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Intuitionistic Fuzzy G-Equivalence Relations and G-Congruences on a Groupoid

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Abstract

We investigate the images and preimages of intuitionistic fuzzy G-equivalence relations and G-congruences under product mappings.

Keywords and phrases : intuitionistic [resp. (λ, μ)- and G-]equivalence relation, product mapping.

0. Intro duction

As a generalization of fuzzy sets defined by Zadeh[15], the notion of intuitionistic fuzzy sets was introduced by Atanassov[1]. Recently, Çoker[4], Hur et al. [8], and Lee and Lee applied intuitionistic fuzzy sets to topology. Also, Banerjee and Basnet[2], and Hur et al.[6,7,10] applied to group theory using intuitionistic fuzzy sets. In particular, Hur et al.[9] applied intuitionistic fuzzy sets to topological group. In 1996, Bustince and Burillo[3] introduced the concept of intuitionistic fuzzy relations and studied some of its properties. In 2003, Deschrijver and Kerre[5] investigated some properties of the composition of intuitionistic fuzzy relations. Furthermore, Hur et al.[11-13] studied various properties of intuitionistic fuzzy relations and intuitionistic fuzzy congruences.

In this paper, we investigate the images and preimages of intuitionistic fuzzy G-equivalence relations and G-congruences under product mappings.

1. Preliminaries

In this section, we will list some concepts and results needed in the later sections.

For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

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Throughout this paper, we will denote the unit interval as I and $X, Y, Z, \cdots$ denote (ordinary) sets.

Definition 1.1[1,4]. Let X be a set. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called a *intuitionistic fuzzy set* (in short, *IFS*) in X if $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, where the mappings $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x \in X$ to A, respectively. In particular, $0_\sim$ and $1_\sim$ denote the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in X defined by $0_\sim(x) = (0, 1)$ and $1_\sim(x) = (1, 0)$ for each $x \in X$, respectively.

We will denote the set of all the IFS_S in X as IFS(X).

Definition 1.2[1]. Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFS_S on X. Then

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
(2) $A = B$ iff $A \subset B$ and $B \subset A$.
(3) $A^c = (\nu_A, \mu_A)$.
(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.
(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

Definition 1.3[4]. Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFS_S in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then

(a) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$.
(b) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

Result 1.A[1, Corollary 2.8]. Let A, B, C, D be IFS_S in X. Then
(1) $A \subset B$ and $C \subset D \Rightarrow A \cup C \subset B \cup D$ and $A \cap C \subset B \cap D$.
(2) $A \subset B$ and $A \subset C \Rightarrow A \subset B \cap C$.
(3) $A \subset B$ and $B \subset C \Rightarrow A \cup B \subset C$.
(4) $A \subset B$ and $B \subset C \Rightarrow A \subset C$.
(5) $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$.
(6) $A \subset B \Rightarrow B^c \subset A^c$.
(7) $(A^c)^c = A$.
(8) $1_\sim{}^c = 0$, $0_\sim{}^c = 1_\sim$.

Definition 1.4[3]. Let X and Y be sets.
(1) $R = (\mu_R, \nu_R)$ is called an *intuitionistic fuzzy relation* from X to Y if $R \in \text{IFS}(X \times Y)$.
(2) $R = (\mu_R, \nu_R)$ is called an *intuitionistic fuzzy relation* (in short, *IFR*) on X if $R \in \text{IFS}(X \times X)$.

We will denote the set of all intuitionistic fuzzy relations on X as IFR(X).

Definition 1.5[5]. Let R be an intuitionistic fuzzy relation from X to Y and let H be an intuitionistic fuzzy relation from Y to Z. Then the *composition* $H \circ R$ of R and H is an intuitionistic fuzzy relation in $X \times Z$ defined as follows: For each $(x, z) \in X \times Z$,

$$\mu_{H \circ R}(x, z) = \bigvee_{y \in Y}[\mu_R(x, y) \wedge \mu_H(y, z)]$$

and

$$\nu_{H \circ R}(x, z) = \bigwedge_{y \in Y}[\nu_R(x, y) \vee \nu_H(y, z)].$$

Definition 1.6[3,5]. Let R be an intuitionistic fuzzy relation on a set X.
(1) R is said to be intuitionistic fuzzy *reflexive* if for each $x \in X$, $\mu_R(x, x) = 1$ and $\nu_R(x, x) = 0$.
(2) R is said to be intuitionistic fuzzy *symmetric* if for each $(x, y) \in X \times X$, $\mu_R(x, y) = \mu_R(y, x)$ and $\nu_R(x, y) = \nu_R(y, x)$.
(3) R is said to be intuitionistic fuzzy *transitive* if $R \circ R \subset R$.
(4) R is called an *intuitionistic fuzzy equivalence relation* on X (in short, *IFER*) if it is reflexive, symmetric and transitive.

We will denote the set of all intuitionistic fuzzy equivalence relations on X as IFE(X).

Result 1.B[13, Proposition 2.2(2)]. If R is an intuitionistic fuzzy G-preorder on a set X, then $R \circ R = R$.

