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Awareness and Knowledge of Pre-Service Teachers on Mathematical Concepts: Arithmetic Series Case Study[1]

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Deep comprehension of basic mathematical notions and concepts is a basic condition of a successful teaching. Some elements of algebraic thinking belong to the elementary school mathematics. The question "What stays the same and what changes?" link arithmetic problems with algebraic conception of variable. We have studied beliefs and comprehensions of future elementary school mathematics teachers on early algebra. Pre-service teachers from three academic pedagogical colleges deal with mathematical problems from the pre-algebra point of view, with the emphasis on changes and invariants. The idea is that the intensive use of non-formal algebra may help learners to construct a better understanding of fundamental ideas of arithmetic on the strong basis of algebraic thinking. In this article the study concerning arithmetic series is described. Considerable number of pre-service teachers moved from formulas to deep comprehension of the subject. Additionally, there are indications of ability to apply the conception of change and invariance in other mathematical and didactical contexts.

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INTRODUCTION: ALGEBRAIC THINKING AND ARITHMETIC PROGRESSIONS IN PRIMARY SCHOOL

Students deal intensively with algebraic notation and formalism, the process starting at secondary school classes. The difficulties of transition from "number language" to the algebraic one have been learned and classified (Kieran, 1989; Steinberg, Sleeman & Ktorza, 1991; Sfard & Linchevski, 1994; Sfard, 1995). The good part of those difficulties is connected with algebra symbolism and "the braking" of arithmetic stereotypes ("brackets first", the meaning of equality). But the most significant is the necessity to work with certain "quantity" that is both unknown and tends to change. In order to help the student, several researches try to introduce some early algebra in primary school (Chappell & Strutchens, 2001; Rivera, 2006). According to NCTM (2000) conception, mathematical discourse in elementary school should include elements of algebraic reasoning. NCTM sees it as a tool for advancing math understanding and as an introduction to study of algebra formalism.

Each one of four components of algebraic thinking at school (i.e. ability to understand patterns and relations; representing and analyzing situations and structures with algebraic symbols; using mathematical models; exploring change in various contexts) refers to algebraic thinking in two interrelated but different aspects. The first one is the intuition and the knowledge on relationship between mathematical objects. The second element is connected with structural thinking (Hoch & Dreyfus, 2006).

It is clear that just the handling of mathematical situation in terms of structure supposes some formal representation and notation. Accordingly, the large part of school educators is confident with "algebraic thinking = symbolism" modus operandi.

Alternatively, "pure" early algebra deals with intuitive reasoning on connections and relations between things rather than formalization. What does it mean "to relate things"? Since one can differ between them, it makes possible to find "the same and not the same". Stavy & Tirosh (2000) had found out the ambiguous role of intuition as an advisor in making right decision in early mathematics (Stavy & Tirosh). It enables to describe the long-term development of construction of mathematical thinking as a process of improving the naive intuition into a well-founded one.

Arithmetic progression of consecutive natural numbers is one of the very first mathematical notions that children acquire. They start their arithmetic intuition while going step by step from counting items to quantity. The primary idea of one-to-one correspondence is accompanied immediately with differentiating of quantities. There is a good portion of algebra in this process. Simple operation of adding next item gives new quantity. Furthermore, the repeating of the same operation provides new quantities with designation them by consecutive natural numbers. It fits the Peano axioms that construct the set of natural numbers with "successor" function. In this process the change is highlighted nevertheless the invariant is also present, since the increment stays the same.

Next time progressions come out in counting by pairs, tens, dozens with the emphasis of constant units which result in equal distances between neighboring terms. The algebraic viewing of progression of natural numbers appears also when one compares between two sums with the same addend in both expressions (23 + 5 *vs.* 23 + 3). Here the invariant operand plays as parameter, and the difference in results is caused purely by the enlarging of the second addend. Intensive learning of first hundred of natural numbers is assisted by structuring and visualization of this set with "Numerical Tables." The regularity in rows and columns of the table invites the problem of calculation of sums of ten-element arithmetic progressions.

Nevertheless, this available mathematical potential of arithmetic sequence and series is not utilized enough in teaching in elementary school. In high school students are involved in algebraic treatment of arithmetic series mostly by application of formulas to derive their sums.

