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On Parameter Estimation of Growth Curves for Technological Forecasting by Using Non-linear Least Squares*

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ABSTRACT

Growth curves including Bass, Logistic and Gompertz functions are widely used in forecasting the market demand. Nonlinear least square method is often adopted for estimating the model parameters but it is difficult to set up the starting value for each parameter. If a wrong starting point is selected, the result may lead to erroneous forecasts. This paper proposes a method of selecting starting values for model parameters in estimating some growth curves by nonlinear least square method through grid search and transformation into linear regression model. Rescaling the market data using the national economic index makes it possible to figure out the range of parameters and to utilize the grid search method. Application to some real data is also included, where the performance of our method is demonstrated.

Keywords: Growth Curves, Starting Value, Nonlinear Least Squares, Grid Search, Market Potential

1. Introduction

The diffusion model is based on an S-shaped curve, which has been long established

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in biological sciences to describe the growth of various phenomena. These S-shaped curves including Bass model [1], Logistic and Gompertz curves have been utilized to model various forms of technology during recent decades. Some growth curves can be easily estimated if market potential is assumed to be known, but clearly this assumption may be unduly restrictive [2, 3].

There are many methods for estimating parameters in a growth curve model. The representative estimation methods are ordinary least squares (OLS) [1, 4], maximum likelihood estimation (MLE) [5, 6], and nonlinear least squares (NLS) [7]. The OLS method may be the simplest. However, if it is not possible to transform into a linear regression model, this method cannot be applicable. The MLE method allows a computation of approximate standard errors for the diffusion model parameters. However, it is sometimes difficult to construct a likelihood function and to find the parameter estimates to maximize the likelihood function since the nonlinearity is involved. The NLS is a straightforward method since our growth curves are nonlinear functions. Moreover, computer packages are more readily available for NLS than for MLE. The NLS approaches are applicable to any diffusion models for which cumulative adoption can be expressed as an explicit function of time [7] although there are still some problems in optimizing the nonlinear function.

To implement the NLS, we must determine the starting value of each model parameter. Since there are no algorithms available that guarantee the global optimum the estimated parameters may depend on the selected starting value for each parameter. One solution to this problem may be the use of multiple starting values, but this requires multiple execution of NLS algorithm. So, choosing the right starting values are very important. We concentrate upon three principal models which have been widely used: Logistic, Gompertz and Bass models. The purpose of this paper is to resolve the problem of selecting starting points which arises in estimating the parameters through the NLS method.

Our approach is to use a grid search. The grid search may not be a simple task since the search space for model parameters is not finite. So, we suggest transforming the market demand into the relative scale using the national economic index such as yearly population or gross domestic product. This transformation requires forecasting the national economic index in predicting future market demand, but this may not be a serious problem since the forecasts for the national economic index are often readily available from the government sources.

The structure of this paper is as follows. In Section 2, we briefly describe the nonlinear least squares method and propose a method of selecting starting values by using grid search. The approach is then applied to two real cases in Section 3. In Section 4, we conclude with some remarks.

2. Determination of Starting Values of Parameters

Let N_t be the cumulative number of adopters or cumulative sales of a product at time t. Then, the general form of growth curve model is expressed as follows:

$$N_t = f(t; \underline{\gamma}) + \varepsilon_t \tag{1}$$

where $f(\cdot)$ represents a particular growth curve as a function of t having parameter vector of γ and ε_t is an error term. If sales at time t, denoted by S_t, is available, then the cumulative sales at time t is given by

$$N_t = \sum_{i=1}^{t} S_i$$

The least squares criterion Q in Eq. (2) must be minimized with respect to the parameter vector γ to obtain the NLS estimates.

$$Min\ \ Q = \sum_{t=1}^{T} [N_t - f(t; \underline{\gamma})]^2$$

where T is the time period in which observations on N_t are available.

