New Weighting Factor of 2D Isotropic-Dispersion Finite Difference Time Domain(ID-FDTD) Algorithm

Meng Zhao · Il-Suek Koh

Abstract

In this paper, a new scheme to calculate the weighting factor of the 2-D isotropic-dispersion finite difference time domain(ID-FDTD) is proposed. The weighting factor in [1] was formulated in free space, so that it may not be optimal in dielectric media. Therefore, the weighting factor was reformulated by considering the material properties and using the least mean square method. As a result, a minimum numerical dispersion error for any dielectric media is guaranteed.

Key words : FDTD, ID-FDTD, Weighting Factor, Numerical Dispersion Error.

Ⅰ. Introduction

Since it was first proposed in 1966 by Kane Yee[2], the finite-difference time-domain(FDTD) method has garnered ever increasing attention. The FDTD scheme has been a very popular and widely-used method, especially during the past several decades. For example, it has been applied in the fields of electromagnetics, biology, materials science and optics, to support scientific research. Due to its low computational complexity, great flexibility and easy implementation[3], FDTD is appropriate for use in various applications. However, standard FDTD has some disadvantages, one of which is known as "numerical dispersion" that causes wave propagation at different phase velocities along different directions. Consequently, the standard FDTD method can not be applied to large-scale or phase-sensitive problems. Some schemes have been proposed to solve this problem[1],[4]. In [1], a new algorithm known as isotropic-dispersion FDTD (ID-FDTD) can solve the numerical dispersion problem without much extra computational complexity. The ID-FDTD scheme replaced the central finite different(FD) scheme, which the standard FDTD method used to approximate the time and spatial derivatives in Maxwell's equation, with a weighted summation of two different FD schemes[1]. By choosing the weighting factor, the ID-FDTD method can minimize the numerical dispersion error over the propagation angles.

However, the weighting factor in [1] was formulated only for free space. Hence, the numerical dispersion error is not minimal in a dielectric medium. Therefore, in this paper, the weighting factor was modified for different materials. Also, the Least-Mean-Square method was applied to minimize the fluctuation of the numerical dispersion error. As a result, the new weighting factor is optimal since it can generate nearly isotropic phase velocities along the propagation angles for any dielectric media. In the next section, the new weighting factor is formulated, and its properties are discussed by comparing it with the old scheme. To verify the new scheme, three scattering examples are considered in Section Ⅲ.

Ⅱ. Formulation

In this section, the new weighting factor will be obtained and the properties investigated. There are two kinds of independent factors in the ID-FDTD scheme, the weighting factor and the scaling factor, both of which are determined based on the dispersion relation[1]. The weighting factor is used to achieve the isotropic numerical dispersion over the propagation angles, while the scaling factor is used to match the numerical phase velocity to the exact value. For consistency, the scaling factor formulated in [5] is used.

2-1 Dispersion Relation

As mentioned above, the weighting factor is determined based on the dispersion relation. Thus, the dispersion relation is the key to formulate an optimal weighting factor. In this paper, a TM_z mode is assumed. The dispersion relation is found in [1]. The dispersion relation of the ID-FDTD scheme for lossy media is represented here as:

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$$\frac{1}{\Delta t^2}\sin\frac{\omega\Delta t}{2}\left[\sin\frac{\omega\Delta t}{2}-j\frac{\sigma\Delta t}{2\varepsilon_0\varepsilon'}\cos\frac{\omega\Delta t}{2}\right]$$
$$=\frac{c_0^2}{\varepsilon'\mu_r}\left[\begin{array}{l}\frac{1}{\Delta x^2}\sin^2\frac{\hat{\gamma}_x\Delta x}{2}\left(1-\alpha\sin^2\frac{\hat{\gamma}_y\Delta y}{2}\right)^2\\+\frac{1}{\Delta y^2}\sin^2\frac{\hat{\gamma}_y\Delta y}{2}\left(1-\alpha\sin^2\frac{\hat{\gamma}_y\Delta y}{2}\right)^2\end{array}\right] \quad (1)$$

