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# On Some Results for Five Mappings using Compatibility of Type($\alpha$) in Intuitionistic Fuzzy Metric Space

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**Abstract**

The object of this paper is to introduce the notion of compatible mapping of type($\alpha$) in intuitionistic fuzzy metric space, and to establish common fixed point theorem for five mappings in intuitionistic fuzzy metric space. Our research are an extension for the results of [1] and [7].

**Key words** : Compatible of type($\alpha$), fixed point, t-norm, t-conorm, intuitionistic fuzzy metric space.

## 1. Introduction

Grabiec[1] obtained the Banach contraction theorem in setting of fuzzy metric spaces introduced by Kramosil and Michalek[3]. Also, Park and Kim[8] proved a fixed point theorem in a fuzzy metric space.

Recently, Park et.al.[11] defined the intuitionistic fuzzy metric space in which it is a little revised in Park[4], and Park et.al.[6] proved a fixed point theorem of Banach for the contractive mapping of a complete intuitionistic fuzzy metric space. Also, Park et. al.[7] obtained a fixed point in $M$-fuzzy metric spaces.

The object of the paper is to introduce the notion of compatible mapping and compatible mapping of type($\alpha$) in intuitionistic fuzzy metric space, and to establish common fixed point theorem for five mappings in this space. These results have been used to obtain generalization of Grabiec's contraction principle. Our research are an extension for the results of [1] and [7].

## 2. Preliminaries

We give some definitions, properties of the intuitionistic fuzzy metric space as following :

**Definition 2.1.** ([12]) A operation $* : [0,1]\times[0,1] \to [0,1]$ is continuous $t$−norm if $*$ is satisfying the following conditions:

(a)$*$ is commutative and associative,
(b)$*$ is continuous,
(c)$a * 1 = a$ for all $a \in [0,1]$,
(d)$a * b \le c * d$ whenever $a \le c$ and $b \le d$ $(a,b,c,d \in [0,1])$.

**Definition 2.2.** ([12])A operation $\diamond : [0,1]\times[0,1] \to [0,1]$ is continuous $t$−conorm if $\diamond$ is satisfying the following conditions:

(a)$\diamond$ is commutative and associative,
(b)$\diamond$ is continuous,
(c)$a \diamond 0 = a$ for all $a \in [0,1]$,
(d)$a \diamond b \ge c \diamond d$ whenever $a \le c$ and $b \le d$ $(a,b,c,d \in [0,1])$.

**Definition 2.3.** ([5])The 5−tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuous $t$−norm, $\diamond$ is a continuous $t$−conorm and $M, N$ are fuzzy sets on $X^2 \times (0,\infty)$ satisfying the following conditions; for all $x, y, z \in X$, such that

(a)$M(x,y,t) > 0$,
(b)$M(x,y,t) = 1 \Longleftrightarrow x = y$,
(c)$M(x,y,t) = M(y,x,t)$,
(d)$M(x,y,t) * M(y,z,s) \le M(x,z,t+s)$,
(e)$M(x,y,\cdot) : (0,\infty) \to (0,1]$ is continuous,
(f)$N(x,y,t) > 0$,
(g)$N(x,y,t) = 0 \Longleftrightarrow x = y$,

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(h)$N(x,y,t) = N(y,x,t)$,
(i)$N(x,y,t) \diamond N(y,z,s) \geq N(x,z,t+s)$,
(j)$N(x,y,\cdot) : (0,\infty) \to (0,1]$ is continuous.

Note that $(M,N)$ is called an intuitionistic fuzzy metric on $X$. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between $x$ and $y$ with respect to $t$, respectively.

**Example 2.4.** ([10])Let $(X,d)$ be a metric space. Denote $a * b = ab$ and $a \diamond b = \min\{1, a+b\}$ for all $a,b \in [0,1]$ and let $M_d, N_d$ be fuzzy sets on $X^2 \times (0,\infty)$ defined as follows :

$$M_d(x,y,t) = \frac{kt^n}{kt^n + md(x,y)},$$
$$N_d(x,y,t) = \frac{d(x,y)}{kt^n + md(x,y)}$$

for $k,m,n \in R^+ (m \geq 1)$. Then $(X, M_d, N_d, *, \diamond)$ is an intuitionistic fuzzy metric space. It is called the intuitionistic fuzzy metric space induced by the metric $d$.

