New Upper Matrix Bounds for the Solution of the Continuous Algebraic Riccati Matrix Equation

Richard Keith Davies, Peng Shi, and Ron Wiltshire

Abstract: In this paper, new upper matrix bounds for the solution of the continuous algebraic Riccati equation (CARE) are derived. Following the derivation of each bound, iterative algorithms are developed for obtaining sharper solution estimates. These bounds improve the restriction of the results proposed in a previous paper, and are more general. The proposed bounds are always calculated if the stabilizing solution of the CARE exists. Finally, numerical examples are given to demonstrate the effectiveness of the present schemes.

Keywords: Continuous algebraic Riccati equation (CARE), eigenvalue bound, iterative algorithm, matrix bound, similarity transformation.

1. INTRODUCTION

Consider the continuous algebraic Riccati equation (CARE):

$$A^T P + PA - PBB^T P = -Q, \tag{1}$$

where $A \in \Re^{n\times n}$ is a constant matrix, $(^T)$ denotes the transpose, $Q \in \Re^{n\times n}$ is a given symmetric positive semidefinite matrix, $B \in \Re^{n\times m}$, and the matrix $P \in \Re^{n\times n}$ is the unique symmetric positive semidefinite solution of the CARE (1). To guarantee the existence of the solution of the CARE, it is assumed that (A,B) is a stabilizable pair and (A,C) is a detectable pair, where $Q = C^T C$ and $C \in \Re^{p\times n}$.

The Continuous Riccati equation is usually employed to solve optimal control, robust control and filter design problems in control theory [27-29]. Analytical solution of this equation is of considerable time consumption and computational complexity, particularly when the dimensions of the system matrices are high. Solution bounds of this equation can ease the computational efforts required to solve this equation and give rough estimates for its actual solution. Furthermore, solution bounds can be applied to deal with practical problems involving the solution of this equation, and often the exact solution of the CARE is not required, but rather bounds of the solution. Therefore, during the past three decades, a number of researches have been presented for deriving solution bounds of this equation [1-20,23,25]. Types of solution bounds derived include bounds for the eigenvalues of the solution and the solution matrix itself. Of these findings, the matrix bounds are the most general and preferable, because they can immediately provide all extremal eigenvalue, summation, trace, product and determinant bounds. Viewing the literature, it appears that most of the upper matrix bounds have been developed under the assumption that the matrix BB^T is nonsingular. Recently, two upper matrix bounds were presented in [20], which may work for the case when BB^T is singular. The restriction for validity of these upper bounds is that there exists a positive constant α which satisfies the following matrix inequality:

$$A + A^T < 2\alpha BB^T. \tag{2}$$

This condition is always met if BB^T is nonsingular, and may, but not always, be met for the case when BB^T is singular. Given below is such an example [3, Example 2] of when the condition (2) cannot be fulfilled:

$$A = \begin{bmatrix} 1 & 3 \\ 0 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \sqrt{6} \end{bmatrix}.$$

For these matrices, it is required to find a positive value of α such that

$$\begin{bmatrix} 2 & 3 \\ 3 & -14 \end{bmatrix} < 2\alpha \begin{bmatrix} 0 & 0 \\ 0 & 6 \end{bmatrix}.$$

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For this case, it is seen that there does not exist any such α that will satisfy (2). As such, the upper bounds proposed in [20] cannot work for this case. In this note, we shall develop two new, similar upper matrix bounds for the solution of the CARE based on the results of [20], together with the idea of a similarity transformation. The derivation of these results also make use of the fact that the stabilizability of the pair (A,B) means there will always exist some matrix K such that $A+BK$ is stable. This is a well-known fact in control theory, and has been used in a number of research papers, such as in [23] and [24]. After the derivation of each bound, an iterative algorithm is developed that can be used to derive tighter upper matrix bounds for the solution of the CARE (1). In comparison, it is seen that the results of [20] are merely special cases of these results. Therefore, this work can be considered to be a generalization of the work of [20]. Finally, we give numerical examples to demonstrate the effectiveness of our results and give comparisons with existing results.