Definition 1.7. Let R be an IFR on a set X and let $(\lambda, \mu) \in (0, 1] \times [0, 1)$. Then R is said to be:
(1) (λ, μ)-*reflexive* if $R(a, a) = (\lambda, \mu)$, and $\mu_R(a, b) \leq \lambda$ and $\nu_R(a, b) \geq \mu$ for any $a, b \in X$.
(2) an *intuitionistic fuzzy* (λ, μ)-*equivalence relation* on X if it is (λ, μ)-reflexive, intuitionistic fuzzy symmetric and transitive.

We will denote the set of all intuitionistic fuzzy (λ, μ)-equivalence relations on X as $\text{IFE}_{(\lambda,\mu)}(X)$.

Definition 1.8[13]. Let R be an intuitionistic fuzzy relation on a set X. Then R is said to be *G-reflexive* if for any $x, y \in X$ with $x \neq y$
(i) $\mu_R(x, x) > 0$ and $\nu_R(x, x) < 1$,
(ii) $\mu_R(x, y) \leq \delta_1(R)$ and $\nu_R(x, y) \geq \delta_2(R)$, where $\delta_1(R) = \bigwedge_{t \in X} \mu_R(t, t)$ and $\delta_2(R) = \bigvee_{t \in X} \nu_R(t, t)$.
An intuitionistic fuzzy G-reflexive and transitive relation on X is called an *intuitionistic fuzzy G-preorder* on S. An intuitionistic fuzzy symmetric G-preorder on X is called an *intuitionistic fuzzy G-equivalence relation* on X. We will denote the set of all intuitionistic fuzzy G-equivalence relations on X as $IFE_G(X)$.

It is clear that if $(\lambda, \mu) = 1_\sim$, then intuitionistic fuzzy (λ, μ)-equivalence relation is an IFER and every intuitionistic fuzzy (λ, μ)-equivalence relation is a G-equivalence relation.

2. Images and preimages of intuitionistic fuzzy equivalence relations and congruences on a groupoid

Definition 2.1[9]. Let R be an intuitionistic fuzzy relation on a semigroup S.
(1) R is said to be *intuitionistic fuzzy left* [resp. *right*] *compatible* if $\mu_R(x, y) \leq \mu_R(tx, ty)$ and $\nu_R(x, y) \geq \nu_R(tx, ty)$ [resp. $\mu_R(x, y) \leq \mu_R(xt, yt)$ and $\nu_R(x, y) \geq \nu_R(xt, yt)$] for any $x, y, t \in S$.
(2) R is said to be *intuitionistic fuzzy compatible* if $\mu_R(x, y) \wedge \mu_R(a, b) \leq \mu_R(xa, yb)$ and $\nu_R(x, y) \vee \nu_R(a, b) \geq \nu_R(xa, yb)$ for any $x, y, a, b \in S$.
(3) R is called an *intuitionistic fuzzy left congruence* [rest. *right congruence* and *congruence*] if it satisfies the following conditions:
(i) It is an intuitionistic fuzzy equivalence relation.
(ii) It is left compatible [resp. right compatible and compatible].

It is clear that R is compatible if and only if it is both left and right compatible.

We will denote the set of all ordinary congruences and the set of all intuitionistic fuzzy congruences on a semigroup S as C(S) and IFC(S) respectively.

Definition 2.2[4]. Let X and Y be nonempty sets and let $f : X \to Y$ be a mapping. Let $A = (\mu_A, \nu_A)$ be an IFS in X and $B = (\mu_B, \nu_B)$ be an IFS in Y. Then

(a) the *preimage* of B under f, denoted by $f^{-1}(B)$, is the IFS in X defined by:

$$f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\nu_B)),$$

where $f^{-1}(\mu_B) = \mu_B \circ f$.

(2) the *image* of A under f, denoted by $f(A)$, is the IFS in Y defined by:

$$f(A) = (f(\mu_A), f(\nu_A)),$$

where for each $y \in Y$

$$f(\mu_A)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{if } f^{-1}(y) = \emptyset, \end{cases}$$

and

$$f(\nu_A)(y) = \begin{cases} \bigwedge_{x \in f^{-1}(y)} \nu_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 1 & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

For sets X_0, X_1, Y_0 and Y_1, let $f_0 : X_0 \to Y_0$ and $f_1 : X_1 \to Y_1$ be mappings. Then $g : X_0 \times X_1 \to Y_0 \times Y_1$ is called the *product-mapping* of f_0 and f_1 if $g(x_0, x_1) = (f_0(x_0), f_1(x_1))$ for each $(x_0, x_1) \in X_0 \times X_1$. In this case, we write $g = f_0 \times f_1$.

In particular, for a mapping $f : X \to Y$, we denote the product mapping $f \times f : X \times X \to Y \times Y$ as f^2.

Proposition 2.3. Let R be an intuitionistic fuzzy compatible relation on a groupoid S and let D be a groupoid. If $f : D \to S$ is a groupoid homomorphism, then $f^{-1^2}(R)$ is an intuitionistic fuzzy compatible relation on D, where $f^{-1^2} = (f \times f)^{-1}$.