**Definition 2.5.** ([10]) Let $X$ be an intuitionistic fuzzy metric space.

(a) $\{x_n\}$ is said to be convergent to a point $x \in X$ by $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} M(x_n,x,t) = 1$, $\lim_{n\to\infty} N(x_n,x,t) = 0$ for all $t > 0$.

(b) $\{x_n\}$ is called a Cauchy sequence if

$$\lim_{n\to\infty} M(x_{n+p},x_n,t) = 1, \quad \lim_{n\to\infty} N(x_{n+p},x_n,t) = 0$$

for all $t > 0$ and $p > 0$.

(c) $X$ is complete if every Cauchy sequence converges in $X$.

In this paper, $X$ is considered to be the intuitionistic fuzzy metric space with the following condition:

$$\lim_{t\to\infty} M(x,y,t) = 1, \quad \lim_{t\to\infty} N(x,y,t) = 0 \qquad (1)$$

for all $x,y \in X$ and $t > 0$.

**Lemma 2.6.** ([6])Let $\{x_n\}$ be a sequence in an intuitionistic fuzzy metric space $X$ with the condition (1). If there exist a number $k \in (0,1)$ such that for all $x,y \in X$ and $t > 0$,

$$M(x_{n+2},x_{n+1},kt) \geq M(x_{n+1},x_n,t),$$
$$N(x_{n+2},x_{n+1},kt) \leq N(x_{n+1},x_n,t) \qquad (2)$$

for all $t > 0$ and $n = 1,2,\cdots$ , then $\{x_n\}$ is a Cauchy sequence in $X$.

*Proof.* From the simple induction with (2), we have

$$M(x_{n+2},x_{n+1},t) \geq M(x_2,x_1,\frac{t}{k^n}),$$
$$N(x_{n+2},x_{n+1},t) \leq N(x_2,x_1,\frac{t}{k^n}) \qquad (3)$$

for all $t > 0$ and $n = 1,2,\cdots$.

Hence, by (1) and (3), we have

$$M(x_n,x_{n+p},t)$$
$$\geq M(x_1,x_2,\frac{t}{pk^{n-1}}) * \cdots\cdots * M(x_1,x_2,\frac{t}{pk^{n+p-2}})$$
$$N(x_n,x_{n+p},t)$$
$$\leq N(x_1,x_2,\frac{t}{pk^{n-1}}) \diamond \cdots\cdots \diamond N(x_1,x_2,\frac{t}{pk^{n+p-2}}).$$

Therefore, from (1), we have

$$\lim_{n\to\infty} M(x_n,x_{n+p},t) \geq 1 * 1 * \cdots\cdots * \geq 1$$
$$\lim_{n\to\infty} N(x_n,x_{n+p},t) \leq 0 \diamond 0 \diamond \cdots\cdots \diamond \leq 0,$$

which implies that $\{x_n\}$ is a Cauchy sequence in $X$. □

**Lemma 2.7.** ([8])Let $X$ be an intuitionistic fuzzy metric space. If there exists a number $k \in (0,1)$ such that for all $x,y \in X$ and $t > 0$,

$$M(x,y,kt) \geq M(x,y,t), \ N(x,y,kt) \leq N(x,y,t),$$

then $x = y$.

## 3. Compatible mapping of type($\alpha$)

In this part, we will introduce the concepts of compatible mappings of type($\alpha$) and give some properties of these mappings for our main results.

**Definition 3.1.** ([7])Let $A,B$ be mappings from intuitionistic fuzzy metric space $X$ into itself. The mappings are said to be compatible if

$$\lim_{n\to\infty} M(ABx_n,BAx_n,t) = 1,$$
$$\lim_{n\to\infty} N(ABx_n,BAx_n,t) = 0$$

for all $t > 0$, whenever $\{x_n\} \subset X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$ for some $x \in X$.

