r$$

for all  $x \in \mathbb{Q}_p$ .

*Proof.* Letting  $y = x$  in (2.1), we get

$$\|f(2x) - 2f(x)\| \leq 2\theta|x|_p^r$$

for all  $x \in \mathbb{Q}_p$ . So

$$\|f(x) - \frac{1}{2}f(2x)\| \leq \theta|x|_p^r$$

for all  $x \in \mathbb{Q}_p$ . Hence

$$(2.3) \quad \left\| \frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \frac{2^{rj}\theta}{2^j}|x|_p^r$$

for all nonnegative integers  $m$  and  $l$  with  $m > l$  and all  $x \in \mathbb{Q}_p$ . It follows from (2.3) that the sequence  $\{\frac{1}{2^n}f(2^n x)\}$  is a Cauchy sequence for all  $x \in \mathbb{Q}_p$ . Since  $B$  is complete, the sequence  $\{\frac{1}{2^n}f(2^n x)\}$  converges. So one can define the mapping  $L : \mathbb{Q}_p \rightarrow B$  by

$$L(x) := \lim_{n \rightarrow \infty} \frac{1}{2^n}f(2^n x)$$

for all  $x \in \mathbb{Q}_p$ .

By (2.1),

$$\begin{aligned} \|L(x+y) - L(x) - L(y)\| &= \lim_{n \rightarrow \infty} \frac{1}{2^n} \|f(2^n x + 2^n y) - f(2^n x) - f(2^n y)\| \\ &\leq \lim_{n \rightarrow \infty} \frac{2^{nr}}{2^n} \theta(|x|_p^r + |y|_p^r) = 0 \end{aligned}$$

for all  $x, y \in \mathbb{Q}_p$ . So

$$L(x+y) = L(x) + L(y)$$

for all  $x, y \in \mathbb{Q}_p$ . Moreover, letting  $l = 0$  and passing the limit  $m \rightarrow \infty$  in (2.3), we get (2.2).

Now, let  $T : \mathbb{Q}_p \rightarrow B$  be another Cauchy additive mapping satisfying (2.2). Then we have

$$\begin{aligned} \|L(x) - T(x)\| &= \frac{1}{2^n} \|L(2^n x) - T(2^n x)\| \\ &\leq \frac{1}{2^n} (\|L(2^n x) - f(2^n x)\| + \|T(2^n x) - f(2^n x)\|) \\ &\leq \frac{4 \cdot 2^{nr} \theta}{(2 - 2^r) 2^n} |x|_p^r, \end{aligned}$$

which tends to zero as  $n \rightarrow \infty$  for all  $x \in \mathbb{Q}_p$ . So we can conclude that  $L(x) = T(x)$  for all  $x \in \mathbb{Q}_p$ . This proves the uniqueness of  $L$ .  $\square$

**THEOREM 2.2.** *Let  $r < \frac{1}{2}$  be a nonnegative real number and  $f : \mathbb{Q}_p \rightarrow B$  a mapping such that*

$$(2.4) \quad \|f(x+y) - f(x) - f(y)\| \leq \theta \cdot |x|_p^r \cdot |y|_p^r$$

for all  $x, y \in \mathbb{Q}_p$ . Then there exists a unique Cauchy additive mapping  $L : \mathbb{Q}_p \rightarrow B$  such that

$$(2.5) \quad \|f(x) - L(x)\| \leq \frac{\theta}{2 - 4^r} |x|_p^{2r}$$

for all  $x \in \mathbb{Q}_p$ .

*Proof.* Letting  $y = x$  in (2.4), we get

$$\|f(2x) - 2f(x)\| \leq \theta |x|_p^{2r}$$

for all  $x \in \mathbb{Q}_p$ . So

$$\|f(x) - \frac{1}{2} f(2x)\| \leq \frac{\theta}{2} |x|_p^{2r}$$

for all  $x \in \mathbb{Q}_p$ . Hence

$$(2.6) \quad \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \frac{4^{rj} \theta}{2^{j+1}} |x|_p^{2r}$$

for all nonnegative integers  $m$  and  $l$  with  $m > l$  and all  $x \in \mathbb{Q}_p$ . It follows from (2.6) that the sequence  $\{\frac{1}{2^n} f(2^n x)\}$  is a Cauchy sequence for all  $x \in \mathbb{Q}_p$ . Since  $B$  is complete, the sequence  $\{\frac{1}{2^n} f(2^n x)\}$  converges. So one can define the mapping  $L : \mathbb{Q}_p \rightarrow B$  by

$$L(x) := \lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n x)$$

for all  $x \in \mathbb{Q}_p$ .