where, c_0, μ_0 and ε_0 are the phase velocity, the permittivity and the permeability of the free space, respectively. μ_r represents the relative permittivity while $\varepsilon_r=\varepsilon'-j\varepsilon''$ denotes the relative permeability. And the conductivity σ is related to ε_r as $\sigma=\omega\varepsilon_0\varepsilon''$. $j=\sqrt{-1}$ and ω is the angular frequency. $\hat{\gamma}_x=\hat{\gamma}\cos\phi$ and $\hat{\gamma}_y=\hat{\gamma}\sin\phi$, here $\hat{\gamma}$ is the numerical wavenumber and ϕ is an azimuth angle (propagation angle). Δx and Δy are cell sizes of the x and y directions, respectively. Δt is a time step. The numerical wavenumber is the solution to (1), which corresponds to the exact wave number of $k=2\pi/\lambda$. Hence, the numerical wave number indicates the phase velocity in a FDTD simulation, which will be different from the exact value.

If square cell is assumed, $\Delta x=\Delta y=\Delta$, the numerical wave number $\hat{\gamma}$ can be solved analytically for ϕ=0° as:

$$\hat{\gamma}=\frac{\Delta}{2}\arcsin\sqrt{\frac{\varepsilon'\mu_r\Delta^2}{c_0^2}\left[\frac{1}{\Delta t^2}\sin^2\frac{\omega\Delta t}{2}-j\frac{\sigma}{4\varepsilon_0\varepsilon'\Delta t}\sin(\omega\Delta t)\right]} \quad (2)$$

2-2 Weighting Factor

The equation, (1) can be converted into a quadratic equation of α as

$$AB(A+B)\alpha^2-4AB\alpha+A+B-\frac{C_2}{C_1}=0 \quad (3)$$

where $A=\sin^2\frac{\hat{\gamma}_x\Delta}{2}$, $B=\sin^2\frac{\hat{\gamma}_y\Delta}{2}$, $C_1=\frac{c_0^2}{\varepsilon'\mu_r\Delta^2}$ and $C_2=\frac{1}{\Delta t^2}\sin^2\frac{\omega\Delta t}{2}-j\frac{\sigma}{4\varepsilon_0\varepsilon'\Delta t}\sin(\omega\Delta t)$.

Fluctuating over the propagation angle ϕ, the old weighting factor α is calculated by solving the quadratic equation (3) and estimated it[1] as its mean value over all propagation angles. A closed-form expression of the old weighting factor can be found in [1].

However, the fluctuation of the old factor is not minimal over the propagation angle. Therefore, the Least Mean Square method is applied to calculate the optimal weighting factor. Then, the following equation is obtained:

$$f=\min.\int_0^{2\pi}\left|\frac{\partial}{\partial\phi}\left[AB(A+B)\alpha^2-4AB\alpha+A+B-\frac{C_2}{C_1}\right]\right|^2 d\phi \quad (4)$$

The minimum value of equation (4) is obtained when the partial derivative with respect to α becomes 0, $\frac{\partial f}{\partial\alpha}=0$. It leads to:

$$\int_0^{2\pi}Re\left\{\left[\frac{\partial(C_+C_\times)}{\partial\phi}\alpha^2-4\alpha\frac{\partial C_\times}{\partial\phi}+\frac{\partial C_+}{\partial\phi}\right]\left[\frac{\partial(C_+C_\times)}{\partial\phi}\alpha-2\frac{\partial C_\times}{\partial\phi}\right]^*\right\}d\phi=0 \quad (5)$$

where $C_+=A+B$, $C_\times=AB$ and $Re[\cdot]$ denotes the real part of a complex number, and "*" represents the complex conjugate. It is easy to observe that equation (5) is a cubic equation of α. After algebraic manipulations, a standard form of a cubic equation can be obtained, which can be expressed as:

$$a\alpha^3+b\alpha^2+c\alpha+d=0 \quad (6)$$

Let $a_1=\hat{\gamma}\frac{\Delta}{2}\cos\phi$, $a_2=\hat{\gamma}\frac{\Delta}{2}\sin\phi$,
$a_+=\sin^2 a_1+\sin^2 a_2$ and $a_\times=\sin^2 a_1\cdot\sin^2 a_2$ then

$$a'=(a_1\sin 2a_2-a_2\sin 2a_1)a_\times+(a_1\sin^2 a_1\cdot\sin 2a_2-a_2\sin^2 a_2\cdot\sin 2a_1)a_+$$

$$a=\int_0^{2\pi}|a'|^2 d\phi$$

$$b=-6\int_0^{2\pi}Re[(a_1\sin^2 a_1\cdot\sin 2a_2-a_2\sin^2 a_2\cdot\sin 2a_1)a'^*]d\phi$$

$$c=\int_0^{2\pi}Re\left[a'(a_1\sin 2a_2-a_2\sin 2a_1)^*+8\left|\begin{array}{l}a_1\sin^2 a_1\sin 2a_2\\-a_2\sin 2a_1\sin^2 a_2\end{array}\right|^2\right]d\phi$$

$$d=-2\int_0^{2\pi}Re\left[\begin{array}{c}(a_1\sin^2 a_1\sin 2a_2-a_2\sin 2a_1\sin^2 a_2)\\(a_1\sin 2a_2-a_2\sin 2a_1)^*\end{array}\right]d\phi$$

Three solutions exist, one of which is real and belongs to the interval 0 to 1, and is chosen as the new weighting factor.

Fig. 1 shows a comparison of the maximum dispersion error over the propagation angles generated by the old and new weighting factors at different dielectric

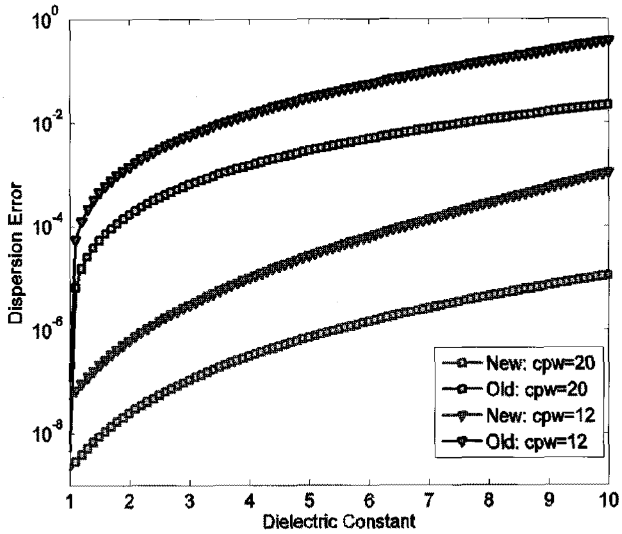


Fig. 1. Comparison of the maximum dispersion error generated by the old and new weighting factors in different dielectric media. Here, S=0.7 is fixed, and the frequency is at 300 MHz.

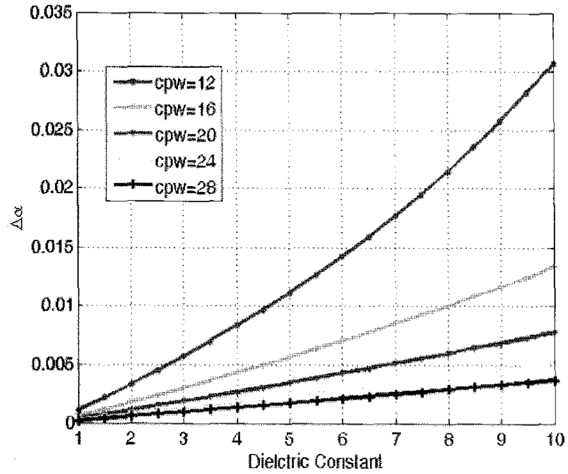


Fig. 2. Variation of the old and new weighting factors in different dielectric media. Here, S=0.7 is fixed, and the frequency is at 300 MHz.