There are many methods to minimize the objective function. The Gauss-Newton method [8] is most popular, which uses a Taylor series expansion to approximate the nonlinear function with linear terms and then employs the OLS to estimate the parameters. The Gauss-Newton method begins with initial or starting values for the regression parameters which may be obtained from previous or related studies, theoretical expectations, or a preliminary search for parameter values. However, different staring values may lead to different estimated parameters since this method does not guarantee the global optimum.

Our present discussion will focus on the Logistic, Gompertz and Bass models as these are the most widely used in practice:

Logistic model $$f(t;\ m,\ \alpha,\ \beta)=\frac{m}{1+\alpha\exp(-\beta\, t)},\quad \alpha,\beta>0 \tag{3}$$

Gompertz model $$f(t;\ m,\ \alpha,\ \beta)=m\exp(-\alpha\exp(-\beta t)),\quad \alpha,\beta>0 \tag{4}$$

Bass model $$f(t;\ m,\ p,\ q)=\frac{m(1-\exp(-(p+q)t))}{1+\frac{q}{p}\exp(-(p+q)t)},\quad p,\ q>0 \tag{5}$$

In the above growth curves m is referred to the market potential and α, β, p and q are parameters to be estimated. Particularly, in Bass model p is called the coefficient of innovation and q is the coefficient of imitation [1].

In this paper, we propose a method of determining starting values of these model parameters. Market potential m is the most important parameter in forecasting. It is easily seen that the Logistic and Gompertz models can be linearly transformed if m is known. That is, if the value of m is perfectly known, then the other parameters are easily estimated.

Our approach is based on the grid search of which outline is as follows: First, we transform the market demand into the relative scale using the national economic index such as population or gross domestic product. This transformation makes the relative demand or market potential have a value ranging from 0 to 1 (or 0 to 100 in percentage). Second, for a fixed value of m ranging from 0 and 100 (or narrower range) we estimate the other model parameters using linear transformation of the model and the OLS. The range from the minimum to the maximum of the estimated value of each parameter will comprise a search space. Third, we perform the grid search. That is, we generate many combinations of parameters from a search space and calculate the sum of squares in Eq. (2) to obtain the combination that yields the minimum. This combination of parameters will be used as the recommended starting values. Using this procedure we need only one execution of NLS algorithm.

2.1 Transformation of data

When the cumulative demand represents sales in domestic market, we recommend that it should be transformed into the relative sales with regard to the gross domestic product (GDP). Let S_t be the sales at time t and GDP_t be the gross domestic product at time t. Then, the relative sales in percentage will be

$$s_t = \frac{S_t}{GDP_t} \cdot 100, \quad t = 1, 2, \cdots, T \tag{6a}$$

and the relative cumulative sales in percentage will be

$$n_t = \sum_{i=1}^{t} s_i \tag{6b}$$

When the cumulative demand represents the cumulative number of subscribers or adopters, we recommend that it be transformed into the proportion relative to the national population. Let N_t be the cumulative number of subscribers at time t and POP_t be the population at time t. Then, the relative subscribers in percentage will be

$$n_t = \frac{N_t}{POP_t} \cdot 100, \quad t = 1, 2, \cdots, T \tag{7}$$

Normally, the market potential of a certain goods cannot exceed a certain proportion of the gross domestic product or the national population. Let $n_{\max}$ denote the maximum value of the proportion or the relative market potential. We may use $n_{\max} = 100$ if there is no information on that and we may use $n_{\min} = 1.1 \cdot n_T$ which is slightly larger than n_T for making the OLS estimation possible (See Eq. (10)). Then, we say that the following holds:

$$n_{\min} \leq m \leq n_{\max} \tag{8}$$

If we divide the above range into k intervals, then we obtain the following (k+1) boundary values for the relative market potential that can be used in pursuing our grid search:

$$m_i = n_T + (i-1)(n_{\max} - n_{\min})/k, \; i = 1, \; \cdots, k+1 \tag{9}$$

2.2 Linear transformation of growth curves

As mentioned earlier, if the market potential m is known, then the parameters α and β for Logistic and Gompertz curves can be estimated by the OLS. Suppose that m_i is selected for the market potential. Then, the other parameters in Logistic curve can be estimated by the following linear regression model as proposed by Mansfield [9]:

$$\ln\frac{m_i - n_t}{n_t} = \ln\alpha_i - \beta_i t, \quad i = 1, 2, \cdots, k+1 \tag{10}$$