By (2.4),

$$\begin{aligned} \|L(x+y) - L(x) - L(y)\| &= \lim_{n \rightarrow \infty} \frac{1}{2^n} \|f(2^n x + 2^n y) - f(2^n x) - f(2^n y)\| \\ &\leq \lim_{n \rightarrow \infty} \frac{4^{nr}}{2^n} \theta \cdot |x|_p^r \cdot |y|_p^r = 0 \end{aligned}$$

for all  $x, y \in \mathbb{Q}_p$ . So

$$L(x+y) = L(x) + L(y)$$

for all  $x, y \in \mathbb{Q}_p$ . Moreover, letting  $l = 0$  and passing the limit  $m \rightarrow \infty$  in (2.6), we get (2.5).

Now, let  $T : \mathbb{Q}_p \rightarrow B$  be another Cauchy additive mapping satisfying (2.5). Then we have

$$\begin{aligned} \|L(x) - T(x)\| &= \frac{1}{2^n} \|L(2^n x) - T(2^n x)\| \\ &\leq \frac{1}{2^n} (\|L(2^n x) - f(2^n x)\| + \|T(2^n x) - f(2^n x)\|) \\ &\leq \frac{2 \cdot 4^{nr} \theta}{(2 - 4^r) 2^n} |x|_p^{2r}, \end{aligned}$$

which tends to zero as  $n \rightarrow \infty$  for all  $x \in \mathbb{Q}_p$ . So we can conclude that  $L(x) = T(x)$  for all  $x \in \mathbb{Q}_p$ . This proves the uniqueness of  $L$ .  $\square$

**THEOREM 2.3.** *Let  $r > 1$  be a real number and  $f : B \rightarrow \mathbb{Q}_p$  a mapping such that*

$$(2.7) \quad |f(x+y) - f(x) - f(y)|_p \leq \theta(\|x\|^r + \|y\|^r)$$

for all  $x, y \in B$ . Then there exists a unique Cauchy additive mapping  $L : B \rightarrow \mathbb{Q}_p$  such that

$$(2.8) \quad |f(x) - L(x)|_p \leq \frac{2\theta}{2^r - 2} \|x\|^r$$

for all  $x \in B$ .

*Proof.* Letting  $y = x$  in (2.7), we get

$$|f(2x) - 2f(x)|_p \leq 2\theta \|x\|^r$$

for all  $x \in B$ . So

$$|f(x) - 2f(\frac{x}{2})|_p \leq \frac{2\theta}{2^r} \|x\|^r$$

for all  $x \in B$ . Hence

$$(2.9) \quad |2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m})|_p \leq \sum_{j=l}^{m-1} \frac{2^{j+1}\theta}{2^{rj+r}} \|x\|^r$$

for all nonnegative integers  $m$  and  $l$  with  $m > l$  and all  $x \in B$ . It follows from (2.9) that the sequence  $\{2^n f(\frac{x}{2^n})\}$  is a Cauchy sequence for all  $x \in B$ . Since  $\mathbb{Q}_p$  is complete, the sequence  $\{2^n f(\frac{x}{2^n})\}$  converges. So one can define the mapping  $L : B \rightarrow \mathbb{Q}_p$  by

$$L(x) := \lim_{n \rightarrow \infty} 2^n f(\frac{x}{2^n})$$

for all  $x \in B$ .

By (2.7),

$$\begin{aligned} |L(x+y) - L(x) - L(y)|_p &= \lim_{n \rightarrow \infty} |2^n (f(\frac{x}{2^n} + \frac{y}{2^n}) - f(\frac{x}{2^n}) - f(\frac{y}{2^n}))|_p \\ &\leq \lim_{n \rightarrow \infty} \frac{2^n}{2^{nr}} \theta (\|x\|^r + \|y\|^r) = 0 \end{aligned}$$

for all  $x, y \in B$ . So

$$L(x+y) = L(x) + L(y)$$

for all  $x, y \in B$ . Moreover, letting  $l = 0$  and passing the limit  $m \rightarrow \infty$  in (2.9), we get (2.8).