**Definition 3.2.** Let $A,B$ be mappings from intuitionistic fuzzy metric space $X$ into itself. The mappings are said to be compatible of type($\alpha$) if

$$\lim_{n\to\infty} M(ABx_n,BBx_n,t) = 1$$
$$\quad \textit{and} \lim_{n\to\infty} M(BAx_n,AAx_n,t) = 1,$$
$$\lim_{n\to\infty} N(ABx_n,BBx_n,t) = 0$$
$$\quad \textit{and} \lim_{n\to\infty} N(BAx_n,AAx_n,t) = 0$$

for all $t > 0$, whenever $\{x_n\} \subset X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$ for some $x \in X$.

**Proposition 3.3.** Let $X$ be an intuitionistic fuzzy metric space and $A, B$ be continuous mappings from $X$ into itself. Then $A$ and $B$ are compatible iff they are compatible of type($\alpha$).

*Proof.* Let $\{x_n\} \subset X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$ for some $x \in X$. Since $A$ is continuous, we have

$$\lim_{n\to\infty} AAx_n = \lim_{n\to\infty} ABx_n = Ax.$$

Also, since $A, B$ are compatible,

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1,$$
$$\lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0$$

for all $t > 0$. From the inequality

$$M(AAx_n, BAx_n, t)$$
$$\geq M(AAx_n, ABx_n, \frac{t}{2}) * M(ABx_n, BAx_n, \frac{t}{2})$$
$$N(AAx_n, BAx_n, t)$$
$$\leq N(AAx_n, ABx_n, \frac{t}{2}) \diamond N(ABx_n, BAx_n, \frac{t}{2}).$$

Therefore

$$\lim_{n\to\infty} M(AAx_n, BAx_n, t) = 1,$$
$$\lim_{n\to\infty} N(AAx_n, BAx_n, t) = 0.$$

Also, we obtain

$$\lim_{n\to\infty} M(BBx_n, ABx_n, t) = 1,$$
$$\lim_{n\to\infty} N(BBx_n, ABx_n, t) = 0.$$

Hence $A, B$ are compatible of type($\alpha$).

Conversely, let $\{x_n\} \subset X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$ for some $x \in X$. Since $B$ is continuous, we have

$$\lim_{n\to\infty} BAx_n = \lim_{n\to\infty} BBx_n = Bx.$$

Also, since $A, B$ are compatible of type($\alpha$), we obtain

$$\lim_{n\to\infty} M(ABx_n, BBx_n, \frac{t}{2})$$
$$= \lim_{n\to\infty} M(BAx_n, AAx_n, \frac{t}{2}) = 1,$$
$$\lim_{n\to\infty} N(ABx_n, BBx_n, \frac{t}{2})$$
$$= \lim_{n\to\infty} N(BAx_n, AAx_n, \frac{t}{2}) = 0$$

for all $t > 0$. From the inequality

$$M(ABx_n, BAx_n, t)$$
$$\geq M(ABx_n, BBx_n, \frac{t}{2}) * M(BBx_n, BAx_n, \frac{t}{2})$$
$$N(ABx_n, BAx_n, t)$$
$$\leq N(ABx_n, BBx_n, \frac{t}{2}) \diamond N(BBx_n, BAx_n, \frac{t}{2}).$$

Therefore

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1,$$
$$\lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0.$$

Hence $A$ and $B$ are compatible. □

**Proposition 3.4.** Let $X$ be an intuitionistic fuzzy metric space and $A, B$ be mappings from $X$ into itself. If $A, B$ are compatible of type($\alpha$) and $Ax = Bx$ for some $x \in X$, then $ABx = BBx = BAx = AAx$.

*Proof.* Let $\{x_n\} \subset X$ defined by $\lim_{n\to\infty} x_n = x$ for some $x \in X$ and $n = 1, 2, \cdots$ and $Ax = Bx$. Then we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = Ax = Bx.$$

Since $A, B$ are compatible of type($\omega$), we obtain

$$M(ABx, BBx, t) = \lim_{n\to\infty} M(ABx_n, BBx_n, t) = 1,$$
$$N(ABx, BBx, t) = \lim_{n\to\infty} N(ABx_n, BBx_n, t) = 0.$$

Therefore $ABx = BBx$.