Similarly, parameters α and β corresponding to m_i in Gompertz curve can be estimated by the following linear regression model as proposed by Young [3]:

$$\ln(\ln\frac{m_i}{n_t}) = \ln\alpha_i - \beta_i t, \quad i = 1, 2, \cdots, k+1 \tag{11}$$

We believe that the true value of each parameter falls into

$$\min[\alpha_1, \alpha_2, \cdots, \alpha_{k+1}] \le \alpha \le \max[\alpha_1, \alpha_2, \cdots, \alpha_{k+1}] \tag{12a}$$

$$\min[\beta_1, \beta_2, \cdots, \beta_{k+1}] \le \beta \le \max[\beta_1, \beta_2, \cdots, \beta_{k+1}] \tag{12b}$$

We treat the Bass curve in a different way since it cannot be linearly transformed even if m is known. Some research shows that the Bass curve behaves similarly as the Logistic curve since the Bass curve is a special case of the extended Logistic curve [10]. So, we first approximate the Bass curve in Eq. (5) as the Logistic curve in Eq. (3) and estimate its parameters using Eq. (10). Then, two other parameters in Bass curve corresponding to m_i are obtained by

$$p_i = \frac{\beta_i}{1+\alpha_i}, \quad q_i = \frac{\alpha_i\beta_i}{1+\alpha_i} \tag{13}$$

Therefore, we believe that the true value of each parameter lies in the following range:

$$\min[p_1, p_2, \cdots, p_{k+1}] \le p \le \max[p_1, p_2, \cdots, p_{k+1}] \tag{14a}$$

$$\min[q_1, q_2, \cdots, q_{k+1}] \le q \le \max[q_1, q_2, \cdots, q_{k+1}] \tag{14b}$$

Bass [1] and Mahajan [11] proposed the estimation method based on the OLS for the following regression model:

$$S_{t+1} = \beta_1 + \beta_2 N_t + \beta_3 N_t^2 \tag{15}$$

where $S_{t+1} = N_{t+1} - N_t$ and $\beta_1 = pm$, $\beta_2 = q - p$, and $\beta_3 = -q/m$. Once the above regression model is estimated, the Bass model parameters are estimated by

$$\hat{m} = (-\hat{\beta}_2 - \sqrt{\hat{\beta}_2^2 - 4\hat{\beta}_1\hat{\beta}_3})/2\hat{\beta}_3, \quad \hat{p} = \hat{\beta}_1/\hat{m}, \quad \hat{q} = -\hat{m}\hat{\beta}_3 \tag{16}$$

These parameter estimates may be adopted as starting values in NLS. However, just looking at the formula in Eq. (16) we see that the market potential cannot be estimated when the term inside the square root is negative. So, the OLS estimates by Bass may not be recommended as starting values in NLS.

2.3 Determining starting values by using grid search

We apply the grid search to determine the starting values of model parameters which yield the minimum least squares in Eq. (2). We first divide the range of each parameter given in Eq. (12) or (14) by r equal intervals to generate $(r+1)^3$ combinations and calculate the following error sum of squares (SSE):

$$SSE(i, j, k) = \sum_{t=1}^{T} (n_t^{(i,j,k)} - n_t)^2, \quad i, j, k = 1, 2, \cdots, r+1 \tag{17}$$

where

$$n_t^{(i,j,k)} = \frac{m_i}{1 + \alpha_j \exp(-\beta_k t)} \quad \text{(for Logistic curve)}$$

$$n_t^{(i,j,k)} = m_i \exp(-\alpha_j \exp(-\beta_k t)) \quad \text{(for Gompertz curve)}$$

$$n_t^{(i,j,k)} = \frac{m_i(1-\exp(-(p_j+q_k)t))}{1+\frac{q_k}{p_j}\exp(-(p_j+q_k)t)} \quad \text{(for Bass curve)}$$

As shown in the case study in Section 3, r = 30 may be sufficient. Then, the recommended set of starting values is the combination that results in the minimum SSE given in Eq. (17). With these starting values in an appropriate algorithm for non-linear least squares the model parameters can be finally estimated. Note that our procedure requires only one time of execution of NLS algorithm. The time required for the grid search may be negligible since it involves only simple operations. Figure 1 shows our estimation process by using gird search and NLS.