We choose the arithmetic series as the mathematical topic for the study, since the subject seems to be well-known to pre-service teachers. They had learned it in frame of high school algebra and have examined the issue at all levels of school matriculation tests. Deeper understanding of the properties of arithmetic progressions and series would be relevant to their future teaching practice as well. Our other aim was to assess transferability of acquired algebraic intuitions onto other subjects than arithmetic series.

THEORETIC BACKGOUND

Research on algebraic thinking of school educators

Awareness of teachers on components and importance of algebraic thinking is a necessary condition of developing this thinking of their students. The content-specific knowledge and beliefs of teachers are generally assumed to play a crucial role in the construction of mathematical thinking of students (Thompson, 1992; Verschaffel, Greer & De Corte, 2000), especially in transition from arithmetic to algebra (Nathan & Koedinger, 2000; Schmidt & Bednarz, 1997). Recent evidence of such an influence has been obtained by van Dooren, Verschaffel & Onghena (2003) throughout their study of impact of pre-service teachers' algebraic knowledge on their evaluation of students' problem solving strategies for novel problems. Future teachers with formal attendance of algebra are hardly capable of being flexible in evaluation of students' solutions. Particularly, they classify non-formal algebra reasoning of students as a "less mathematical one".

Most elementary school mathematics teachers have a lack of experience in dealing with algebra. Learning algebra is in the periphery of training of elementary school teachers in Israel. Accordingly, they accept the issue as a technique one or as fine extra-curricula enrichment. A sample study on algebraic thinking of pre-service teachers (Sinitsky & Guberman, 2006) shows:

- Almost all pre-service teachers have no satisfactory knowledge on non-formal components of algebraic thinking;
- Future elementary school teachers don't aware on any relation of algebra and subject matter of elementary school mathematics they intend to teach;
- The term "non-formal algebra" is accepted in this population in the only sense of "preparation to real algebra learning."

We suppose that systematic application of the algebraic idea of interconnection of changes and invariants is a fruitful tool to develop mathematical comprehension. Pre- and in-service teachers of mathematics in pre-high school seem to be a natural reference group to assess the validity of this conjecture. They typically have relevant mathematical basis to solve and discuss the problems of school mathematics of different level for several grades. On the other hand, their expertise in algebra is mostly technical and formal one. It is reasonable to introduce them to a non-formal algebra by problem solving activities on elementary school and higher level.

To advance this concept, we elaborate a number of assignments related to arithmetic series. The choice of problems for these activities is dictated by audience we deal with. Each task consists of cluster of problems varying in difficulty.

Pre-algebra in the study of arithmetic series

Construction of significant algebraic thinking is a long-term process. It is desirable to start it as early as possible. Such early algebra aims to help students in transition from arithmetic objects to algebraic ones. On the other hand, it should stimulate argumentation abilities of students beyond the learning itself and lead to comprehension of mathematical facts and procedures.

The algebraic concept of pattern, change and invariance is one of the central ideas of mathematical knowledge. It is treated intensively by researchers and is used as a powerful tool in problem solving (*e.g.* Engel, 1998; Mason, Burton & Stacey, 1982; Pólya, 1988; Schoenfeld, 1985). Conversely, this concept is almost omitted both in school mathematics and in future teachers training in Israel. The notion of variation and change arises mostly in the learning of function in formal algebra. Accordingly, it seems to the students as contra-version of constant quantity (Usiskin, 1988).

However, the introduction of algebraic ideas is essential for learning of some topics of early school mathematics. The algebraic viewing and intuition help to understand basic notions and algorithms of arithmetic such as parity and divisibility, addition and multiplication algorithms, ways of comparison of fractions, *etc*. This approach does not rely on algebraic notation and does not need formal algebraic technique.

Since the most famous anecdote on young Gauss' summation of first hundred natural numbers (Burton, 1989, p.80–81), the arithmetic series is well-known example of successful treatment of numerical problem by non-formal algebraic tools. In the wide context of school mathematics, the arithmetic series are connected with other topics – such as divisibility problems, mean value calculations and combinatorial problems of decompositions of a given natural number. The complex learning of this issue integrates several components of algebraic thinking and reasoning.