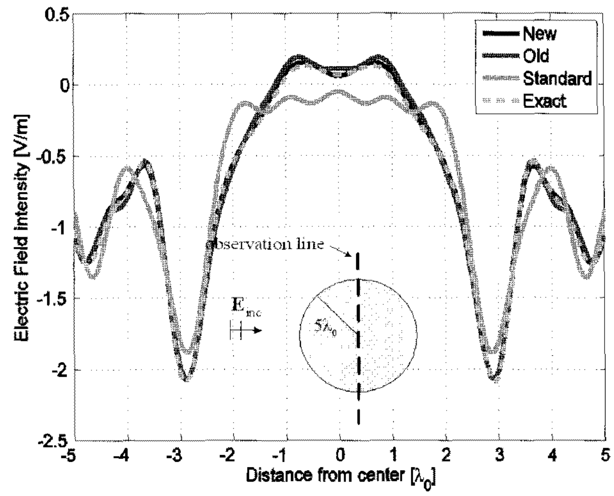
constants. The Courant limit S is fixed to be 0.7. Two cell sizes of 12 and 20 CPW (cells per wavelength) are considered. For free space, dielectric constant =1, the new weighting factor can reduce the dispersion error by about 50 %. When the dielectric constant reached a larger value, the error generated by the old weighting factor drastically increased. For a dielectric medium, the new scheme can improve the isotropy of dispersion over 1,000 times.

Fig. 2 shows the variation of the old and new weighting factors at different dielectric constants. The variation $\Delta\alpha$ is defined as $\Delta\alpha = \alpha_{new} - \alpha_{old}$. The Courant limit S is fixed to be 0.7. Since the old weighting factor is calculated for the free space, it is fixed when S and CPW are constant. Therefore, the variation should become larger as the dielectric constant becomes larger, as seen in the Fig. 2.

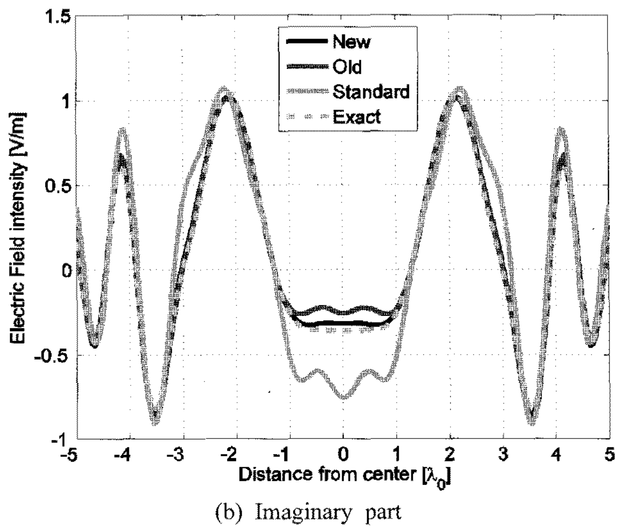
Ⅲ. Numerical Results

To verify the new scheme, three scattering problems were considered. An incident TM_z wave that was scattered by a 2D dielectric cylinder was assumed. The dielectric constant inside the cylinder was set to be 2, and outside was the free space. The scattered electric fields were calculated by the ID-FDTD schemes using the new/old weighting factors, the standard FDTD scheme and the exact eigen-series solution[6]. The ID-FDTD scheme utilized the weighting factor to achieve the isotropic numerical dispersion over the propagation angles and the scaling factor to match the numerical phase velocity to the exact value[1]~[5]. In addition, the square cell was used and the Courant limit S was fixed to be 0.7 for the simulations.