3. Applications

3.1 Annual sales of printers in Korea

Annual sales data of computer printers from 1989 to 1998 are collected as in Table 1. We will use the data from 1986 to 1993 (T = 8) for estimation of model parameters and the rest for the model validation.

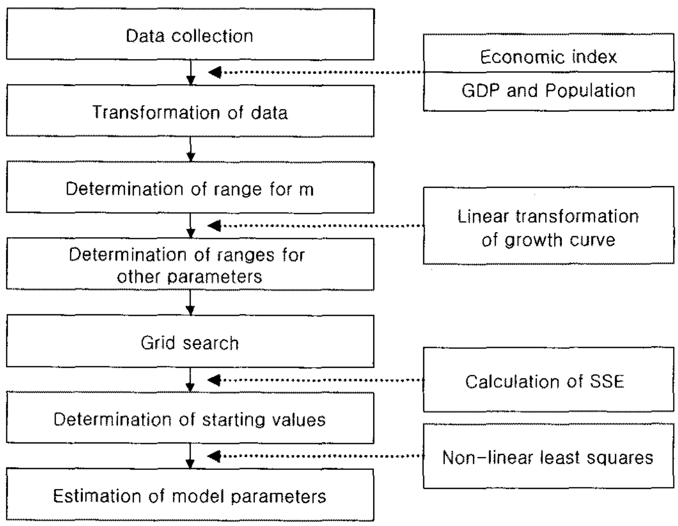

Figure 1. The proposed estimation process by using gird search and NLS

Table 1. Annual sales of printers and the proportions to GDP

year		Annual sales in 10^6 Won (S_t)	GDP in 10^6 Won (GDP_t)	Proportion in percentage (s_t)	Cumulative proportion in percentage (n_t)
1986	1	75,063	94,861,700	0.0791	0.0791
1987	2	78,265	111,197,700	0.0704	0.1495
1988	3	131,858	132,111,800	0.0998	0.2493
1989	4	172,562	148,197,000	0.1164	0.3658
1990	5	221,115	178,796,800	0.1237	0.4894
1991	6	283,877	216,510,900	0.1311	0.6205
1992	7	363,636	245,699,600	0.1480	0.7685
1993	8	418,563	277,496,500	0.1508	0.9194
1994	9	507,845	323,407,100	0.1570	1.0764
1995	10	685,199	377,349,800	0.1816	1.2580
1996	11	529,040	418,479,000	0.1264	1.3844
1997	12	370,772	453,276,400	0.0818	1.4662
1998	13	250,229	444,366,500	0.0563	1.5225

Source: Bureau of Statistics in Korea.

We expect that the annual sales cannot exceed 10 percent of GDP in the long run. Also, we see that $n_T = 0.9194$ for 1993. So, the interval for the market potential of printers is chosen as $(0.9194 \times 1.1, 10)$. The interval is then divided by ten ($k = 10$) and eleven values for the market potential are investigated as in Table 2. This table shows the parameter estimates for the Logistic curve using simple regression model of Eq. (10) according to eleven selected market potentials.

Table 2. Parameters for Logistic curve according to market potentials (Printer)

Market potential		m_1	m_2	m_3	m_i	m_9	m_{10}	m_{11}
		1.0113	1.9102	2.8091	···	8.2023	9.1011	10.0000
Estimates from linear model	a	22.2736	27.4782	38.5662	···	108.2359	119.9190	131.6067
	β	0.6327	0.4249	0.3905	···	0.3536	0.3520	0.3506
SSE		0.0063	0.0123	0.0246	···	0.0500	0.0516	0.0530

It can be seen from Table 2 that the three parameters for Logistic curve are within the following ranges summarized in Table 3.