Proof. It is clear that $f^{-1^2}(R) \in IFR(D)$. Let $a, b, c, d \in D$. Then

$$\begin{aligned} \mu_{f^{-1^2}(R)}(ac, bd) &= \mu_R((f \times f)(ac, bd)) \\ &= \mu_R(f(ac), f(bd)) \\ &= \mu_R(f(a)f(c), f(b)f(d)) \text{ (Since } f \text{ is a groupoid homomorphism)} \\ &\geq \mu_R(f(a), f(b)) \wedge \mu_R(f(c), f(d)) \text{ (Since } R \text{ is intuitionistic fuzzy compatible)} \\ &= \mu_R(f \times f)(a, b) \wedge \mu_R(f \times f)(c, d) \\ &= \mu_{f^{-1^2}(R)}(a, b) \wedge \mu_{f^{-1^2}(R)}(c, d) \end{aligned}$$

and

$$\begin{aligned} \nu_{f^{-1^2}(R)}(ac, bd) &= \nu_R(f(ac), f(bd)) \\ &= \nu_R(f(a)f(c), f(b)f(d)) \\ &\leq \nu_R(f(a), f(b)) \vee \nu_R(f(c), f(d)) \\ &= \nu_{f^{-1^2}(R)}(a, b) \vee \nu_{f^{-1^2}(R)}(c, d). \end{aligned}$$

Hence $f^{-1^2}(R)$ is an intuitionistic fuzzy compatible relation on D. □

Proposition 2.4. Let R be an intuitionistic fuzzy relation on a groupoid D and let S be a groupoid. If $f : D \to S$ is a groupoid homomorphism, then $f^2(R)$ is an intuitionistic fuzzy compatible relation on S.

Proof. It is clear that $f^2(R) \in$ IFR(S). Let $u, v, w, r \in S$.

Case(i) : $f^{-1^2}(u, v) = \emptyset$ or $f^{-1^2}(w, r) = \emptyset$. Then $f^2(R)(u, v) = (0, 1)$ or $f^2(R)(w, r) = (0, 1)$. Thus

$$\mu_{f^2(R)}(uw, vr) \geq 0 = \mu_{f^2(R)}(u, v) \wedge \mu_{f^2(R)}(w, r)$$

and

$$\nu_{f^2(R)}(uw, vr) \leq 1 = \nu_{f^2(R)}(u, v) \vee \nu_{f^2(R)}(w, r).$$

Case(ii) : Suppose $f^{-1^2}(u, v) \neq \emptyset$ and $f^{-1^2}(w, r) \neq \emptyset$. Then clearly, $f^{-1^2}(uw, vr) \neq \emptyset$. Thus

$$\begin{aligned} \mu_{f^2(R)}(uw, vr) &= \bigvee_{(x,x') \in f^{-1^2}(uw,vr)} \mu_R(x, x') \\ &\geq \bigvee_{(ac,bd) \in f^{-1^2}(uw,vr)} \mu_R(ac, bd) \\ &= \bigvee_{(f(ac),f(bd))=(uw,vr)} \mu_R(ac, bd) \\ &= \bigvee_{(f(a)f(c),f(b)f(d))=(uw,vr)} \mu_R(ac, bd) \text{ (Since } f \text{ is a groupoid homo-morphism)} \\ &\geq \bigvee_{(f(a),f(b))=(u,v),(f(c),f(d))=(w,r)} [\mu_R(a, b) \wedge \mu_R(c, d)] \text{ (Since } R \text{ is intuitionistic fuzzy compatible)} \\ &= \{\bigvee_{(a,b) \in f^{-1^2}(u,v)} \mu_R(a, b)\} \wedge \{\bigvee_{(c,d) \in f^{-1^2}(w,r)} \mu_R(c, d)\} \\ &= \mu_{f^2(R)}(u, v) \wedge \mu_{f^2(R)}(w, r) \end{aligned}$$

and

$$\begin{aligned} \nu_{f^2(R)}(uw, vr) &= \bigwedge_{(x,x') \in f^{-1^2}(uw,vr)} \nu_R(x, x') \\ &\leq \bigwedge_{(ac,bd) \in f^{-1^2}(uw,vr)} \nu_R(ac, bd) \\ &= \bigwedge_{(f(ac),f(bd))=(uw,vr)} \nu_R(ac, bd) \\ &= \bigwedge_{(f(a)f(c),f(b)f(d))=(uw,vr)} \nu_R(ac, bd) \\ &\leq \bigwedge_{(f(a),f(b))=(u,v),(f(c),f(d))=(w,r)} [\nu_R(a, b) \vee \nu_R(c, d)] \\ &= \{\bigwedge_{(a,b) \in f^{-1^2}(u,v)} \nu_R(a, b)\} \vee \{\bigwedge_{(c,d) \in f^{-1^2}(w,r)} \nu_R(c, d)\} \\ &= \nu_{f^2(R)}(u, v) \vee \nu_{f^2(R)}(w, r). \end{aligned}$$

Hence $f^2(R)$ is an intuitionistic fuzzy compatible relation on S. □

Proposition 2.5. Let $f : X \to Y$ be a mapping. If R is an intuitionistic fuzzy (λ, μ)-equivalence relation on Y, then $f^{-1^2}(R)$ is an intuitionistic fuzzy (λ, μ)-equivalence relation on X.