The following symbol conventions are used in this note. $\Re$ denotes the real number field; the inequality $A>(\geq)B$ means $A-B$ is a positive (semi)definite matrix; $\lambda_i(A)$ denotes the i th eigenvalue of a symmetric matrix A for $i=1,2,\ldots n$ whereas $\lambda_i(A)$ is arranged in the nonincreasing order (i.e., $\lambda_1(A)\geq\lambda_2(A)\geq\ldots\geq\lambda_n(A)$); $\mu(A)$ denotes the matrix measure of matrix A and $Re(\lambda(A))$ is the real part of an eigenvalue of matrix A; the identity matrix with appropriate dimensions is represented by I.

Before developing the main results, we shall review the following useful lemmas:

Lemma 1 [21]: For any symmetric matrices $X,Y\in\Re^{n\times n}$ and $1\leq i,\ j\leq n$, the following inequality holds:

$$\lambda_{i+j-1}(X+Y)\leq\lambda_j(X)+\lambda_i(Y),\quad i+j\leq n+1.$$

Lemma 2 [21]: For any symmetric matrix $X\in\Re^{n\times n}$, the following inequality holds:

$$X\leq\lambda_1(X)I.$$

Lemma 3 [21]: For any matrix $A\in\Re^{n\times m}$ and any positive semidefinite matrices $X,Y\in\Re^{n\times n}$ such that $X\geq Y>(\geq)0$, it holds that $A^TXA\geq A^TYA$, with strict inequality if X and Y are positive definite and A is of full rank.

2. MAIN RESULTS

In this section we shall develop the main results of this paper. First, we shall derive an upper matrix bound for the CARE, followed by the development of an iterative algorithm which can be used to derive tighter upper matrix bounds for the CARE. Then, we shall derive a second upper matrix bound for the solution of the CARE, also accompanied by a second iterative algorithm which can also be used to derive more precise upper matrix bounds for the solution of the CARE.

Theorem 1: Let P be the positive semidefinite solution of the CARE (1) and let $\overline{P}$ be the positive semidefinite solution of the modified CARE (8). Then P and $\overline{P}$ have the upper bounds

$$\overline{P}\leq W^{-T}(\rho[(W+I)^T(W+I)+I]+M^{-T}(Q+K^TK)M^{-1})W^{-1}\equiv\overline{P}_{U1} \tag{3}$$

and

$$P\leq M^TW^{-T}(\rho[(W+I)^T(W+I)+I]+M^{-T}(Q+K^TK)M^{-1})W^{-1}M\equiv P_{U1} \tag{4}$$

respectively, where the positive constant ρ and matrix W are defined, respectively, by

$$\rho\equiv\frac{\lambda_1[W^{-T}M^{-T}(Q+K^TK)M^{-1}W^{-1}]}{1-\lambda_1\{W^{-T}[(W+I)^T(W+I)+I]W^{-1}\}} \tag{5}$$

$$W\equiv M(A+BK)M^{-1}-I \tag{6}$$

and the matrix $K\in\Re^{m\times n}$ is chosen to stabilize $A+BK$, and the nonsingular matrix $M\in\Re^{n\times n}$ is chosen such that

$$M(A+BK)M^{-1}+M^{-T}(A+BK)M^T<0. \tag{7}$$

Proof: Using the similarity transformation, we can define the following similarity variables:

$$\overline{A}=MAM^{-1},\quad \overline{B}=MB,$$
$$\overline{Q}=M^{-T}QM^{-1},\quad \overline{P}=M^{-T}PM^{-1},$$

where M is a nonsingular matrix. Using these similarity variables, the CARE (1) is equivalent to the following modified CARE:

$$\overline{A}^T+\overline{P}\overline{A}-\overline{P}\overline{B}\overline{B}^T\overline{P}=-\overline{Q}. \tag{8}$$

By use of the matrix identity

$$\overline{P}\overline{B}\overline{B}^T\overline{P}=[KM^{-1}+\overline{B}^T\overline{P}]^T[KM^{-1}+\overline{B}^T\overline{P}]-M^{-T}K^TKM^{-1}-M^{-T}K^T\overline{B}^T\overline{P}-\overline{P}\overline{B}KM^{-1},$$

where $K\in\Re^{m\times n}$, the modified CARE (8) becomes

$$\overline{P}(\overline{A}+\overline{B}KM^{-1})+(\overline{A}+\overline{B}KM^{-1})^T\overline{P}$$

$$
\begin{aligned}
&+\overline{Q}+M^{-T}K^{T}KM^{-1}\\
&=[KM^{-1}+\overline{B}^{T}\overline{P}]^{T}[KM^{-1}+\overline{B}^{T}\overline{P}].
\end{aligned}
\tag{9}
$$