Table 3. Interval of each parameter for Logistic curve (Printer)

Parameter	Interval range	
	minimum	maximum
m	1.0113	10.0000
a	22.2736	131.6067
β	0.3506	0.6327

The same procedure applies to other growth curves and the results are shown in Table 4 for Gompertz curve and in Table 5 for Bass model.

Table 4. Interval of each parameter for Gompertz curve (Printer)

Parameter	Interval range	
	minimum	maximum
m	1.0113	10.0000
a	3.8530	5.1015
β	0.0995	0.4310

Table 5. Interval of each parameter for Bass model (Printer)

Parameter	Interval range	
	minimum	maximum
m	1.0113	10.0000
p	0.0026	0.0272
q	0.3480	0.6055

To perform the grid search, each range of three parameters is divided by r equal intervals so that a total of $(r+1)^3$ combinations of parameters can be evaluated with the criterion of Eq. (17). A question may arise regarding to the selection of r. So, we perform experiments according to various values of r and observe the trend of the minimum SSE achieved form the grid search, which result is shown in Table 6.

It is seen in Table 6 that SSE is not significantly improved between $r = 30$ and $r = 45$ in our three models. Therefore, we propose that $r = 30$ would be sufficient. The total number of grid search that we perform for each model is $(30 + 1)^3 \cong 30000$ and the combination of three parameters yielding the minimum *SSE* is chosen as our recommended starting values for the model parameters. Then, the NLS algorithm is run using these recommended starting values. Table 7 shows the recommended starting value (SV) and finally estimated value (FV) by the NLS for each parameter in three models.

Table 6. Minimum SSE achieved by the choice of r

r	Logistic curve	Gompertz curve	Bass model
5	0.00626	0.00227	0.01129
10	0.00584	0.00227	0.01129
15	0.00470	0.00088	0.00593
20	0.00454	0.00110	0.00780
25	0.00398	0.00052	0.00825
30	**0.00358**	**0.00051**	**0.00593**
35	0.00340	0.00059	0.00689
40	0.00314	0.00030	0.00699
45	0.00295	0.00038	0.00593

Table 7. Recommended starting values and final estimates of parameters (Printer)

Growth curves		m	α or p	β or q	SSE
Logistic	SV	1.3109	22.2736	0.5011	0.00358
	FV	1.2216	18.2171	0.4961	0.00140
Gompertz	SV	1.9102	3.8946	0.2100	0.00051
	FV	1.9145	3.8081	0.2049	0.00026
Bass	SV	1.6106	0.0272	0.3480	0.00593
	FV	2.3117	0.0282	0.2052	0.00012

Note: SV: recommended starting value, FV: final estimated value of parameter.

Note that for all three curves there is little difference between a recommended starting value and the finally estimated parameter value. Among the three models Bass model leads to the minimum SSE. Figure 2 shows predicted values from each model as compared with the actual sales.

As seen in Figure 2 three curves predict sales similarly and fairly well up to year 1993 whose data have been used in the models. However, each model predicts somewhat differently for years between 1994 and 1998. Table 8 shows the root mean squared error (RMS) of each model, which is the square root of the SSE divided by the number of periods involved.