Proof. For each $a \in X$, let $f(a) = u$. Then

$$\begin{aligned} f^{-1^2}(R)(a, a) &= (\mu_R(f^2(a, a)), \nu_R(f^2(a, a)) \\ &= (\mu_R(f(a), f(a)), \nu_R(f(a), f(a))) \\ &= (\mu_R(u, u), \nu_R(u, u)) \\ &= (\lambda, \mu). \text{ (Since } R \text{ is } (\lambda, \mu)\text{-reflexive)} \end{aligned}$$

For any $a, b \in X$, let $f(a) = u$ and $f(b) = v$. Since R is (λ, μ)- reflexive,

$$\mu_{f^{-1^2}(R)}(a,b) = \mu_R(f^2(a,b)) = \mu_R(f(a), f(b)) = \mu_R(u,v) \leq \lambda$$

and

$$\nu_{f^{-1^2}(R)}(a,b) = \nu_R(f^2(a,b)) = \nu_R(f(a), f(b)) = \nu_R(u,v) \geq \mu.$$

Thus $f^{-1^2}(R)$ is (λ, μ)-reflexive. Moreover,

$$\begin{aligned} f^{-1^2}(R)(a,b) &= (\mu_R(f^2(a,b)), \nu_R(f^2(a,b))) \\ &= (\mu_R(f(a), f(b)), \nu_R(f(a), f(b))) \\ &= (\mu_R(u,v), \nu_R(u,v))) \\ &= (\mu_R(v,u), \nu_R(v,u))) \text{ (Since } R \text{ is intuitionistic fuzzy symmetric)} \\ &= (\mu_R(f^2(a,b)), \nu_R(f^2(a,b))) \\ &= f^{-1^2}(R)(b,a). \end{aligned}$$

So $f^{-1^2}(R)$ is intuitionistic fuzzy symmetric. On the other hand,

$$\begin{aligned} \mu_{f^{-1^2}(R) \circ f^{-1^2}(R)}(a,b) &= \bigvee_{x \in X}[\mu_{f^{-1^2}(R)}(a,x) \wedge \mu_{f^{-1^2}(R)}(x,b)] \\ &= \bigvee_{x \in X}[\mu_R(f^2((a,x)) \wedge \mu_R(f^2(x,b))] \\ &= \bigvee_{x \in X}[\mu_R(f(a), f(x)) \wedge \mu_R(f(x), f(b))] \\ &= \bigvee_{x \in X}[\mu_R(u, t_x)) \wedge \mu_R(t_x, v))] \\ &\leq \bigvee_{w \in Y}[\mu_R(u,w) \wedge \mu_R(w,v)] \\ &= \mu_{R \circ R}(u,v) \\ &\leq \mu_R(u,v) \text{ (Since } R \text{ is intuitionistic fuzzy transitive)} \\ &= \mu_R(f^2(a,b)) \\ &= \mu_{f^{-1^2}(R)}(a,b) \end{aligned}$$

and

$$\begin{aligned} \nu_{f^{-1^2}(R) \circ f^{-1^2}(R)}(a,b) &= \bigwedge_{x \in X}[\nu_{f^{-1^2}(R)}(a,x) \vee \nu_{f^{-1^2}(R)}(x,b)] \\ &= \bigwedge_{x \in X}[\nu_R(f^2((a,x)) \vee \nu_R(f^2(x,b))] \\ &= \bigwedge_{x \in X}[\nu_R(u, t_x)) \vee \nu_R(t_x, v))] \\ &\geq \bigwedge_{w \in Y}[\nu_R(u,w) \vee \nu_R(w,v)] \\ &= \nu_{R \circ R}(u,v) \\ &\geq \nu_R(u,v) \\ &= \nu_R(f^2(a,b)) \\ &= \nu_{f^{-1^2}(R)}(a,b). \end{aligned}$$

Thus $f^{-1^2}(R)$ is intuitionistic fuzzy transitive. This completes the proof. □

The following is the immediate result of Propositions 2.3 and 2.5.

Corollary 2.5. Let R be an intuitionistic fuzzy (λ, μ)-congruence on a groupoid S. If $f : D \to S$ is a groupoid homomorphism, then $f^{-1^2}(R)$ is an intuitionistic fuzzy (λ, μ)-congruence on D.

Definition 2.6. Let $f : X \to Y$ be a mapping and let $R \in \text{IFR}(X)$. Then R is said to be:

(1) *intuitionistic f-invariant* if $f^2(a,b) = f^2(a_1, b_1)$ implies $R(a,b) = R(a_1, b_1)$.

(2) *weakly intuitionistic f-invariant* if $f^2(a,b) = f^2(a_1, b)$ implies $R(a,b) = R(a_1, b)$.

It is clear that if R is intuitionistic f-invariant, then R is weakly intuitionistic f-invariant, but not conversely.

Example 2.6. Let $X = \{a, b, c, d\}$, $Y = \{x, y\}$ and let $f : X \to Y$ be the mapping defined as follows:

$$f(a) = f(b) = f(c) = x \quad \text{and} \quad f(d) = y.$$

Consider the intuitionistic fuzzy relation on X defined as follows:

R	a	b	c	d
a	(r_1, s_1)	(r_2, s_2)	(r_3, s_3)	(r_4, s_4)
b	(r_2, s_2)	(r_1, s_1)	(r_2, s_2)	(r_4, s_4)
c	(r_2, s_2)	(r_2, s_2)	(r_3, s_3)	(r_4, s_4)
d	(r_3, s_3)	(r_1, s_1)	(r_3, s_3)	(r_1, s_1),

where $r_i, s_i \in I$, $r_i + s_i \leq 1$ and $(r_i, s_i) \neq (r_j, s_j)$ for $i \neq j$. Then we can easily see that R is intuitionistic weakly f-invariant but not intuitionistic f-invariant.