First, scattering from a lossless cylinder was considered, whose radius was $5\lambda_0$. Here, λ_0 was the wavelength in free space, and the cell size of $\Delta = \lambda_0/20$ was used. Fig. 3 shows the real and imaginary parts of the scattered electric fields calculated by the different methods mentioned above. The observation line was located inside the cylinder, at $(0, -5\lambda_0 \sim 5\lambda_0)$, as seen in Fig. 3. Both the old and new ID-FDTD schemes generated much more accurate results than the standard method. However, due to the dispersion error, the discrepancy became larger when the observation line approached the central parts using the old ID-FDTD scheme.



(a) Real part



(b) Imaginary part

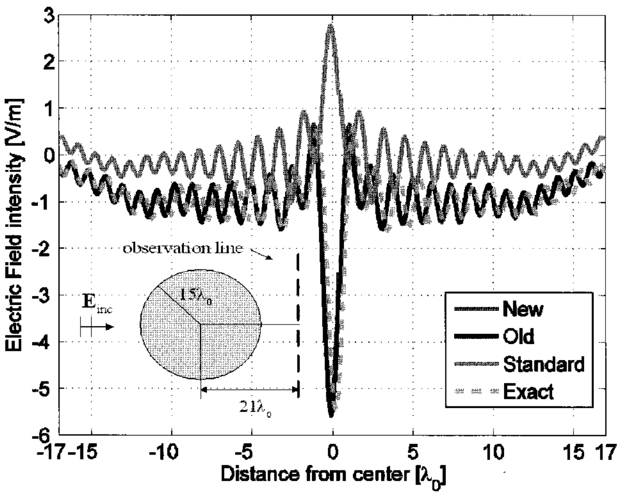
Fig. 3. Plots of the scattered electric field observed inside a lossless cylinder. The radius and permittivity of the dielectric cylinder were $5\lambda_0$ and ε_r=2.0, respectively. S=0.7, $\Delta = \lambda_0/20$ and frequency f=300 MHz were assumed.

The second example replaced the small cylinder with a larger one whose radius was $15\lambda_0$. The dielectric constant and the cell size remained the same as in the previous case. The observation line was located $2\lambda_0$ away from the cylinder, at $(21\lambda_0, -17\lambda_0 \sim 17\lambda_0)$. Apparently, the larger cylinder should generate more dispersion error. As seen in Fig. 4, the standard method was not accurate at all for this case. The result achieved using the old ID-FDTD scheme became more erroneous, especially in the imaginary part. However, the new scheme kept isotropy very well. The error was not obviously increased using the new scheme.

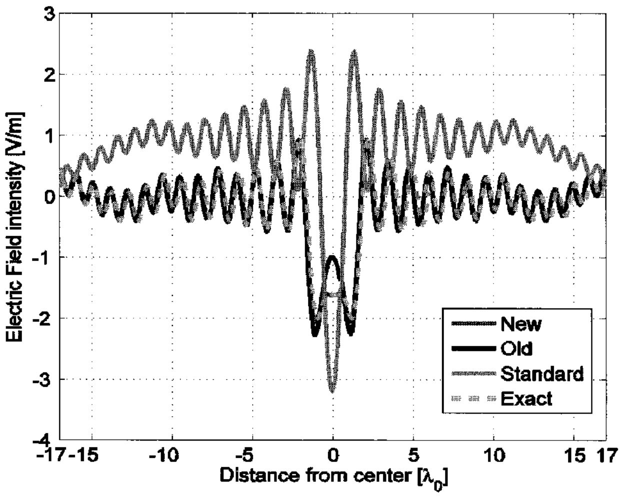
Lastly, scattering from a lossy cylinder was considered. Instead of the lossless cylinder, a lossy one with the same radius was used. The dielectric constant was $\varepsilon_r = 2.0 - j0.01$. The scattered electric field was observed outside the cylinder, $2\lambda_0$ away from the cylinder at $(7\lambda_0, -5\lambda_0 \sim 5\lambda_0)$. The same cell size was used for this case. The results generated by both the old and new ID-FDTD scheme maintained very high accuracy. Due to the conductivity, the dispersion error could be slightly larger than in the lossless case by using the old scheme, while the error of the new scheme was as small as in the previous examples.