With the aid of the identity

$$
\begin{aligned}
W^{T}\overline{P}W=(W+I)^{T}\overline{P}(W+I)-\overline{P}(\overline{A}+\overline{B}KM^{-1})\\
-(\overline{A}+\overline{B}KM^{-1})^{T}\overline{P}+\overline{P},
\end{aligned}
$$

where the matrix W is defined by (6), (9) can be rewritten as

$$
\begin{aligned}
&(W+I)^{T}\overline{P}(W+I)+\overline{P}+M^{-T}(Q+K^{T}K)M^{-1}\\
&=W^{T}\overline{P}W+[KM^{-1}+\overline{B}^{T}\overline{P}]^{T}[KM^{-1}+\overline{B}^{T}\overline{P}]\\
&\geq W^{T}\overline{P}W.
\end{aligned}
\tag{10}
$$

Since Re $(\lambda(M(A+BK)M^{-1}))\leq\mu(M(A+BK)M^{-1})$ $=\frac{1}{2}\lambda_1(M(A+BK)M^{-1}+M^{-T}(A+BK)M^{T})$, one can see that satisfaction of condition (7) ensures the nonsingularity of W. Using Lemma 3, (10) becomes

$$
\begin{aligned}
\overline{P}\leq W^{-T}[(W+I)^{T}\overline{P}(W+I)+\overline{P}\\
+M^{-T}(Q+K^{T}K)M^{-1}]W^{-1}.
\end{aligned}
\tag{11}
$$

Utilizing Lemma 2, (11) becomes

$$
\begin{aligned}
\overline{P}\leq W^{-T}[(W+I)^{T}(W+I)+I]W^{-1}\lambda_1(\overline{P})\\
+W^{-T}[M^{-T}(Q+K^{T}K)M^{-1}]W^{-1}.
\end{aligned}
\tag{12}
$$

Application of Lemma 1 to (12) leads to

$$
\begin{aligned}
&\lambda_1(\overline{P})\\
&\leq\lambda_1\{W^{-T}[(W+I)^{T}(W+I)+I]W^{-1}\lambda_1(\overline{P})\\
&\quad+W^{-T}[M^{-T}(Q+K^{T}K)M^{-1}]W^{-1}\}\\
&\leq\lambda_1\{W^{-T}[(W+I)^{T}(W+I)+I]W^{-1}\}\lambda_1(\overline{P})\\
&\quad+\lambda_1\{W^{-T}[M^{-T}(Q+K^{T}K)M^{-1}]W^{-1}\}.
\end{aligned}
\tag{13}
$$

Using the well-known fact that $\lambda_i(XY)=\lambda_i(YX)$ for any $X,Y\in\Re^{n\times n}$ and $i=1,2,\ldots,n$, we have

$$
\begin{aligned}
&\lambda_1\{W^{-T}[(W+I)^{T}(W+I)+I]W^{-1}\}\\
&\quad=\lambda_1\{[(W+I)^{T}(W+I)+I](W^{T}W)^{-1}\}\\
&\quad=\lambda_1\{[W^{T}W+W+W^{T}+2I](W^{T}W)^{-1}\}.
\end{aligned}
$$

If $W^{T}W+W+W^{T}+2I<W^{T}W$, then it is seen that

$$
\lambda_1\{W^{-T}[(W+I)^{T}(W+I)+I]W^{-1}\}<1,
$$

which is equivalent to the condition (7). Under the satisfaction of condition (7), (13) implies

$$
\begin{aligned}
\lambda_1(\overline{P})&\leq\frac{\lambda_1[W^{-T}M^{-T}(Q+K^{T}K)M^{-1}W^{-1}]}{1-\lambda_1\{W^{-T}[(W+I)^{T}(W+I)+I]W^{-1}\}}\\
&\equiv\rho.
\end{aligned}
\tag{14}
$$

Substituting (14) into (12) into leads to the upper bound (3) for the solution of the modified CARE (8). Now, since $\overline{P}=M^{-T}PM^{-1}$ and $\overline{P}\leq\overline{P}_{U1}$, a further application of Lemma 3 to the upper bound (3) leads to the upper bound (4) for the solution of the CARE (1). This completes the proof of the theorem. □

Having developed Theorem 1, the following iterative algorithm can be used to get together upper matrix bounds for the solution of the CARE (1), by first getting tighter upper matrix bounds for the solution of the modified CARE (8).