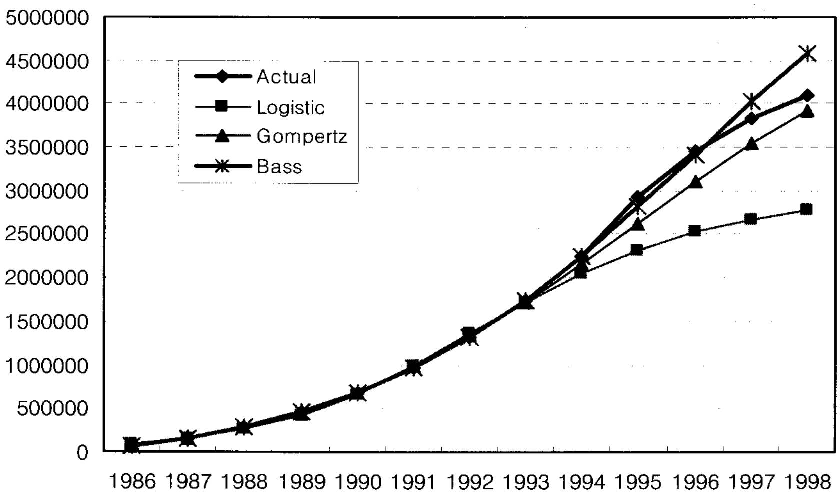

Figure 2. Actual cumulative sales and predicted values of Printer

Table 8. Forecasting RMS of each model (Printer)

	Logistic Curve	Gompertz Curve	Bass Curve
RMS during 1986~1993	0.01321	0.00574	0.00387
RMS during 1994~1998	0.24501	0.07514	0.05355

Table 9. Performance of 216 NLS runs as compared with ours (Printer)

	Same SSE	Smaller SSE	Larger SSE	No of local optima	SSE from our method	Worst SSE
Logistic	33.33%	0%	66.67%	5	0.00140	0.01641
Gompertz	100%	0%	0%	1	0.00026	0.00026
Bass	93.06%	0%	6.94%	4	0.00012	0.62741

In order to investigate how different starting values in NLS lead to different results we perform further experiments. We first widen the search space so that the market potential (m) varies up to 50 and divide the interval of each parameter into five. Then we execute NLS algorithm at $(5 + 1)^3 = 216$ combinations of parameters and compare the SSEs with the SSE from our procedure using the recommended starting point. Table 9 summarizes the result, which shows the percentage of the total runs yielding the same, smaller, and larger SSEs as compared to our procedure. In Table 9 we see that there exist several local optimal points having larger SSEs in Logistic and Bass models. Particularly, if we select wrong starting values in Logistic model, there is about 73% of chance that we may reach a local optimum. In Bass model a wrong starting value may lead to an extremely poor result.

3.2 Number of registered host computers in Korea

The number of registered host computers in Korea according to years (from years 1993 to 2000) is reported as in Table 10. We will use data from 1993 to 1997 for estimating the models and the rest for the model validation.

As for this case we again assume that the relative market potential cannot exceed 10 percent of the national population in the long run. So, the search interval for m would be [0.3135, 10] for every model. The procedure of determining the search space for other model parameters is same as for the Printer case and the result is summarized in Table 11.

Table 10. Number of registered host computers

year		Number of Host Computers (N_t)	Population (POP_t)	Proportion to Population (%) (n_t)
1993	1	7,650	44,195,000	0.017
1994	2	13,856	44,642,000	0.031
1995	3	36,644	45,093,000	0.081
1996	4	73,191	45,545,000	0.161
1997	5	131,005	45,991,000	0.285
1998	6	202,510	46,430,000	0.436
1999	7	296,300	46,858,000	0.632
2000	8	460,974	47,293,000	0.975

Source: the Bureau of Statistics in Korea.

Table 11. Search interval of each parameter (Host computer)

	Logistic curve		Gompertz curve		Bass curve	
	α	β	α	β	p	q
minimum	91.037	0.731	5.998	0.150	0.00060	0.731
maximum	1223.571	1.254	10.0777	0.807	0.01363	1.241

The number of intervals (r) dividing the search space of each parameter is chosen as 30, which is shown to be sufficient in Printer case. The recommended starting values and final estimated values of model parameters are shown in Table 12.

Table 12. Recommended starting values and final estimates of parameters (Host Computer)

Growth curves		m	α or p	β or q	SSE
Logistic	SV	0.6362	91.0372	0.8533	0.000066
	FV	0.6048	94.5321	0.8858	0.000030
Gompertz	SV	6.4482	7.3576	0.1715	0.000040
	FV	5.3992	7.1985	0.1790	0.000036
Bass	SV	0.9591	0.0080	0.7307	0.000067
	FV	0.8510	0.0093	0.7337	0.000049

Note: SV: recommended starting value, FV: final estimated value of parameter.