By the same method as in the proof of Theorem 2.1, one can prove the uniqueness of  $L$ .  $\square$

**THEOREM 2.4.** *Let  $r > \frac{1}{2}$  be a real number and  $f : B \rightarrow \mathbb{Q}_p$  a mapping such that*

$$(2.10) \quad |f(x+y) - f(x) - f(y)|_p \leq \theta \cdot \|x\|^r \cdot \|y\|^r$$

for all  $x, y \in B$ . Then there exists a unique Cauchy additive mapping  $L : B \rightarrow \mathbb{Q}_p$  such that

$$(2.11) \quad |f(x) - L(x)|_p \leq \frac{\theta}{4^r - 2} \|x\|^{2r}$$

for all  $x \in B$ .

*Proof.* Letting  $y = x$  in (2.10), we get

$$|f(2x) - 2f(x)|_p \leq \theta \|x\|^{2r}$$

for all  $x \in B$ . So

$$|f(x) - 2f(\frac{x}{2})|_p \leq \frac{\theta}{4^r} \|x\|^{2r}$$

for all  $x \in B$ . Hence

$$(2.12) \quad |2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m})|_p \leq \sum_{j=l}^{m-1} \frac{2^j \theta}{4^{rj+r}} \|x\|^{2r}$$

for all nonnegative integers  $m$  and  $l$  with  $m > l$  and all  $x \in B$ . It follows from (2.12) that the sequence  $\{2^n f(\frac{x}{2^n})\}$  is a Cauchy sequence for all  $x \in B$ . Since  $\mathbb{Q}_p$  is complete, the sequence  $\{2^n f(\frac{x}{2^n})\}$  converges. So one can define the mapping  $L : B \rightarrow \mathbb{Q}_p$  by

$$L(x) := \lim_{n \rightarrow \infty} 2^n f(\frac{x}{2^n})$$

for all  $x \in B$ .

By (2.10),

$$\begin{aligned} |L(x+y) - L(x) - L(y)|_p &= \lim_{n \rightarrow \infty} |2^n (f(\frac{x}{2^n} + \frac{y}{2^n}) - f(\frac{x}{2^n}) - f(\frac{y}{2^n}))|_p \\ &\leq \lim_{n \rightarrow \infty} \frac{2^n}{4^{nr}} \theta \cdot \|x\|^r \cdot \|y\|^r = 0 \end{aligned}$$

for all  $x, y \in B$ . So

$$L(x+y) = L(x) + L(y)$$

for all  $x, y \in B$ . Moreover, letting  $l = 0$  and passing the limit  $m \rightarrow \infty$  in (2.12), we get (2.11).

By the same method as in the proof of Theorem 2.2, one can prove the uniqueness of  $L$ .  $\square$

### 3. Stability of the Jensen mapping over the $p$ -adic field $\mathbb{Q}_p$

In this section, we prove the Hyers–Ulam–Rassias stability of the Jensen functional equation over the  $p$ -adic field  $\mathbb{Q}_p$ .

**THEOREM 3.1.** *Let  $r < 1$  be a nonnegative real number and  $f : \mathbb{Q}_p \rightarrow B$  a mapping satisfying  $f(0) = 0$  such that*

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\| \leq \theta(|x|_p^r + |y|_p^r)$$

for all  $x, y \in \mathbb{Q}_p$ . Then there exists a unique Jensen additive mapping  $L : \mathbb{Q}_p \rightarrow B$  such that

$$(3.1) \quad \|f(x) - L(x)\| \leq \frac{3 + 3^r}{3 - 3^r} \theta |x|_p^r$$

for all  $x \in \mathbb{Q}_p$ .

*Proof.* By the same reasoning as in the proofs of Theorem 1 in [18] and Theorem 2.1, there is a unique Jensen additive mapping  $L : \mathbb{Q}_p \rightarrow B$  satisfying (3.1). The Jensen additive mapping  $L : \mathbb{Q}_p \rightarrow B$  is given by

$$L(x) := \lim_{n \rightarrow \infty} \frac{1}{3^n} f(3^n x)$$

for all  $x \in \mathbb{Q}_p$ . □

**THEOREM 3.2.** Let  $r < \frac{1}{2}$  be a nonnegative real number and  $f : \mathbb{Q}_p \rightarrow B$  a mapping satisfying  $f(0) = 0$  such that

$$\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\| \leq \theta \cdot |x|_p^r \cdot |y|_p^r$$

for all  $x, y \in \mathbb{Q}_p$ . Then there exists a unique Jensen additive mapping  $L : \mathbb{Q}_p \rightarrow B$  such that

$$(3.2) \quad \|f(x) - L(x)\| \leq \frac{1 + 3^r}{3 - 3^{2r}} \theta |x|_p^{2r}$$

for all  $x \in \mathbb{Q}_p$ .