Algorithm 1:

Step 1: Set $\overline{R}_0\equiv\overline{P}\leq\overline{P}_{U1}$

Step 2: Compute

$$
\begin{aligned}
\overline{R}_{k+1}=W^{-T}[(W+I)^{T}\overline{R}_k(W+I)+\overline{R}_k\\
+M^{-T}(Q+K^{T}K)M^{-1}]W^{-1},\ k=0,1,\ldots.
\end{aligned}
\tag{15}
$$

The matrices $\overline{R}_k$ are also upper solution bounds of the modified CARE (8). Then, $R_k=M^{T}\overline{R}_kM$ are upper bounds for the solution of the CARE (1).

Proof: Set $k=0$ in (15) to get:

$$
\begin{aligned}
\overline{R}_1=W^{-T}[(W+I)^{T}\overline{R}_0(W+I)+\overline{R}_0\\
+M^{-T}(Q+K^{T}K)M^{-1}]W^{-1}.
\end{aligned}
\tag{16}
$$

Using Lemma 2, we have from (16) that

$$
\begin{aligned}
\overline{R}_1\leq W^{-T}([(W+I)^{T}(W+I)+I]\lambda_1(\overline{R}_0)\\
+M^{-T}(Q+K^{T}K)M^{-1})W^{-1}.
\end{aligned}
\tag{17}
$$

By the definition of $\overline{R}_0\equiv\overline{P}_{U1}$ and the upper bound (3), we have

$$
\begin{aligned}
&\lambda_1(\overline{R}_0)\\
&\leq\lambda_1\{W^{-T}(\rho[(W+I)^{T}(W+I)+I]\\
&\quad+M^{-T}(Q+K^{T}K)M^{-1})W^{-1}\}\\
&\leq\lambda_1\{W^{-T}[(W+I)^{T}(W+I)+I]W^{-1}\}\rho\\
&\quad+\lambda_1\{[W^{-T}M^{-T}(Q+K^{T}K)M^{-1}]W^{-1}\}\\
&=\left\{1-\frac{\lambda_1[W^{-T}M^{-T}(Q+K^{T}K)M^{-1}M^{-1}]}{\rho}\right\}\rho\\
&\quad+\lambda_1[W^{-T}M^{-T}(Q+K^{T}K)M^{-1}M^{-1}]\\
&=\rho,
\end{aligned}
\tag{18}
$$

where Lemma 1, the condition (7) and (5) have been

employed. Substituting (18) into (17) gives

$$\begin{aligned}\overline{R}_1 &\le W^{-T}(\rho[(W+I)^T(W+I)+I] \\ &\quad + M^{-T}(Q+K^TK)M^{-1})W^{-1} \equiv \overline{R}_0.\end{aligned} \tag{19}$$

Now assume that $\overline{R}_k \le \overline{R}_{k-1}$. Then

$$\begin{aligned}\overline{R}_{k+1} &= W^{-T}[(W+I)^T\overline{R}_k(W+I)+\overline{R}_k \\ &\quad + M^{-T}(Q+K^TK)M^{-1}]W^{-1} \\ &\le W^{-T}[(W+I)^T\overline{R}_{k-1}(W+I)+\overline{R}_{k-1} \\ &\quad + M^{-T}(Q+K^TK)M^{-1}]W^{-1} \equiv \overline{R}_k.\end{aligned}$$

Using mathematical induction, one can reach the conclusion that

$$\overline{R}_k \le \overline{R}_{k-1} \le \ldots \le \overline{R}_1 \le \overline{R}_0. \tag{20}$$

Using Lemma 3 with the fact that $\overline{R}_k = M^{-T}\overline{R}_k M^{-1}$, (19) is equivalent to

$$R_k \le R_{k-1} \le \ldots \le R_1 \le R_0.$$

This completes the proof of the algorithm.

A second upper matrix bound is obtained as follows.