We see from Table 12 that the finally estimated parameter values are very close to the starting values that are determined from our procedure. Figure 3 shows the predicted number of registered host computers according to each model considered.

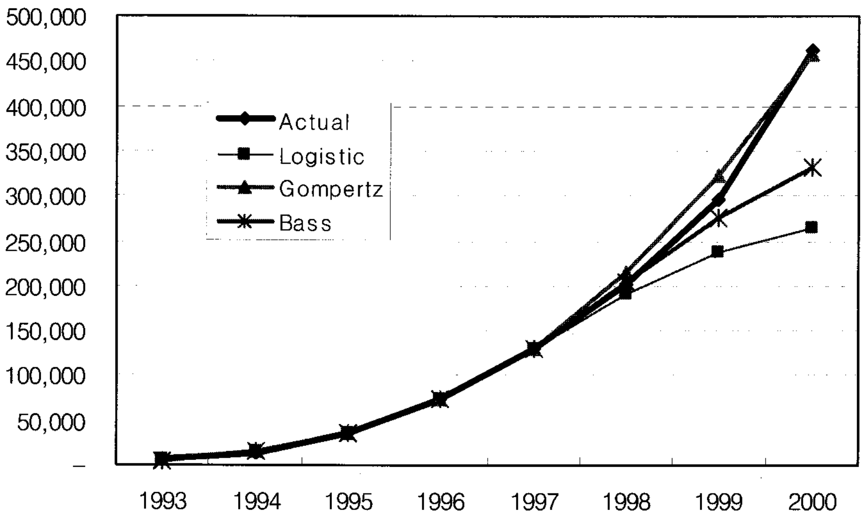

Figure 3. The actual and predicted number of host computers

Each model predicts similarly up to year 1997 whose data are used in estimation, but it predicts differently for years between 1998 and 2000. Table 13 shows the RMS of each model. In this case Gompertz curve seems to perform well.

Table 13. Forecasting RMS of each model (Host computer)

	Logistic Curve	Gompertz Curve	Bass Curve
RMS during 1993~1997	0.00244	0.00269	0.00314
RMS during 1998~2000	0.25015	0.03706	0.15822

To test the performance of starting values as we did in the Printer case we execute a total of 216 runs of NLS algorithm with different starting values within the expanded search space and compare the SSEs with the SSE from our procedure using the recommended starting point. Table 14 summarizes the result, which shows the percentage of the total runs yielding the same, smaller, and larger SSEs as compared to our procedure. Here again there are no runs found which have smaller SSE than from our proposed procedure. We observe that among the three models Logistic curve is most sensitive to the selection of staring values.

Table 14. Performance of 216 NLS runs as compared with ours (Host computer)

	Same SSE	Smaller SSE	Larger SSE	No of local optima	SSE from our method	Worst SSE
Logistic	9.1%	0%	90.9%	7	0.000030	0.000540
Gompertz	85.6%	0%	14.4%	6	0.000036	0.000064
Bass	78.6%	0%	21.4%	8	0.000049	0.000089

4. Conclusions

This paper proposes a method of selecting starting values for model parameters through search and transformation into linear regression model. To determine starting value, rescaling the market data using an appropriate national economic index makes it possible to figure out the range of parameters and to utilize the grid search method. The proposed procedure requires only one time execution of the NLS algorithm by using a "right" starting value, which may save the computational time tremendously. From two case studies we observe that our procedure always leads to the least SSE and that Logistic curve is most sensitive to the selection of starting values of parameters.

We expect that our procedure can be applied to other growth curves that are not considered in this paper with some modifications. We have some limitation in applying to a model that may not be easily transformed into a linear form, in which case we need a suitable approximation for functional form. Application to a more complex model such as multi-generation model or repeated purchase model would be a further interesting problem.

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