Similarly, we have $BAx = AAx$. Since $Ax = Bx$, $BBx = BAx$. Hence $ABx = BBx = BAx = AAx$. □

**Proposition 3.5.** Let $X$ be an intuitionistic fuzzy metric space and $A, B$ be mappings from $X$ into itself. If $A, B$ are compatible of type($\alpha$) and $\{x_n\} \subset X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$ for some $x \in X$, then

(a)$\lim_{n\to\infty} BAx_n = Ax$ if $A$ is continuous at $x \in X$,

(b)$ABx = BAx$ and $Ax = Bx$ if $A$ and $B$ are continuous at $x \in X$.

*Proof.* (a)Since $A$ is continuous at $x$ and $\lim_{n\to\infty} Ax_n = x$, $\lim_{n\to\infty} AAx_n = Ax$. Since $A, B$ are compatible of type($\alpha$), we have

$$\lim_{n\to\infty} M(BAx_n, AAx_n, t) = 1,$$
$$\lim_{n\to\infty} N(BAx_n, AAx_n, t) = 0$$

for all $t > 0$. From (d) of Definition 2.3,

$$\lim_{n\to\infty} M(BAx_n, Ax, t)$$
$$\geq \lim_{n\to\infty} M(BAx_n, AAx_n, \frac{t}{2})$$
$$* \lim_{n\to\infty} M(AAx_n, Ax, \frac{t}{2}) \geq 1$$
$$\lim_{n\to\infty} N(BAx_n, Ax, t)$$
$$\leq \lim_{n\to\infty} N(BAx_n, AAx_n, \frac{t}{2})$$
$$\diamond \lim_{n\to\infty} N(AAx_n, Ax, \frac{t}{2}) \leq 0.$$

Hence $\lim_{n\to\infty} BAx_n = Ax$.

(b)Since $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$ and $A, B$ are continuous at $x \in X$, we have, by (a), $\lim_{n\to\infty} ABx_n = Ax$ and $\lim_{n\to\infty} BAx_n = Bx$.

Thus, from the uniqueness of the limit, $Ax = Bx$. By above Proposition 3.4, $ABx = BAx$. □

## 4. Some results for five mappings using compatibility of type($\alpha$)

Now, we will prove some fixed point theorems for five mappings satisfying some conditions.

**Theorem 4.1.** Let $X$ be a complete intuitionistic fuzzy metric space with $t*t \geq t$, $t\diamond t \leq t$ for all $t \in [0,1]$ and satisfy the condition (1). Let $A, B, S, T$ and $P$ be mappings from $X$ into itself such that

(a)$P(X) \subset AB(X)$, $P(X) \subset ST(X)$,

(b)There exist $k \in (0,1)$ such that for all $x, y \in X$, $\beta \in (0,2)$ and $t > 0$,

$$\begin{aligned}
&M(Px, Py, kt)\\
&\geq M(ABx, Px, t) * M(STy, Py, t) *\\
&\quad *M(STy, Px, \beta t) * M(ABx, Py, (2-\beta)t)\\
&\quad *M(ABx, STy, t),\\
&N(Px, Py, kt)\\
&\leq N(ABx, Px, t) \diamond N(STy, Py, t) \diamond\\
&\quad \diamond N(STy, Px, \beta t) \diamond N(ABx, Py, (2-\beta)t)\\
&\quad \diamond N(ABx, STy, t),
\end{aligned}$$

(c)$PB = BP$, $PT = TP$, $AB = BA$ and $ST = TS$,

(d)$A$ and $B$ are continuous,

(e)$P$ and $AB$ are compatible of type($\alpha$),

(f)$M(x, STx, t) \geq M(x, ABx, t)$, $N(x, STx, t) \leq N(x, ABx, t)$ for all $x \in X$ and $t > 0$.

Then $A, B, S, T$ and $P$ have a common fixed point in $X$.