Theorem 2.7. Let $f : X \to Y$ be a mapping. If R is an intuitionistic fuzzy weakly intuitionistic f-invariant symmetric relation on X with $R \circ R = R$, then R is intuitionistic f-invariant.

Proof. For any $a, a_1, b, b_1 \in X$ and any $u, v \in Y$, suppose $f^2(a,b) = f^2(a_1, b_1) = (u,v)$. Let $x \in X$. Then there exists a unique $t_x \in Y$ such that $f^2(x,x) = (t_x, t_x)$, $f^2(a,x) = (u, t_x) = f^2(a_1, x)$ and $f^2(x,b) = (t_x, v) = f^2(x, b_1)$. Since R is weakly intuitionistic f-invariant, $R(a,x) = R(a_1, x)$ and $R(x,b) = R(x, b_1)$. On the other hand,

$$\begin{aligned} R(a,b) &= (R \circ R)(a,b) \\ &= (\bigvee_{x \in X}[\mu_R(a,x) \wedge \mu_R(x,b)], \bigwedge_{x \in X}[\nu_R(a,x) \vee \nu_R(x,b)]) \\ &= (\bigvee_{x \in X}[\mu_R(a_1,x) \wedge \mu_R(x,b_1)], \bigwedge_{x \in X}[\nu_R(a_1,x) \vee \nu_R(x,b_1)]) \\ &= (\mu_{R \circ R}(a_1, b_1), \nu_{R \circ R}(a_1, b_1)) \\ &= R \circ R(a_1, b_1) \\ &= R(a_1, b_1). \end{aligned}$$

Hence R is intuitionistic f-invariant. □

The following is the immediate result of Result 1.B and Theorem 2.7.

Corollary 2.7. Let $f : X \to Y$ be a mapping. If R is an intuitionistic fuzzy (λ,μ)-equivalence (or G-equivalence) relation on X which is weakly intuitionistic f-invariant, then R is intuitionistic f-invariant.

Proposition 2.8. Let $f : X \to Y$ be a surjective mapping. If R is an intuitionistic fuzzy (λ,μ)-equivalence relation on X which is weekly intuitionistic f-invariant, then $f^2(R)$ is an intuitionistic fuzzy (λ,μ)-equivalence relation on Y.

Proof. Let $u \in Y$. Since f is a surjective mapping, there exists $a, a' \in X$ such that $f(a) = u = f(a')$. By Corollary 2.7, R is intuitionistic f-invariant. Then

$$\begin{aligned} f^2(R)(u,u) &= (\textstyle\bigvee_{(x,x')\in f^{-1^2}(u,u)} \mu_R(x,x'), \\ &\quad \textstyle\bigwedge_{(x,x')\in f^{-1^2}(u,u)} \nu_R(x,x')) \\ &= (\textstyle\bigvee_{f^2(x,x')=(u,u)} \mu_R(x,x'), \\ &\quad \textstyle\bigwedge_{f^2(x,x')=(u,u)} \nu_R(x,x')) \\ &= (\textstyle\bigvee_{f^2(x,x')=f^2(a,a)} \mu_R(x,x'), \\ &\quad \textstyle\bigwedge_{f^2(x,x')=f^2(a,a)} \nu_R(x,x')) \\ &= (\mu_R(a,a), \nu_R(a,a)) \\ &= R(a,a) \\ &= (\lambda,\mu). \text{ (Since } R \text{ is } (\lambda,\mu)\text{-reflexive)} \end{aligned}$$

Let $u, v \in Y$. Then there exist $a, b \in X$ such that $f^2(a,b) = (u,v)$ and $f^2(b,a) = (v,u)$. Then

$$\begin{aligned} \mu_{f^2(R)}(u,v) &= \textstyle\bigvee_{(x_1,x_2)\in f^{-1^2}(u,v)} \mu_R(x_1,x_2) \\ &= \textstyle\bigvee_{f^2(x_1,x_2)=(u,v)} \mu_R(x_1,x_2) \\ &= \textstyle\bigvee_{f^2(x_1,x_2)=f^2(a,b)} \mu_R(x_1,x_2) \\ &= \mu_R(a,b) \text{ (Since } R \text{ is intuitionistic } f\text{-invariant)} \\ &\le \lambda \text{ (Since } R \text{ is } (\lambda,\mu)\text{-reflexive)} \end{aligned}$$

and

$$\begin{aligned} \nu_{f^2(R)}(u,v) &= \textstyle\bigwedge_{(x_1,x_2)\in f^{-1^2}(u,v)} \nu_R(x_1,x_2) \\ &= \textstyle\bigwedge_{f^2(x_1,x_2)=(u,v)} \nu_R(x_1,x_2) \\ &= \textstyle\bigwedge_{f^2(x_1,x_2)=f^2(a,b)} \nu_R(x_1,x_2) \\ &= \nu_R(a,b) \\ &\ge \mu. \end{aligned}$$

So $f^2(R)$ is (λ,μ)-reflexive. Moreover, $f^2(R)$ is intuitionistic fuzzy symmetric.