Ⅳ. Conclusions

In this paper, the weighting factor for 2D ID-FDTD

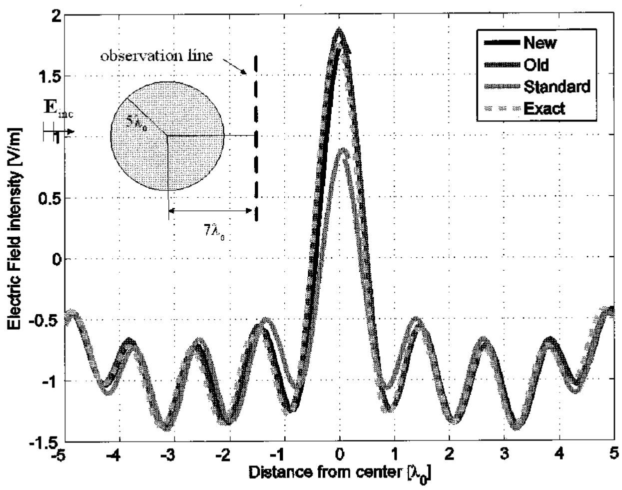


(a) Real part

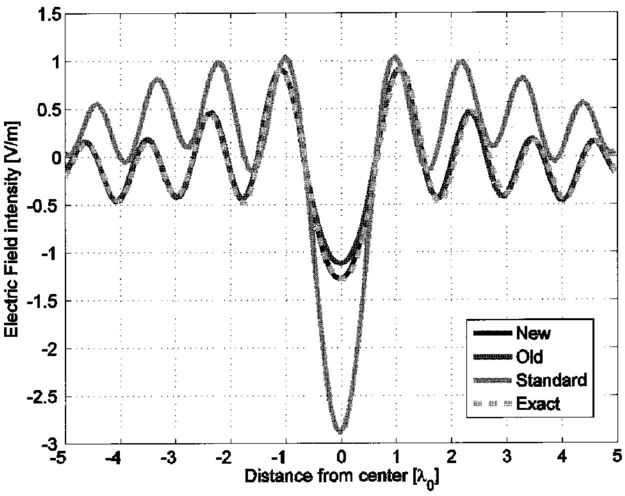


(b) Imaginary part

Fig. 4. Plots of the scattered electric field observed outside a lossless cylinder. The radius and permittivity of the dielectric cylinder were $15\lambda_0$ and ε_r=2.0, respectively. S=0.7, $\Delta = \lambda_0/20$ and frequency f=300 MHz were assumed.



(a) Real part



(b) Imaginary part

Fig. 5. Plots of the scattered electric field observed outside a lossy cylinder. The radius and permittivity of the dielectric cylinder were $5\lambda_0$ and ε_r= $2.0 - j0.01$, respectively. S=0.7, $\Delta = \lambda_0/20$ and frequency f=300 MHz were assumed.

was modified and reformulated. The new one was a function of permittivity, conductivity, cell size, time step, etc. By applying the Least Mean Square method, it fluctuated minimally over the propagation angles. Three scattering examples for lossless and lossy dielectric cylinders were given to demonstrate the validity of the new scheme. For all cases, the new weighting factor showed better isotropic dispersion and excellent accuracy for the scattering problem, as expected.

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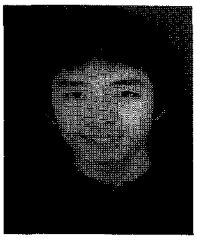

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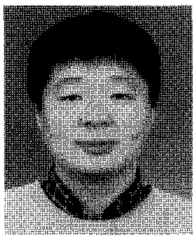

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