*Proof.* By the same reasoning as in the proofs of Theorem 1 in [18] and Theorem 2.2, there is a unique Jensen additive mapping  $L : \mathbb{Q}_p \rightarrow B$  satisfying (3.2). The Jensen additive mapping  $L : \mathbb{Q}_p \rightarrow B$  is given by

$$L(x) := \lim_{n \rightarrow \infty} \frac{1}{3^n} f(3^n x)$$

for all  $x \in \mathbb{Q}_p$ . □

**THEOREM 3.3.** Let  $r > 1$  be a real number and  $f : B \rightarrow \mathbb{Q}_p$  a mapping satisfying  $f(0) = 0$  such that

$$|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)|_p \leq \theta (\|x\|^r + \|y\|^r)$$

for all  $x, y \in B$ . Then there exists a unique Jensen additive mapping  $L : B \rightarrow \mathbb{Q}_p$  such that

$$(3.3) \quad |f(x) - L(x)|_p \leq \frac{3^r + 3}{3^r - 3} \theta \|x\|^r$$

for all  $x \in B$ .

*Proof.* By the same reasoning as in the proofs of Theorem 6 in [18] and Theorem 2.3, there is a unique Jensen additive mapping  $L : B \rightarrow \mathbb{Q}_p$  satisfying (3.3). The Jensen additive mapping  $L : B \rightarrow \mathbb{Q}_p$  is given by

$$L(x) := \lim_{n \rightarrow \infty} 3^n f\left(\frac{x}{3^n}\right)$$

for all  $x \in B$ . □

**THEOREM 3.4.** *Let  $r > \frac{1}{2}$  be a real number and  $f : B \rightarrow \mathbb{Q}_p$  a mapping satisfying  $f(0) = 0$  such that*

$$\left|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right|_p \leq \theta \cdot \|x\|^r \cdot \|y\|^r$$

for all  $x, y \in B$ . Then there exists a unique Jensen additive mapping  $L : B \rightarrow \mathbb{Q}_p$  such that

$$(3.4) \quad |f(x) - L(x)|_p \leq \frac{3^r + 1}{3^{2r} - 3} \theta \|x\|^{2r}$$

for all  $x \in B$ .

*Proof.* By the same reasoning as in the proofs of Theorem 6 in [18] and Theorem 2.4, there is a unique Jensen additive mapping  $L : B \rightarrow \mathbb{Q}_p$  satisfying (3.4). The Jensen additive mapping  $L : B \rightarrow \mathbb{Q}_p$  is given by

$$L(x) := \lim_{n \rightarrow \infty} 3^n f\left(\frac{x}{3^n}\right)$$

for all  $x \in B$ . □

## References

- [1] M. Amyari and M.S. Moslehian, *Approximate homomorphisms of ternary semigroups*, Letters Math. Phys. **77** (2006), 1–9.
- [2] L. M. Arriola and W.A. Beyer, *Stability of the Cauchy functional equation over  $p$ -adic fields*, Real Analysis Exchange **31** (2005/2006), 125–132.
- [3] D. Boo, S. Oh, C. Park and J. Park, *Generalized Jensen's equations in Banach modules over a  $C^*$ -algebra and its unitary group*, Taiwanese J. Math. **7** (2003), 641–655.
- [4] P. Czerwik, *On stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Hamburg **62** (1992), 59–64.
- [5] P. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific Publishing Company, New Jersey, Hong Kong, Singapore and London, 2002.
- [6] C. J. Everett and S. M. Ulam, *On some possibilities of generalizing the Lorentz group in the special relativity theory*, J. Comb. Theory **1** (1966), 248–270.
- [7] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci. **14** (1991), 431–434.