Let $x \in X$. Then there exists a unique $t_x \in Y$ such that $f^2(a,x) = (u,t_x)$, $f^2(x,b) = (t_x,v)$ and $f^2(x,x) = (t_x,t_x)$. Thus

$$\begin{aligned} \mu_{f^2(R)}(u,v) &= \mu_R(a,b) \text{ (Since } R \text{ is intuitionistic } f\text{-invariant)} \\ &\ge \mu_{R\circ R}(a,b) \text{ (Since } R \text{ is intuitionistic fuzzy transitive)} \\ &= \textstyle\bigvee_{x\in X}[\mu_R(a,x) \wedge \mu_R(x,b)] \\ &= \textstyle\bigvee_{x\in X}[\mu_{f^2(R)}(u,t_x) \wedge \mu_{f^2(R)}(t_x,v)] \\ &\quad \text{(Since } R \text{ is intuitionistic } f\text{-invariant)} \\ &= \textstyle\bigvee_{w\in Y}[\mu_{f^2(R)}(u,w) \wedge \mu_{f^2(R)}(w,v)] \\ &= \textstyle\bigvee_{f^2(R)\circ f^2(R)}(u,v) \end{aligned}$$

and

$$\begin{aligned} \nu_{f(R)}(u,v) &= \nu_R(a,b) \\ &\le \nu_{R\circ R}(a,b) \\ &= \textstyle\bigwedge_{x\in X}[\nu_R(a,x) \vee \nu_R(x,b)] \\ &= \textstyle\bigwedge_{x\in X}[\nu_{f^2(R)}(u,t_x) \vee \nu_{f^2(R)}(t_x,v)] \\ &= \textstyle\bigwedge_{w\in Y}[\nu_{f^2(R)}(u,w) \vee \nu_{f^2(R)}(w,v)] \\ &= \textstyle\bigwedge_{f^2(R)\circ f^2(R)}(u,v). \end{aligned}$$

So $f^2(R)$ is intuitionistic fuzzy transitive. This completes the proof. □

The following is the immediate result of Proposition 2.10.

Corollary 2.8. Let $f : X \to Y$ be a bijective mapping. If R is an intuitionistic fuzzy (λ,μ)-equivalence relation on X, then $f^2(R)$ is an intuitionistic fuzzy (λ,μ)-equivalence relation on Y.

Definition 2.9. Let R be an IFR on a groupoid S. Then R is called an *intuitionistic fuzzy* (λ,μ)-*congruence* on S if it is a compatible intuitionistic fuzzy (λ,μ)-equivalence relation.

Combining Propositions 2.4 and 2.9, we have the following result.

Proposition 2.10. Let $f : D \to S$ be a groupoid epimorphism. If R is an intuitionistic fuzzy (λ,μ)-congruence on D which is weakly intuitionistic f-invariant, then $f^2(R)$ is an intuitionistic fuzzy (λ,μ)-congruence on S.

The following is the immediate result of Proposition 2.10.

Corollary 2.10. Let $f : D \to S$ be a groupoid isomorphism. If R is an intuitionistic fuzzy (λ,μ)-congruence on D, then $f^2(R)$ is an intuitionistic fuzzy (λ,μ)-congruence on S.

3. Images and preimages of intuitionistic fuzzy G-equivalence relations and G-congruences on a groupoid

Definition 3.1[10]. Let R be an IFR on a groupoid S. Then R is called an *intuitionistic fuzzy G-congruence* on S if it is a compatible intuitionistic fuzzy G-equivalence relation.

Proposition 3.2. Let $f : X \to Y$ be a surjective mapping. If R is an intuitionistic fuzzy G-equivalence relation on X which is weakly intuitionistic f-invariant, then $f^2(R)$ is an intuitionistic fuzzy G-equivalence relation on Y with $\delta(f^2(R)) = \delta(R)$, where $\delta(R) = (\delta_1(R), \delta_2(R))$.

Proof. Let $u \neq v \in Y$. Since f is a surjective mapping, there exist a', $a \neq b \in X$ such that $f(a) = u = f(a')$ and $f(b) = v$. Then

$$\begin{aligned}\mu_{f^2(R)}(u,u) &= \bigvee_{(x,x')\in f^{-1^2}(u,u)} \mu_R(x,x')\\ &= \bigvee_{f^2(x,x')=f(a,a)} \mu_R(x,x')\\ &= \mu_R(a,a) \text{ (Since } R \text{ is intuitionistic } f\text{-invariant by Corollary 2.7)}\\ &> 0 \text{ (Since } R \text{ is } G\text{-reflexive)}\end{aligned}$$

and

$$\begin{aligned}\nu_{f^2(R)}(u,u) &= \bigwedge_{(x,x')\in f^{-1^2}(u,u)} \nu_R(x,x')\\ &= \bigwedge_{f^2(x,x')=f(a,a)} \nu_R(x,x')\\ &\le \nu_R(a,a)\\ &< 1.\end{aligned}$$