The idea of interrelation between changes and invariants is the central point in the learning of arithmetic series. We attempt to recognize a number of types of these links with the activities about arithmetic series. It occurs that they can appear also in other mathematical situations of different level, from very initial to high mathematics. Each type of "what is the same – what is changed" relation provides further comprehension of arithmetic sequences and supplies extra ways to compute related series.

Presented below is a partial list with topic-oriented examples:

- For arithmetic series, the sum of couples of elements with equal distance from the progression ends is invariant (as in Gauss's case, 1+100=2+99=...=50+51). This fact can be derived from the analytical description of the progression. But at informal level, there is discovery of the fact that the sum of increasing (or decreasing) numbers consists of constant addends. This property also remains with any variation of constant "jump" between adjacent terms of series. The standard formula for arithmetic series calculation exploits this invariant of any arithmetic series.
- The dual viewing of the same invariant is achieved by re-organizing the given series. The rewriting of a given sum in a reverse order varies the structure of the expression but does not change the series value (1+2+...+99+100= 100+99+...+2+1). In contrary with the previous finding, one does arrange the suitable pairing here. Additionally, the construction of pairs of constant sum may be easy visualized.
- Both previous schemas deal with the given arithmetic series. An additional way refers to well known idea of substituting the problem by another one (Pólya, 1988). Instead of n-length arithmetic progression with various terms, the sequence of n equal numbers is introduced. The suitable choice of the number provides the initial progression can be reconstructed from this sequence (4,4,4,4,4 ⟶ 2,3,4,5,6).

This transformation keeps the sum of the sequence. Furthermore, there is a possibility to see the situation by a dual way: how can given quantity be split into the sum of subsequent addends?

- The sum of any arithmetic progression can be calculated on the basis of the sum of consecutive natural numbers (as the sum 2+3+...+100+101 can be derived from the sum 1+2+...+99+100). Here is another version of transformation of series. In this case the sum certainly does change, but some components of the sum remain the same. It would be like changes of all the terms (e.g., 1⟶ 2, 2 ⟶ 3,..., 100 ⟶ 101) or the altering in only one term (101 in the second series instead of 1 in the first one) with invariance of the rest of addends. The dynamic construction of the series turns the static situation to a dynamic one and provides some didactic benefit (Nesher, Greeno & Riley, 1982). More generally, it is the way to derive any new problem into a solved one and to use the ready solution with suitable variation.
- It seems that arithmetic series is suitable issue to discuss the algebraic idea of variants and invariants in non-formal language "what is change – what stay the same through the change". It is close to school curricula and familiar in some scale to every pre-service teacher. At the same time, the issue of arithmetic series is sufficiently rich with various aspects of non-formal algebra and enables many analogies and generalizations.

A CASE STUDY: ARITHMETIC SERIES

Purposes and didactic tools

In our study we examine the change in the awareness of non-formal algebra for pre-service elementary school teachers of mathematics and influence of this change on beliefs, algebra comprehension, problem solving strategies and teaching practice. We expect that systematic use of pre-algebra argumentation in problem solving situations followed by further discussions on algebraic thinking should improve algebraic thinking and reasoning of future teachers. It is logical to suggest that such a progress will improve a comprehension of ideas of elementary school mathematics and provide a breakthrough in teaching strategies and evaluation of students' reasoning. Following after idea of Jaworsky (2005), we create learning situations with a clear opportunity to discover concepts in mathematics teaching.

The subject matter of consequences and especially arithmetic series is a prominent candidate to bridge arithmetic and algebra (Lee, 1996, p. 103; Zazkis & Liljedahl, 2002). Fortunately (but not occasionally), this topic provides a wide range of links between change and invariance. For problem solving of arithmetic series and related tasks, one can apply various algebraic ideas in non-formal manner. Because of our aspiration to see the conception of variants and invariants as one of global ideas of elementary mathematics, the purposes of the study are:

1. How (and in which scale) does systematic use of non-formal algebra discourse in problem solving on arithmetic series can improve comprehension of this mathematical subject itself?
2. Is there any transferability in the handling of non-formal algebra viewing and argumentation? In other words, can one detect any change in awareness and/or beliefs concerning other subjects beyond arithmetic series?