Theorem 2: Let P and $\overline{P}$ be the positive semidefinite solution of the CARE (1) and modified CARE (8) respectively. For any matrices K and M chosen to fulfil the condition (7), $\overline{P}$ and P have the respective upper bounds

$$\begin{aligned}\overline{P} &\le W^{-T}[\rho(W+2I)^T(W+2I) \\ &\quad + 2M^{-T}(Q+K^TK)M^{-1}]W^{-1} \equiv \overline{P}_{U2}\end{aligned} \tag{21}$$

and

$$\begin{aligned}P &\le M^TW^{-T}[\rho(W+2I)^T(W+2I) \\ &\quad + 2M^{-T}(Q+K^TK)M^{-1}]W^{-1}M \equiv P_{U2},\end{aligned} \tag{22}$$

where the positive constant ρ and matrix W are defined by (5) and (6) respectively.

Proof: Using the definition of W from (5), (9) can be rewritten as

$$\begin{aligned}&(W+I)^T\overline{P}+\overline{P}(W+I)+M^{-T}(Q+K^TK)M^{-1} \\ &=[KM^{-1}+\overline{B}^T\overline{P}]^T[KM^{-1}+\overline{B}^T\overline{P}].\end{aligned} \tag{23}$$

Multiplying both sides of (22) by 2 and adding $W^T\overline{P}W$ to both sides gives:

$$\begin{aligned}&W^T\overline{P}W+2W^T\overline{P}+2\overline{P}W+4P \\ &\quad +2M^{-T}(Q+K^TK)M^{-1} \\ &=W^T\overline{P}W+[KM^{-1}+\overline{B}^T\overline{P}]^T[KM^{-1}+\overline{B}^T\overline{P}] \\ &\ge W^T\overline{P}W.\end{aligned} \tag{24}$$

By realizing that

$$\begin{aligned}&W^T\overline{P}W+2W^T\overline{P}+2\overline{P}W+4P \\ &=(W+2I)^T\overline{P}(W+2I)\end{aligned}$$

(23) becomes

$$\begin{aligned}W^T\overline{P}W &\le (W+2I)^T\overline{P}(W+2I) \\ &\quad +2M^{-T}(Q+K^TK)M^{-1}.\end{aligned} \tag{25}$$

By use of Lemma 3, (24) becomes

$$\begin{aligned}\overline{P} &\le W^{-T}[(W+2I)^T\overline{P}(W+2I) \\ &\quad +2M^{-T}(Q+K^TK)M^{-1}]W^{-1}.\end{aligned} \tag{26}$$

Applying Lemma 2 to (25) results in

$$\begin{aligned}\overline{P} &\le W^{-T}[\lambda_1(\overline{P})(W+2I)^T(W+2I) \\ &\quad +2M^{-T}(Q+K^TK)M^{-1}]W^{-1}.\end{aligned} \tag{27}$$

Utilizing Lemma 1, (26) becomes

$$\begin{aligned}&\lambda_1(\overline{P}) \\ &\le \lambda_1\{W^{-T}[\lambda_1(\overline{P})(W+2I)^T(W+2I) \\ &\quad +2M^{-T}(Q+K^TK)M^{-1}]W^{-1}\} \\ &\le \lambda_1(\overline{P})\lambda_1\{W^{-T}[\lambda_1(\overline{P})(W+2I)^T(W+2I)W^{-1}\} \\ &\quad +2\lambda_1\{W^{-T}[M^{-T}(Q+K^TK)M^{-1}]W^{-1}\}.\end{aligned} \tag{28}$$

Along the lines of Theorem 1, it can be seen that under the satisfaction of condition (7), $\lambda_1\{W^{-T}(W+2I)^T(W+2I)W^{-1}\} < 1$ and (27) then implies that

$$\begin{aligned}\lambda_1(\overline{P}) &\le \frac{2\lambda_1[W^{-T}M^{-T}(Q+K^TK)M^{-1}W^{-1}]}{1-\lambda_1\{W^{-T}(W+2I)^T(W+2I)W^{-1}\}} \\ &= \frac{\lambda_1[W^{-T}M^{-T}(Q+K^TK)M^{-1}W^{-1}]}{1-\lambda_1\{W^{-T}[(W+I)^T(W+I)+I]W^{-1}\}} \\ &\equiv \rho,\end{aligned} \tag{29}$$

where ρ is defined by (5). Substituting (28) into (26) leads to the upper bound (20) for the solution of the modified CARE (8). Application of Lemma 3 to (20) results in the upper bound (21) for the solution of the CARE (1). This finishes the proof of the theorem.