Also,

$$\begin{aligned}\mu_{f^2(R)}(u,v) &= \bigvee_{(x,x')\in f^{-1^2}(u,v)} \mu_R(x,x')\\ &= \bigvee_{f^2(x,x')=f(u,v)} \mu_R(x,x')\\ &= \mu_R(a,b) \text{(Since } R \text{ is intuitionistic } f\text{-invariant by Corollary 2.7)}\\ &\le \delta_1(R) = \bigwedge_{x\in X} \mu_R(x,x) \text{ (Since } R \text{ is } G\text{-reflexive)}\\ &= \bigwedge_{x\in X} \mu_{f^2(R)}(t_x,t_x)\\ &= \bigwedge_{w\in Y} \mu_{f^2(R)}(w,w)\\ &= \delta_1(f^2(R))\end{aligned}$$

and

$$\begin{aligned}\nu_{f^2(R)}(u,v) &= \bigwedge_{(x,x')\in f^{-1^2}(u,v)} \nu_R(x,x')\\ &= \bigwedge_{f^2(x,x')=f(\mu,\nu)} \nu_R(x,x')\\ &= \nu_R(a,b)\\ &\ge \delta_2(R) = \bigvee_{x\in X} \nu_R(x,x)\\ &= \bigvee_{x\in X} \nu_{f^2(R)}(t_x,t_x)\\ &= \bigvee_{w\in Y} \nu_{f^2(R)}(w,w)\\ &= \delta_2(f^2(R)).\end{aligned}$$

Thus $f^2(R)$ is G-reflexive with $\delta(f^2(R)) = \delta(R)$. Intuitionistic fuzzy symmetry and transitivity of $f^2(R)$ can be proved, as shown in the proof of Proposition 2.10. This completes the proof. □

The following is the immediate result of Proposition 3.2.

Corollary 3.2. Let $f : X \to Y$ be a bijective mapping. If R is an intuitionistic fuzzy G-equivalence relation on X, then $f^2(R)$ is an intuitionistic fuzzy G-equivalence relation on Y with $\delta^2(f(R)) = \delta(R)$.

The following can be deduced from Propositions 3.2 and 2.4.

Proposition 3.3. Let $f : D \to S$ be a groupoid epimorphism. If R is an intuitionistic fuzzy G-congruence on D which is weakly f-invariant, then $f^2(R)$ is an intuitionistic fuzzy G-congruence on S with $\delta(f^2(R)) = \delta(R)$.

The following is the immediate result of Proposition 3.3.

Corollary 3.3. Let $f : D \to S$ be a groupoid isomorphism. If R is an intuitionistic fuzzy G-congruence on D, then $f^2(R)$ is an intuitionistic fuzzy G-congruence on S with $\delta(f^2(R)) = \delta(R)$.

Definition 3.4. Let $R \in \mathrm{IFR}(Y)$ and let $f : X \to Y$ be a mapping. Then R is said to be *intuitionistic f-stable* if $f^2(a,b) = (u,u)$ where $a \neq b \in X$ and $u \in Y$, implies that $\mu_R(f^2(a,b)) \le \mu_R(f^2(x,x))$ and $\nu_R(f^2(a,b)) \ge \nu_R(f^2(x,x))$ for each $x \in X$.

Example 3.4. Let $X = \{a,b,c\}$ and let $Y = \{u,v,w,r\}$. Consider the mapping $f : X \to Y$ defined as follow: $f(a) = f(c) = u$ and $f(b) = v$.

We define two complex mappings $R = (\mu_R, \nu_R)$, $H = (\mu_H, \nu_H) : Y \times Y \to I \times I$ as follows:

$R(u,u) = (\frac{1}{3}, \frac{2}{3})$, $R(v,v) = R(w,w) = R(r,r) = (\frac{1}{2}, \frac{1}{2})$,
$R(s,t) = (\frac{1}{4}, \frac{1}{2})$ for each $s \neq t \in Y$

and

$H(u,v) = H(u,w) = H(v,v) = H(w,w) = (\frac{1}{3}, \frac{2}{3})$,
$H(u,u) = H(r,r) = (\frac{1}{4}, \frac{2}{3})$,
$H(v,u) = H(w,u) = H(v,w) = H(w,v) = H(r,u) = H(u,r) = H(v,r) = H(r,v) = H(w,r) = H(r,w) = (\frac{1}{5}, \frac{1}{3})$.

Then

$\mu_R(f^2(a,c)) = \mu_R(u,u) = \frac{1}{3} \le \mu_R(f^2(x,x))$ for each $x \in X$

and

$\nu_R(f^2(a,c)) = \nu_R(u,u) = \frac{2}{3} \ge \nu_R(f^2(x,x))$ for each $x \in X$.

Also,

$\mu_H(f^2(a,c)) = \mu_H(u,u) = \frac{1}{4} \le \mu_H(f^2(x,x))$ for each $x \in X$

and

$\nu_H(f^2(a,c)) = \nu_H(u,u) = \frac{2}{3} \ge \nu_H(f^2(x,x))$ for each $x \in X$.

So R and H are intuitionistic f-stable. It may be noted that R is an intuitionistic fuzzy G-equivalence relation on Y, whereas H is not an intuitionistic fuzzy G-equivalence relation on Y.