*Proof.* Because of $P(X) \subset AB(X)$, for fixed $x_0 \in X$, we can choose a point $x_1 \in X$ such that $Px_0 = ABx_1$. Also, since $P(X) \subset ST(X)$, we can choose $x_2 \in X$ for this point $x_1$ such that $Px_1 = STx_2$. Inductively construct sequence $\{y_n\} \subset X$ such that $y_{2n} = Px_{2n} = ABx_{2n+1}$, $y_{2n+1} = Px_{2n+1} = STx_{2n+2}$ for $n = 1, 2, \cdots$. By (b), we have

$$\begin{aligned}
&M(y_{2n+1}, y_{2n+2}, kt) = M(Px_{2n+1}, Px_{2n+2}, kt)\\
&\geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t)\\
&\quad *M(y_{2n+1}, y_{2n+1}, t) * M(y_{2n}, y_{2n+2}, (1+q)t)\\
&\quad *M(y_{2n}, y_{2n+1}, t)\\
&\geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t)\\
&\quad *M(y_{2n}, y_{2n+1}, qt),\\
&N(y_{2n+1}, y_{2n+2}, kt) = N(Px_{2n+1}, Px_{2n+2}, kt)\\
&\leq N(y_{2n}, y_{2n+1}, t) \diamond N(y_{2n+1}, y_{2n+2}, t)\\
&\quad \diamond N(y_{2n+1}, y_{2n+1}, t) \diamond N(y_{2n}, y_{2n+2}, (1+q)t)\\
&\quad \diamond N(y_{2n}, y_{2n+1}, t)\\
&\leq N(y_{2n}, y_{2n+1}, t) \diamond N(y_{2n+1}, y_{2n+2}, t)\\
&\quad \diamond N(y_{2n}, y_{2n+1}, qt)
\end{aligned}$$

for all $t > 0$ and $\beta = 1 - q$ with $q \in (0,1)$.

Since $*$ is continuous and $M(x, y, \cdot)$, $N(x, y, \cdot)$ are continuous, let $q \to 1$ in above equation, we obtain

$$\begin{aligned}
&M(y_{2n+1}, y_{2n+2}, kt)\\
&\geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t), \qquad (4)\\
&N(y_{2n+1}, y_{2n+2}, kt)\\
&\leq N(y_{2n}, y_{2n+1}, t) \diamond N(y_{2n+1}, y_{2n+2}, t).
\end{aligned}$$

Also, we have

$$\begin{aligned}
&M(y_{2n+2}, y_{2n+3}, kt)\\
&\geq M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+2}, y_{2n+3}, t) \quad (5)\\
&N(y_{2n+2}, y_{2n+3}, kt)\\
&\leq N(y_{2n+1}, y_{2n+2}, t) \diamond N(y_{2n+2}, y_{2n+3}, t).
\end{aligned}$$

From (4) and (5),

$$\begin{aligned}
&M(y_{2n+1}, y_{2n+2}, kt)\\
&\geq M(y_n, y_{n+1}, t) * M(y_{n+1}, y_{n+2}, t),\\
&N(y_{2n+1}, y_{2n+2}, kt)\\
&\leq N(y_n, y_{n+1}, t) \diamond N(y_{n+1}, y_{n+2}, t)
\end{aligned}$$

for $n = 1, 2, \cdots$.

Then for positive integers $n$ and $p$,

$$\begin{aligned}
&M(y_{n+1}, y_{n+2}, kt)\\
&\geq M(y_n, y_{n+1}, t) * M(y_{n+1}, y_{n+2}, \frac{t}{k^p})\\
&N(y_{n+1}, y_{n+2}, kt)\\
&\leq N(y_n, y_{n+1}, t) \diamond N(y_{n+1}, y_{n+2}, \frac{t}{k^p}).
\end{aligned}$$

Hence, since $\lim_{n\to\infty} M(y_{n+1}, y_{n+2}, kt) = 1$, $\lim_{n\to\infty} N(y_{n+1}, y_{n+2}, kt) = 0$, we have

$$\begin{aligned}
&M(y_{n+1}, y_{n+2}, kt) \geq M(y_n, y_{n+1}, t),\\
&N(y_{n+1}, y_{n+2}, kt) \leq N(y_n, y_{n+1}, t).
\end{aligned}$$