Framework of the study

Mathematical problems on arithmetic series have been elaborated in order to explain the links inside the given series and the relations between different ones. On the basis of the set of these tasks, assignments for students' activities have been prepared. Each activity typically starts with problem which is hopefully known to the student and continues to more complicated and less-familiar questions. In order to bring future teachers closer to development algebraic reasoning in school, we effort to design assignment in the "both for teachers and for students" style (Sinitsky, 2008) that invites analogies and applications to teaching in school.

The immediate goal of these activities was examining the ability of pre-service teachers to distinguish changes and invariants in sets of arithmetic sequences and to use this knowledge for computation of arithmetic series in algebraic style without algebraic formalism.

The assignments have been organized by three manners:

- Part of the assignments was a set of connected problems on arithmetic series which invites the students to use the same or similar non-formal algebra argumentation to solve the problems;
- Another portion of assignments invites some combination of various algebraic ideas to solve different problems of the assignment (including variety of ways to solve the same problem);
- In some assignments, the didactic aspects have been posed in explicit way. For example, students have been invited to compare and to evaluate two different solving strategies for the same problem on series.

Typical fragments of assignments for the study are shown in APPENDIX 1.

The study was conducted in three academic pedagogical colleges in Israel. The participants were 38 female students of Math Teaching in Elementary School, in their first

(19), second (11) and forth (8) years of study. For all the students the study has been integrated into learning course that deals with algebra or into workshop on problems solving. The tasks on arithmetic series were either a part of assignments on several subjects or separate ones. Students have solved the problems in a classroom and at home and after that they have discussed their solutions and solving strategies. Their teachers have paid a special attention for justification and for comparison of ways of solutions.

The data have been collected from pre- and post-tests that include mathematical tasks and attitudes questionnaires. The study also included lesson observations and interviews with eight students.

All mathematical problems have been presented in several slightly different editions adjusted for the audience, framework of the courses and the stage of study. For instance, almost the same task to find arithmetic series of first even numbers has been formulated in two versions. It was explicit writing of the terms ("To find the sum of first fifty even numbers 2+4+6+...+100") for pre-test – vs. word-type problem ("To find the sum of 40 first even numbers") for in-course assignment.

Throughout the learning courses each activity has included a discussion on different methods to solve a problem with pre-algebraic argumentation. We have attempted to provide friendly environment for discussions that invite ideas exchange.

RESULTS AND DISCUSSION

Specific knowledge: calculation of arithmetic series

Both in pre- and post-tests students have been asked to compute four arithmetic series preferably without use of formal algebraic formula. Additionally both in pre- and post-tests, we posed a question on relation between series. In pre-test the series were equal-length sums: the sum of 50 first natural numbers, the sum of natural numbers from 2 up to 51, the sum of 50 first even natural numbers and the sum of natural numbers from 51 to 100. Each of them has been introduced in standard way, for example 2+4+6+...+100. In two versions of post-test three arithmetic series in each one were proposed, two of them with an explicit expression and the last one in verbal form ("sum of 20 first multiples of 4" or "sum of 20 first odd numbers").

Surprisingly, in pre-test 55% of future teachers responded in form "There is some formula, but I don't remember it". In a few cases, they also try to correspond some series elements to the formula components ("$a_1 = 2$, $d = 2$, $a_n = 100$, and further we need to use a formula that I don't keep in mind"). Among all study population, only two students (5% of total subjects) have obtained correct answers for all the series by the application of relevant formula. About 40% of pre-service teachers calculated correctly one or two of four given series with typical mistakes in counting the number of addends. No one has used any checking procedure: for instance, sum of even numbers was frequently "by occurrence" an odd number.