- [8] P. Găvruta, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
- [9] F. Q. Gouvêa, *p-adic Numbers*, Springer-Verlag, Berlin, 1997.
- [10] K. Hensel, *Theorie der Algebraischen Zahlen*, Tuebner, Leipzig and Berlin, 1908.
- [11] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.
- [12] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.
- [13] D. H. Hyers, G. Isac and Th. M. Rassias, *On the asymptoticity aspect of Hyers–Ulam stability of mappings*, Proc. Amer. Math. Soc. **126** (1998), 425–430.
- [14] D. H. Hyers and Th. M. Rassias, *Approximate homomorphisms*, Aequationes Math. **44** (1992), 125–153.
- [15] G. Isac and Th. M. Rassias, *Stability of  $\psi$ -additive mappings: Applications to nonlinear analysis*, Internat. J. Math. Math. Sci. **19** (1996), 219–228.
- [16] K. Jun and H. Kim, *Ulam stability problem for quadratic mappings of Euler–Lagrange*, Nonlinear Anal.–TMA **61** (2005), 1093–1104.
- [17] K. Jun and H. Kim, *On the generalized A-quadratic mappings associated with the variance of a discrete type distribution*, Nonlinear Anal.–TMA **62** (2005), 975–987.
- [18] K. Jun and Y. Lee, *A generalization of the Hyers–Ulam–Rassias stability of Jensen’s equation*, J. Math. Anal. Appl. **238** (1999), 305–315.
- [19] S. Jung, *On the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **204** (1996), 221–226.
- [20] S. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Florida, 2001.
- [21] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras*, Academic Press, New York, 1983.
- [22] A. Khrennikov, *p-adic Valued Distributions in Mathematical Physics*, Kluwer Academic Publishers, Dordrecht, 1994.
- [23] A. Khrennikov, *Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models*, Kluwer Academic Publishers, Dordrecht, 1997.
- [24] C. Park, *On the stability of the linear mapping in Banach modules*, J. Math. Anal. Appl. **275** (2002), 711–720.
- [25] C. Park, *Lie  $*$ -homomorphisms between Lie  $C^*$ -algebras and Lie  $*$ -derivations on Lie  $C^*$ -algebras*, J. Math. Anal. Appl. **293** (2004), 419–434.
- [26] C. Park, *Homomorphisms between Lie  $JC^*$ -algebras and Cauchy–Rassias stability of Lie  $JC^*$ -algebra derivations*, J. Lie Theory **15** (2005), 393–414.
- [27] C. Park, *Homomorphisms between Poisson  $JC^*$ -algebras*, Bull. Braz. Math. Soc. **36** (2005), 79–97.
- [28] C. Park, *Cauchy–Rassias stability of Cauchy–Jensen additive mappings in Banach spaces*, Acta Math. Sinica (to appear).
- [29] C. Park, J. Park and J. Shin, *Hyers–Ulam–Rassias stability of quadratic functional equations in Banach modules over a  $C^*$ -algebra*, Chinese Ann. Math. **24** (2003), 261–266.
- [30] J. M. Rassias, *On approximation of approximately linear mappings by linear mappings*, Bull. Sci. Math. **108** (1984), 445–446.

- [31] J. M. Rassias, *Solution of a problem of Ulam*, J. Approx. Theory **57** (1989), 268–273.
- [32] J. M. Rassias, *On the stability of the Euler–Lagrange functional equation*, Chinese J. Math. **20** (1992), 185–190.
- [33] J. M. Rassias and M.J. Rassias, *On the Ulam stability for Euler–Lagrange type quadratic functional equations*, Austral. J. Math. Anal. Appl. **2** (2005), 1–10.
- [34] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
- [35] Th. M. Rassias, *Problem 16; 2*, Report of the 27<sup>th</sup> International Symp. on Functional Equations, Aequationes Math. **39** (1990), 292–293; 309.
- [36] Th. M. Rassias, *The problem of S.M. Ulam for approximately multiplicative mappings*, J. Math. Anal. Appl. **246** (2000), 352–378.
- [37] Th. M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251** (2000), 264–284.
- [38] Th. M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000), 23–130.
- [39] Th. M. Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2003.
- [40] Th. M. Rassias and P. Šemrl, *On the behaviour of mappings which do not satisfy Hyers–Ulam stability*, Proc. Amer. Math. Soc. **114** (1992), 989–993.
- [41] Th. M. Rassias and P. Šemrl, *On the Hyers–Ulam stability of linear mappings*, J. Math. Anal. Appl. **173** (1993), 325–338.
- [42] F. Skof, *Proprietà locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.
- [43] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.

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