By Lemma 2.6, $\{y_n\}$ is a Cauchy sequence in $X$ and since $X$ is complete, $\{y_n\}$ converges to a point $x \in X$. Since $\{Px_n\}$, $\{ABx_{2n+1}\}$, $\{STx_{2n+2}\}$ are subsequences of $\{y_n\}$, $\lim_{n\to\infty} Px_n = x = \lim_{n\to\infty} ABx_{2n+1} = \lim_{n\to\infty} STx_{2n+2}$. Also, since $A, B$ are continuous and $P, AB$ are compatible of type($\alpha$), by Proposition 3.5(a), we have $\lim_{n\to\infty} PABx_{2n+1} = ABx$ and $\lim_{n\to\infty}(AB)^2x_{2n+1} = ABx$. By (b) with $\beta = 1$, we obtain

$$\begin{aligned}
&M(PABx_{2n+1}, Px_{2n+2}, kt)\\
&\geq M((AB)^2x_{2n+1}, PABx_{2n+1}, t)\\
&\quad *M(STx_{2n+2}, Px_{2n+2}, t)\\
&\quad *M(STx_{2n+2}, PABx_{2n+1}, t)\\
&\quad *M((AB)^2x_{2n+1}, Px_{2n+2}, t)\\
&\quad *M((AB)^2x_{2n+1}, STx_{2n+2}, t),\\
&N(PABx_{2n+1}, Px_{2n+2}, kt)\\
&\leq N((AB)^2x_{2n+1}, PABx_{2n+1}, t)\\
&\quad \diamond N(STx_{2n+2}, Px_{2n+2}, t)\\
&\quad \diamond N(STx_{2n+2}, PABx_{2n+1}, t)\\
&\quad \diamond N((AB)^2x_{2n+1}, Px_{2n+2}, t)\\
&\quad \diamond N((AB)^2x_{2n+1}, STx_{2n+2}, t)
\end{aligned}$$

which implies that

$$\begin{aligned}
&M(ABx, x, kt)\\
&= \lim_{n\to\infty} M(PABx_{2n+1}, Px_{2n+2}, kt)\\
&\geq 1 * 1 * M(x, ABx, t) * M(ABx, x, t)\\
&\quad *M(ABx, x, t)\\
&\geq M(ABx, x, t),\\
&N(ABx, x, kt)\\
&= \lim_{n\to\infty} N(PABx_{2n+1}, Px_{2n+2}, kt)\\
&\leq 0 \diamond 0 \diamond N(x, ABx, t) \diamond N(ABx, x, t)\\
&\quad \diamond N(ABx, x, t)\\
&\leq N(ABx, x, t).
\end{aligned}$$

Hence, by Lemma 2.7, $ABx = x$.

Also, by (f), since $M(x, STx, t) \geq M(x, ABx, t) = 1$ and $N(x, STx, t) \leq N(x, ABx, t) = 0$ for all $t > 0$, we get $STx = x$.

By (b) with $\beta = 1$, we have

$$\begin{aligned}
&M(PABx, Px, kt)\\
&\geq M((AB)^2x_{2n+1}, PABx_{2n+1}, t) * M(STx, Px, t)\\
&\quad *M(STx, PABx_{2n+1}, t) * M((AB)^2x_{2n+1}, Px, t)\\
&\quad *M((AB)^2x_{2n+1}, STx, t),\\
&N(PABx, Px, kt)\\
&\leq N((AB)^2x_{2n+1}, PABx_{2n+1}, t) \diamond N(STx, Px, t)\\
&\quad \diamond N(STx, PABx_{2n+1}, t) \diamond N((AB)^2x_{2n+1}, Px, t)\\
&\quad \diamond N((AB)^2x_{2n+1}, STx, t).
\end{aligned}$$

Thus

$$\begin{aligned}
&M(ABx, Px, kt)\\
&= \lim_{n\to\infty} M(PABx_{2n+1}, Px, kt)\\
&\geq 1 * 1 * 1 * M(ABx, Px, t) * 1\\
&\geq M(ABx, Px, t),\\
&N(ABx, Px, kt)\\
&= \lim_{n\to\infty} N(PABx_{2n+1}, Px, kt)\\
&\leq 0 \diamond 0 \diamond 0 \diamond N(ABx, Px, t) \diamond 0\\
&\leq N(ABx, Px, t).
\end{aligned}$$

Therefore by Lemma 2.7, $ABx = Px = x$.