31% of students replied pre-test questions concerning the relation between different arithmetic series. They have recognized a formal similarity ("For all of them, we need a formula to calculate"), have compared the number of addends ("There are the same number of addends") or have declared the difference of series a posteriori ("The series are different, we have obtained different results"). There were three responses on non-formal connection between given series. One student has observed correct relation between addends of the sums 1+2+...+50 and 2+4+...+100 but didn't use it to find second series. Another reply reported on connection between the sums 1+2+...+50 and 2+3+...+51 – and used it to compute the second series. In the third relevant response student made mistake and saw change only in first and last addends. Consequently, to calculate second series she added 2 to the first sum.

In post-test all students tried to compute sums for three given series. The 73% of respondents carried out suitable computation and obtained correct answers. No any difficulty has been observed for series that have been described verbally. When the last addend of the sum has been omitted, 50% of students have preferred to write out all the addends of the sum. Nevertheless, also after that redundant step they applied relevant calculating strategies and avoided straightforward summation. No one of the students did use a formula for series either with even or with odd number of addends. All the students provided some explanation (mostly schematic) and justification for their ways of computation.

Solving strategies of students

The attempt of each student to detail and to clarify her way of solving is an unambiguous result of her participation in the study. Students' strategies to compute series corresponded to several aspects of algebraic thinking at informal level.

In order to calculate the first series of the task, the participants have used the following strategies:

- To obtain the equal sums in suitable pairs they have rewrote the given sum in the inverse order. It was some variation of the given number expression that keeps the value and enables to discover of desired invariant (see Figure 1).

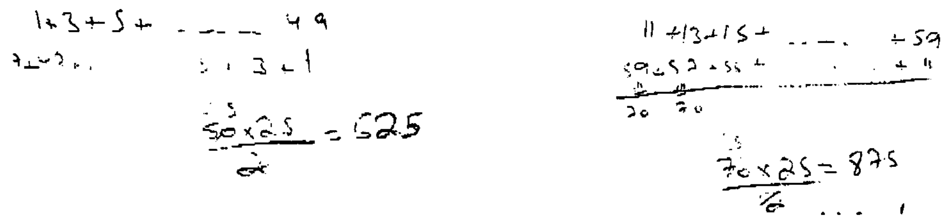

Figure 1. Arithmetic series as a half-sum of two copies of the given series

- Students have constructed suitable pairs of terms with the same sum. After that they have counted/ calculated the number of pairs and found out the product. When this way led them to the alone addend (because of odd number of terms in the given sum), they added it to the derived product (Figure 2).

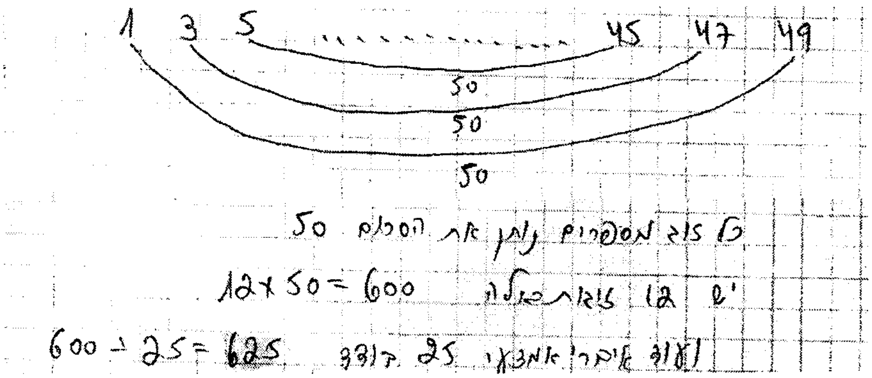

Figure 2. Arithmetic series as a sum of equal pairs, with an increment

- In some reports we found "splitting" of the given series into smaller fragments. This formalism has not been discussed with students previously. The sum of first 5 addends was "the basic one" to derive all the rest. In this way, the difference between partial sums has been obtained as the sum of variations of each one of five members of basic series (see Figure 3)