Proposition 3.5. Let $f : X \to Y$ be a mapping and let R be an intuitionistic fuzzy G-equivalence relation on Y which is intuitionistic f-stable. Then $f^{-1^2}(R)$ is an intuitionistic fuzzy G-equivalence relation on X with $\delta_1(f^{-1^2}(R)) \ge \delta_1(R)$ and $\delta_2(f^{-1^2}(R)) \le \delta_2(R)$. Furthermore, if f is surjective, then $\delta(f^{-1^2}(R)) = \delta(R)$.

Proof. Let $a \in X$. Then

$$\mu_{f^{-1^2}(R)}(a,a) = \mu_R(f^2(a,a)) = \mu_R(f(a),f(a)) > 0 \text{ (Since } R \text{ is } G\text{-reflexive)}$$

and

$$\nu_{f^{-1^2}(R)}(a,a) = \nu_R(f^2(a,a)) = \nu_R(f(a),f(a)) < 1.$$

Let $a \neq b \in X$. Then there exist $u, v \in Y$ such that $f(a) = u$ and $f(b) = v$.

Case(i): Suppose $u = v$. Then

$$\begin{aligned}\mu_{f^{-1^2}(R)}(a,b) &= \mu_R(f^2(a,b)) \\ &= \mu_R(u,u) \\ &\leq \mu_R(f^2(x,x)) \text{ for each } x \in X \text{ (Since } R \text{ is intuitionistic } f\text{-stable)} \\ &= \mu_{f^{-1^2}(R)}(x,x) \text{ for each } x \in X\end{aligned}$$

and

$$\begin{aligned}\nu_{f^{-1^2}(R)}(a,b) &= \nu_R(f^2(a,b)) \\ &= \nu_R(u,u) \\ &\geq \nu_R(f^2(x,x)) \text{ for each } x \in X \\ &= \nu_{f^{-1^2}(R)}(x,x) \text{ for each } x \in X.\end{aligned}$$

Case(ii) : Suppose $u \neq v$. Then

$$\begin{aligned}\mu_{f^{-1^2}(R)}(a,b) &= \mu_R(f^2(a,b)) \\ &= \mu_R(u,v) \\ &\leq \delta_1(R) \\ &= \textstyle\bigwedge_{w\in Y} \mu_R(w,w) \text{ (Since } R \text{ is } G\text{-reflexive)} \\ &\leq \textstyle\bigwedge_{x\in X} \mu_R(f^2(x,x)) \\ &= \textstyle\bigwedge_{x\in X} \mu_{f^{-1^2}(R)}(x,x) \\ &= \delta_1(f^{-1^2}(R))\end{aligned}$$

and

$$\begin{aligned}\nu_{f^{-1^2}(R)}(a,b) &= \nu_R(f^2(a,b)) \\ &= \nu_R(u,v) \\ &\geq \delta_2(R) = \textstyle\bigvee_{w\in Y} \nu_R(w,w) \\ &\geq \textstyle\bigvee_{x\in X} \nu_R(f^2(x,x)) \\ &= \textstyle\bigvee_{x\in X} \nu_{f^{-1^2}(R)}(x,x) \\ &= \delta_2(f^{-1^2}(R)).\end{aligned}$$

In all, $f^{-1^2}(R)$ is G-reflexive with $\delta_1(f^{-1^2}(R)) \geq \delta_1(R)$ and $\delta_2(f^{-1^2}(R)) \leq \delta_2(R)$. As in the proof of Proposition 2.5, we can show that $f^{-1^2}(R)$ is intuitionistic fuzzy symmetric and transitive. Moreover, it is clear that if f is surjective, then $\delta(f^{-1^2}(R)) = \delta(R)$. This completes the proof. □

The following is the immediate result of Proposition 3.5.

Corollary 3.5. Let $f : X \to Y$ be an injective mapping and let R be an intuitionistic fuzzy G-equivalence relation on Y. Then $f^{-1^2}(R)$ is an intuitionistic fuzzy G-equivalence relation on X with $\delta_1(f^{-1^2}(R)) \geq \delta_1(R)$ and $\delta_2(f^{-1^2}(R)) \leq \delta_2(R)$. Furthermore, if f is surjective, then $\delta(f^{-1^2}(R)) = \delta(R)$.

The following can be deduced from Propositions 3.5 and 2.4.

Proposition 3.6. Let $f : D \to S$ be a groupoid homomorphism. If R is an intuitionistic fuzzy G-congruence on S which is intuitionistic f-stable, then $f^{-1^2}(R)$ is an intuitionistic fuzzy G-congruence on D with $\delta_1(f^{-1^2}(R)) \geq \delta_1(R)$ and $\delta_2(f^{-1^2}(R)) \leq \delta_2(R)$. Furthermore, if f is surjective, then $\delta(f^{-1^2}(R)) = \delta(R)$.

The following is the immediate result of Proposition 3.6.

Corollary 3.6. Let $f : D \to S$ be a groupoid monomorphism. If R is an intuitionistic fuzzy G-congruence on S, then $f^{-1^2}(R)$ is an intuitionistic fuzzy G-congruence on D with $\delta_1(f^{-1^2}(R)) \geq \delta_1(R)$ and $\delta_2(f^{-1^2}(R)) \leq \delta_2(R)$. Furthermore, if f is surjective, then $\delta(f^{-1^2}(R)) = \delta(R)$.

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