Now we will show that $Bx = x$. By (b) with $\beta = 1$ and (c), we obtain

$$\begin{aligned}
&M(Bx, x, kt)\\
&= M(BPx, Px, kt)\\
&= M(PBx, Px, kt)\\
&\geq M(ABBx, PBx, t) * M(STx, Px, t)\\
&\quad *M(STx, PBx, t) * M(ABBx, Px, t)\\
&\quad *M(ABBx, STx, t)\\
&= 1 * 1 * M(x, Bx, t) * M(Bx, x, t) * M(Bx, x, t)\\
&\geq M(Bx, x, t),\\
&N(Bx, x, kt)\\
&= N(BPx, Px, kt)\\
&= N(PBx, Px, kt)\\
&\leq N(ABBx, PBx, t) \diamond N(STx, Px, t)\\
&\quad \diamond N(STx, PBx, t) \diamond N(ABBx, Px, t)\\
&\quad \diamond N(ABBx, STx, t)\\
&= 0 \diamond 0 \diamond N(x, Bx, t) \diamond N(Bx, x, t) \diamond N(Bx, x, t)\\
&\leq N(Bx, x, t)
\end{aligned}$$

which implies that $Bx = x$. Since $ABx = x$, hence $Ax = x$. Now, we will prove that $Tx = x$. By (b) with $\beta = 1$ and (c), we get

$$\begin{aligned}
&M(Tx, x, kt)\\
&= M(TPx, Px, kt)\\
&= M(Px, TPx, kt)\\
&= 1 * 1 * M(Tx, x, t) * M(x, Tx, t)\\
&\quad *M(x, Tx, t)\\
&\geq M(Tx, x, t),\\
&N(Tx, x, kt)\\
&= N(TPx, Px, kt)\\
&= N(Px, TPx, kt)\\
&= 0 \diamond 0 \diamond N(Tx, x, t) \diamond N(x, Tx, t)\\
&\quad \diamond N(x, Tx, t)\\
&\leq N(Tx, x, t),
\end{aligned}$$

which implies that $Tx = x$. Since $STx = x$, we have $Sx = STx = x$. Therefore, we obtain $Ax = Bx = Sx = Tx = Px = x$, that is, $x$ is common fixed point of $A, B, S, T$ and $P$.

Finally, the uniqueness of the fixed point of $A, B, S, T$ and $P$ follows easily from (b). Hence $x$ is unique common fixed point of the five mappings $A, B, S, T$ and $P$. □

**Corollary 4.2.** Let $X$ be a complete intuitionistic fuzzy metric space with $t * t \geq t$, $t \diamond t \leq t$ for all $t \in [0, 1]$ and satisfy the condition (1). Let $A, B$ and $P$ be mappings from $X$ into itself such that

(g)$P(X) \subset A(X)$, $P(X) \subset S(X)$,

(h)There exist $k \in (0, 1)$ such that for all $x, y \in X$, $\beta \in (0, 2)$ and $t > 0$,

$$\begin{aligned} &M(Px, Py, kt)\\ &\geq M(Ax, Px, t) * M(Sy, Py, t) * M(Ax, Sy, \beta t) *\\ &\qquad *M(Ax, Py, (2-\beta)t) * M(Sy, Px, t),\\ &N(Px, Py, kt)\\ &\leq N(Ax, Px, t) \diamond N(Sy, Py, t) \diamond N(Ax, Sy, \beta t) \diamond\\ &\qquad \diamond N(Ax, Py, (2-\beta)t) \diamond N(Sy, Px, t) \end{aligned}$$

(i)$A$ is continuous,

(j)$P$ and $A$ are compatible of type($\alpha$),

(k)$M(x, Sx, t) \geq M(x, Ax, t)$, $N(x, Sx, t) \leq N(x, Ax, t)$ for all $x \in X$ and $t > 0$.

Then $A, S$ and $P$ have a common fixed point in $X$.