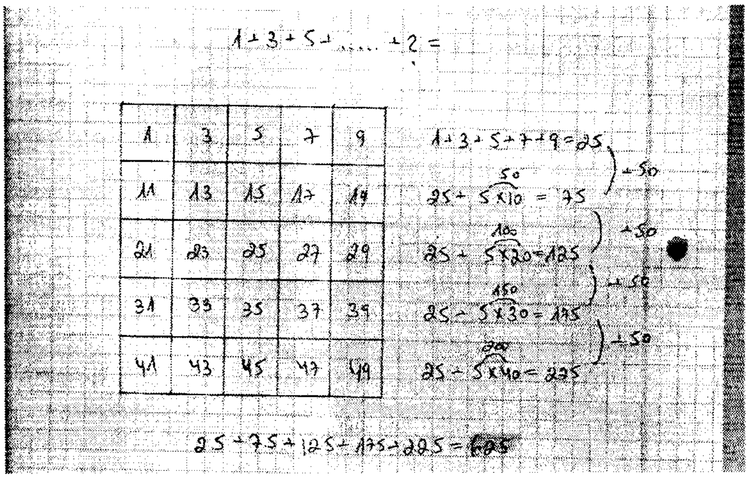

Figure 3. Calculation of arithmetic series by fragmentation. See also the pointing of the same value of change between the sums

As another variation of the same strategy one can see in Figure 4 an additional splitting of partial sums.

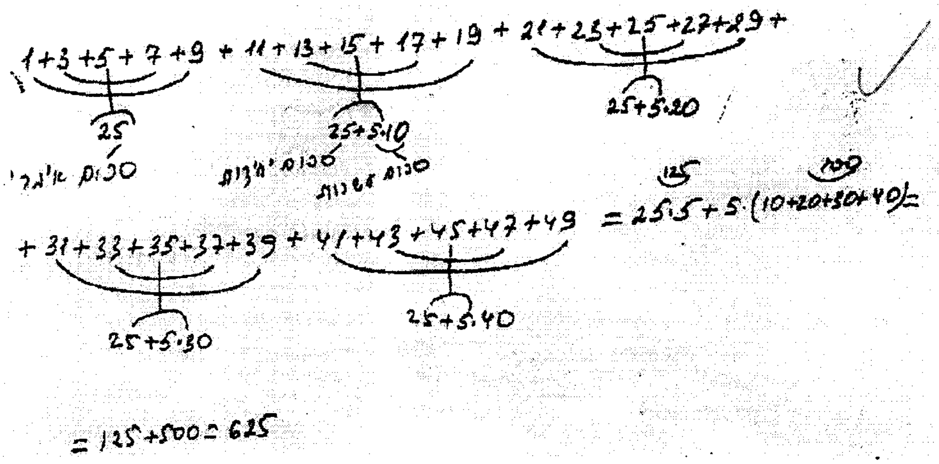

Figure 4. Calculation of arithmetic series by fragmentation. Student has designated "the sum of units" and "the sum of tens" and after that has reorganized partial sums

Some students provided multiple ways to calculate the same series: for example, the solutions that are presented in Figure 2 and Figure 3 belong to the same student.

To calculate next series of the same task, students have applied one of two main strategies (with almost equal distribution of the participants between two schemas):

- We detected an application of the same method that has been used for first series. Here one can see the transfer of the reasoning to the very similar problem. Even so, there is a comprehension of the fact that the basic invariant of arithmetic series (i.e. sum of suitable pairs of addends) stands also with a "shift" of a sequence. Several students saw and explained in relevant terms the common and the difference between new series and the first one, but didn't utilize these links for computation.

"This sequence includes the same number of terms, both of them are constructed from odd numbers and provide the same number of pairs when calculate the series. Each term of the second series is 10 greater, sum of each pair is 20 greater, and the sum is 250 greater."

Just the last passage proves that there is comparison of results rather than strategy to solve.

- Pre-service teachers have tried to connect between different series. They have described this relation in terms of "What is the change? What does not change?" and have applied it to solve the problem. Despite the unlike look, these solutions are based on the same aspect of view of variants and invariants.

A: "Let us write out and compare the members of both sequences: <u>1, 3, 5, 7, 9</u>, 11,..., 39, 41, 43, 45, 47, 49 and 11, 13, 15, 17, 19, 21, 23,...41, 43, 45, 47,49, <u>51, 53, 55, 57, 59</u>" [in the original text all the members are written]. The underlined numbers make the difference of the sequences. By comparison of these numbers we can obtain the difference in sums:

$$(51-1)+(53-3)+(55-5)+(57-7)+(59-9)=50+50+50+50+50=50\times 5=250."$$

B: "These sums have the same number of addends. Each element of the second sum is 10 greater then the appropriate element of the first one. So, the total will be $10\times 25=250$ greater."