Having derived a second upper matrix bound for the solution of the CARE (1), we can propose the following iterative algorithm to derive tighter upper matrix bounds for the solution of the CARE (1).

Algorithm 2:

Step 1: Set $\overline{S}_0 \equiv \overline{P}_{U2}$.

Step 2: Compute

$$\bar{S}_{k+1} = W^{-T}[(W+2I)^T \bar{S}_k (W+2I) + 2M^{-T}(Q+K^TK)M^{-1}]W^{-1}, \quad k=0,1,\ldots.$$

Then, $\bar{S}_k$ are also upper solution bounds of the modified CARE (8). Then, $S_k = M^T \bar{S}_k M$ are upper solution bounds of the CARE (1).

Remark 1: From (22) and (4), we have

$$\begin{aligned} P_{U2} &= M^T W^{-T}[\rho(W+I)^T(W+I) \\ &\quad + M^{-T}(Q+K^TK)M^{-1} + \rho(W+W^T+2I) \\ &\quad + M^{-T}(Q+K^TK)M^{-1}]W^{-1}M \\ &= P_{U1} + M^T W^{-T}[\rho(W+W^T+2I) \\ &\quad + M^{-T}(Q+K^TK)M^{-1}]W^{-1}M. \end{aligned}$$

As such, if $\rho(W+W^T+2I)+M^{-T}(Q+K^TK)M^{-1} \geq 0$ then P_{U1} is tighter than P_{U2}, whereas if $\rho(W+W^T+2I)+M^{-T}(Q+K^TK)M^{-1} \leq 0$ then P_{U2} is tighter than P_{U1}.

Remark 2: In fact, there always exists matrices K and D that satisfy the condition (7), so the upper bounds of this work are always calculable if the solution of the CARE exists. Using Lemma 3, it can be seen that condition (7) and the following condition

$$P_M(A+BK)+(A+BK)^T P_M < 0, \tag{30}$$

where $P = M^T M$, are equivalent. Since the pair (A,B) are assumed to be stabilizable, there will always exist a matrix K stabilizing $A+BK$. Then, since $A+BK$ is stable, there always exist a positive definite matrix P_M yielding (29) by the Lyapunov Theorem. As such, there will always exist matrices K and M rendering the conditions (7) and (29). Therefore, the bounds proposed here can always work if the stabilizing solution of the CARE (1) exists. We may use the following procedure to test the satisfaction of condition (29), and hence condition (7):

Step 1: Choose a matrix $K \in \Re^{m\times n}$ so that $A+BK$ is stable, i.e., so that $\mathrm{Re}\,\lambda_i(A+BK) < 0$ $\forall i$. There are a number of methods in the literature that can be used to construct K to arbitrarily assign the eigenvalues of $A+BK$, see for example [26].

Step 2: Choose a symmetric positive definite matrix P_M so that (29) is satisfied.

Step 3: From knowledge of P_M, a possibility for M is $M = P_M^{1/2}$. The square root of P_M may be found be several methods, see for example [21] and the references therein.

Having chosen a matrix K that stabilizes $A+BK$, an alternative to Step 2 of the above procedure is to solve a Lyapunov equation of the form

$$P_M(A+BK)+(A+BK)^T P_M = -kI,$$

where k is any positive constant and P_M is the solution matrix.

Remark 3: When $M = I$ and $K = -\alpha B^T$, where α is a positive constant, the upper bounds (4) and (22) derived in this note decompose into the upper bounds presented in [20], as do the iterative algorithms. The theorems and algorithms presented above are based on [20] and modified to cover the case that the restriction (3) of [20] is not valid. Application of this modification leads to less restrictive, more general results. The tightness between the bounds reported in [20] and those presented here depend, respectively, on the choices of the matrices K and M for our bounds and the value of the positive constant α for the bounds of [20].