*Proof.* Let $I_X$ be the identity mapping on $X$. Then the proof follows from Theorem 4.1 with $B = T = I_X$ □

**Corollary 4.3.** Let $X$ be a complete intuitionistic fuzzy metric space with $t * t \geq t$, $t \diamond t \leq t$ for all $t \in [0, 1]$ and satisfy the condition (1). Let $P$ be a mapping from $X$ into itself such that

(h)There exist $k \in (0, 1)$ such that for all $x, y \in X$, $\beta \in (0, 2)$ and $t > 0$,

$$\begin{aligned} &M(Px, Py, kt)\\ &\geq M(x, Px, t) * M(y, Py, t) * M(y, Px, \beta t) *\\ &\qquad *M(x, Py, (2-\beta)t) * M(x, y, t),\\ &N(Px, Py, kt)\\ &\leq N(x, Px, t) \diamond N(y, Py, t) \diamond N(y, Px, \beta t) \diamond\\ &\qquad \diamond N(x, Py, (2-\beta)t) \diamond N(x, y, t) \end{aligned}$$

for all $x, y \in X$, $\beta \in (0, 2)$ and $t > 0$. Then $A, S$ and $P$ have a common fixed point in $X$.

*Proof.* The proof follows from Theorem 4.1 with $A = B = S = T = I_X$ □

**Corollary 4.4.** (Extension of Banach contraction theorem)Let $X$ be a complete intuitionistic fuzzy metric space with $t * t \geq t$, $t \diamond t \leq t$ for all $t \in [0, 1]$ and satisfy the condition (1). Let $P$ be a mapping from $X$ into itself such that there exist $k \in (0, 1)$ such that

$$M(Px, Py, kt) \geq M(x, y, t), \quad N(Px, Py, kt) \leq N(x, y, t)$$

for all $x, y \in X$ and $t > 0$. Then $P$ has a fixed point in $X$.

## References


[1] Grabiec, M., 1988. Fixed point in fuzzy metric spaces. Fuzzy Sets and Systems 27, 385–389.

[2] George, A., Veeramani, P., 1994. On some results in fuzzy metric spaces. Fuzzy Sets and Systems 64, 395–399.

[3] Kramosil,J., Michalek J., 1975. Fuzzy metric and statistical metric spaces. Kybernetica 11, 326–334.

[4] Park, J.H., 2004. Intuitionistic fuzzy metric spaces. Chaos Solitons & Fractals 22(5), 1039–1046.

[5] Park, J.H., Park, J.S., Kwun, Y.C., 2006. A common fixed point theorem in the intuitionistic fuzzy metric space. Advances in Natural Comput. Data Mining(Proc. 2nd ICNC and 3rd FSKD), 293–300.

[6] Park, J.H., Park, J.S., Kwun, Y.C., 2007. Fixed point theorems in intuitionistic fuzzy metric space(I). JP J. fixed point Theory & Appl. 2(1), 79–89.

[7] Park, J.H., Park, J.S., Kwun, Y.C., 2007. Fixed points $M$-fuzzy metric spaces. Advanced in soft computing. 40, 206–215.

[8] Park, J.S., Kim, S.Y., 1999. A fixed point theorem in a fuzzy metric space. F.J.M.S. 1(6), 927–934.

[9] Park J.S., Kang H.J., 2007. Common fixed point theorem for a sequence of mappings in intuitionistic fuzzy metric space. Internat. J. KFIS. 7(1), 30–33.

[10] Park, J.S., Kwun, Y.C., 2007. Some fixed point theorems in the intuitionistic fuzzy metric spaces. F.J.M.S. 24(2) 227–239.

[11] Park, J.S., Kwun, Y.C., Park, J.H., 2005. A fixed point theorem in the intuitionistic fuzzy metric spaces. F.J.M.S. 16(2), 137–149.

[12] Schweizer, B., Sklar, A., 1960. Statistical metric spaces. Pacific J. Math. 10, 314–334.

[13] Vasuki, R., 1998. A common fixed point theorem in a fuzzy metric space. Fuzzy Sets and Systems 97, 395–397.

[14] Zadeh, L.A., 1965. Fuzzy sets. Inform. and Control 8, 338–353.

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