Comprehension of early algebra and teaching practice

During the whole study we received evidence that pre-service teachers improved their awareness on non-formal algebra argumentation in solving mathematical problems. At the observations of discussions during the lessons we detected numerous attempts to recognize relation of change and invariance. Students attempted to connect different tasks and to find invariant components inside the same problem. They mostly successfully applied their knowledge in problem solving activities. The ability of pre-service teachers to solve the problems on arithmetic series is significantly improved.

However, it is evident that formation of algebraic thinking is a long-term process. At this stage we studied the changes in attitudes of pre-service teachers concerning algebraic thinking and reasoning in elementary school after they learned and dealt with pre-algebra argumentation and problem solving. For this purpose, students have been interviewed and asked with questionnaire on the issue. The fragment of attitude questionnaire is presented in APPENDIX 2.

At the beginning of the study, only 38% of future teachers chose the comprehension of relations between quantities as a main algebraic component of elementary school mathematics. Another good part (42%) supposed formal basic rules to be the substantial component of elementary school algebraic thinking. Although the "proper" stand was in a large scale a declarative one. The large part of students omitted the open questions on algebraic thinking activities. Another part of pre-service teachers gave an example of specific algebraic (linear or quadratic) equation.

Not surprisingly, nobody sampled some algebraic thinking activity for elementary school (although half of respondents mentioned general numbers properties and problems with unknown as important algebra-related topics of elementary school mathematics).

After involving in problem solving activities on arithmetic series in early algebra manner, students have responded in absolutely different way. Beyond the formal suitable choice for both attitudes questions (94% and 100% respectively), pre-service teachers were able to justify and to exemplify their answers. Even more significant, future teachers have reported in very emotional and personal oriented manner on their comprehension of the issue and its relevance for elementary school.

> B: "Previously I only knew substitution into the given formulas and thought that is algebra. All the problems on series we have deal with have an algebraic nature. They are relevant also for elementary school. In my lessons for the first grade [*during teaching practice in school as a part of pre-service training program*], I have entered the exercises on variation and invariant. For example, I asked students to compose the number 9 from two addends, to write the sums and to compare parts of these sums. In the second grade, I wrote the equality 90+69=80+? - and we have discussed what is the same and how to keep the sum by the change of addends."

> C (*Oral communication*): "Algebraic thinking includes ability to see the similar and different and to solve new problem on the basis of the previous solution. There is rather comprehension than technique. I learned to solve the problems with the non-procedural way. I think that the same way I can talk with children in the classroom. For example, I never solved the arithmetic exercises "in the row" – only long multiplication and similar. Before this study, I never thought that it is possible to play with components of the arithmetic exercise and to achieve the answer by this way. If I'll prepare proper exercises, I'll use the variation and invariant in order to teach my students to think and not to solve automatically without any comprehension"

CONCLUSION

In this article we have evaluated the change in the awareness of concepts of non-formal algebra thinking and influence of this change on beliefs on algebra, problem solving strategies and teaching practice for pre-service mathematics teachers. Arithmetic series has been chosen as a sample mathematical topic.

Before the study, the knowledge of future teachers on early algebra was very partial and mostly formal. The study demonstrates that systematic use of algebraic conception results in improving the understanding of pre-algebra. It comes up with non-formal and multiple ways to solve problems on arithmetic series and with explanations of solving strategies. Future teachers have begun to construct examples for teaching in elementary school that invite algebraic thinking and argumentation in terms of change, comparison and invariants.

Therefore, we suppose that systematic use of non-formal algebra in teachers' education might be an effective tool to bring non-formal algebraic thinking in elementary school.

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APPENDIX 1:

EXTRACTS FROM THE ASSIGNMENTS ON ARITHMETIC SERIES WITH DIFFERENT TYPES OF "CHANGE-INVARIANCE" RELATIONS

Extract from Problem 1 (variations within the given quantity).

1. Sums of equal addends and sums of consequent addends.
- Each even number is the sum of two equal addends. Also the converse is right. Complete the generalization of this proposition: "The natural number can be presented as a sum of k equal addends if and only if ..."
- According to previous paragraph, the sum of three equal numbers is the multiple of 3. What can you say on divisibility by 3 of the sum of three consecutive natural numbers? Try to prove the conjecture.
- Construct the analogies of the statement of the previous paragraph for more than three consecutive addends and evaluate the validity of these statements.

2. The way to represent the number as a sum of consequent addends.
- Get any multiple of 3 as a sum of three consecutive numbers. Is there any connection between this presentation and the decomposition of the same number into sum of three equal addends?
- What is the way to transform the sum of three equal addends into the sum of three consecutive addends?
- Apply the similar method for the sum of 5 addends.

3. Have you found any restriction for the method? Check the situation for odd and for even number of equal addends.

Extract from Problem 2 (seeking for unchanged sums).

Problem posing: How can one derive the sum of arithmetic progression without the use of the formula? In other words, what is the origin of the formula for arithmetic series?

1. Following young Gauss, find the sum of first 100 natural numbers.
- Display this sum as a sum of 50 pairs with the same sum each one. What is the reason for invariant of these sums?
- Calculate the sum.
- For explanation purposes, rewrite the given series in the reverse order and place the new sum under the given one. Are both sums the same? Why?

2. Which number can you add to the sequence in order to keep both the regularity and the sum?

3. (The "Hundred Table" is 10 by 10 matrix of consecutive natural numbers from 1 to 100).

- Choose 10 arbitrary numbers from "Hundred Table" taking only one number from each row and one number from each column.
- Find the sum of these numbers. Can you propose any additional way for this computation? Try to use an arithmetic series?
- Construct another sum of ten numbers from the table with the same restriction. What did you find?
- Explain why the sum remains the same.

Extract from Problem 3 (the same through the change).

As a starting point, calculate the sum of first ten natural numbers.

1. Comparison of the sum 2+3+4+...+11 with the previous one.

Dan says: "I see that it is sufficient to change the only first addend, in the first sum in order to receive the second one"

Ron says: "I can receive the second sum from the previous one by the same change for all the addends. They are all vary but by the same way"

- Is it possible that both of students are right? Detail the method of each one of them.
- Calculate the new sum as a result of the change of the given sum.

2. Calculation of the sum of any ten consecutive addends.

- Compare the series 11+12+13+...+20 with the series 1+2+3+...+10. What is the connection between these series? Formulate this fact in terms of change and invariant.
- Calculate the series 11+12+13+...+20.
- Generalize the procedure to calculate the arithmetic series of any ten consecutive natural numbers.

3. Propose two different procedures to find arithmetic series of 20 consecutive addends.

4. Compare the arithmetic series of the 20 first even natural numbers with the sum of the 20 first natural numbers. Calculate the series 2+4+6+...+40.

APPENDIX 2:

ATTITUDES QUESTIONNAIRE ON ALGEBRA IN ELEMENTARY SCHOOL (A FRAGMENT)

- The main component of algebraic thinking in primary school is (choose the best suitable answer):

a) The skill to use the language of letters and unknowns in order to find quantities and links over them;
b) The awareness of basic rules to deal with algebraic expressions, the ability to apply them to solve equations, inequalities and word problems;
c) The ability to describe relations between quantities by means of formulas and functions;
d) The comprehension of relations between different quantities and their properties in various situations and the ability to compare them at non-formal and formal language.

- Amongst the elementary school mathematic topics related to algebraic thinking, the important ones are (*you can choose more than one answer*):

a) The formulas to calculate perimeters, areas and volumes in geometry;
b) The problems with unknowns – as they are presented in some textbooks;
c) The general properties of numbers (magnitude estimation, parity etc) and of operations over them;
d) Algebra is not a specific subject matter in elementary school, so there are topics in elementary school mathematics that might be connected with algebraic thinking.

- Give an example of activity related to algebraic thinking.
- Give an example of activity related to algebraic thinking that is suitable for elementary school.