Remark 4: The following upper matrix bound for the CARE (1) was derived in [19]:

$$P \leq E^{-1}[E(Q+A^TU_1A)E]^{1/2}E^{-1} \equiv P_{U3}, \tag{31}$$

where the positive definite matrix U_1 is chosen such that $E \equiv (BB^T - U_1^{-1})^{1/2} > 0$. With a suitable choice of U_1, it was shown in [19] that the bound (30) is tighter than the parallel results proposed in [16] and [17], and that the corresponding eigenvalue bounds are also sharper as a result. To ease the calculation of bound (30), some choices of U_1 were listed in [19], together with the range of tuning parameters involved. These choices are re-listed in the table at the end of this paper. As it was mentioned above, most of the existing upper matrix bounds for the CARE have to assume that the matrix BB^T is nonsingular for them to be able to work. Recently, two upper matrix bounds were derived in [20], which may work for the case when BB^T is singular, but are only valid under a condition that is more conservative than the fundamental existence conditions for the solution of the CARE. Our upper bounds can always work if the solution of the CARE exists. It appears that the tightness between most of the existing results and those presented here cannot be compared mathematically. They do, however, provide a supplement to each other.

Remark 5: In [22], an iterative technique was proposed to solve the CARE (1). This technique shall be re-stated as follows:

Step 1: Choose a positive definite matrix P_0 such that $A - BB^T P_0$ is stable.

Step 2: Let P_N be the solution of the following Lyapunov-type matrix equation:

$$P_N(A-BB^TP_{N-1})+(A-BB^TP_{N-1})^TP_N$$
$$=-(Q+P_{N-1}BB^TP_{N-1}),\ N=1,2,\ldots.$$

Then, $\lim_{N\to\infty}P_N=P$, where P is the solution of the CARE (1). In [23], it is seen that if P_U is an upper bound of P, then the matrix $A-BB^TP_U$ is stable, so we could choose the presented upper bounds P_{U1} or P_{U2} as the initial matrix P_0 and solve the CARE (1) by this algorithm. This is yet another application of the solution bounds, and this can also reduce the possible conservativeness of the bounds in sense of tightness.

Remark 6: From (9), (10) and the standard results of optimal control theory, it is seen that the choice of the matrix K which results in the optimal upper bound for the solution of the CARE is $K=-B^TP$. One way to deal with the possible conservativeness of the tightness of the proposed upper bounds is to utilize the algorithm of Remark 5, using the upper bounds presented in this note as the starting point.

3. NUMERICAL EXAMPLES

In this section, we shall give two numerical examples to demonstrate the effectiveness of the proposed results of this paper. The first example will concentrate on the case BB^T is singular, whilst the second example will focus on the case BB^T is nonsingular. In both examples, use is made of the algorithm stated in

Example 1: BB^T is singular [3, Example 2]
Consider the CARE (1) with:

$$A=\begin{bmatrix}1 & 3\\ 0 & -7\end{bmatrix},\ B=\begin{bmatrix}0\\ \sqrt{6}\end{bmatrix},\ Q=\begin{bmatrix}8 & 0\\ 0 & 8\end{bmatrix}.$$

Then the unique positive definite solution of the CARE (1) is:

$$P_{\text{exact}}=\begin{bmatrix}8 & 2\\ 2 & 1\end{bmatrix}.$$

Since the matrix BB^T is singular for this example, the upper matrix bounds reported in [16,17,19] cannot work for this case. Also, the bounds presented in [20] cannot work here, because the restriction (2) cannot be met for any value of the positive constant α. However, our results can be applied to this case.

With $K=\begin{bmatrix}-\frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}}\end{bmatrix}$ and $M=\begin{bmatrix}2.3942 & 0.8172\\ 0.8172 & 1.0322\end{bmatrix}$, the upper bounds P_{U1} and P_{U2} are found by Theorems 1 and 2 to be:

$$P_{U1}=\begin{bmatrix}40.5290 & 19.2792\\ 19.2792 & 13.3086\end{bmatrix},$$
$$P_{U2}=\begin{bmatrix}20.2089 & 11.9373\\ 11.9373 & 10.1383\end{bmatrix}.$$

Here, we have that P_{U2} is the tighter bound. Using 2 iterations of Algorithm 2, we can derive the following tighter upper matrix bounds for the solution of the CARE (1):

$$S_1=\begin{bmatrix}15.6829 & 7.3026\\ 7.3026 & 5.2766\end{bmatrix},$$
$$S_2=\begin{bmatrix}15.5580 & 7.0187\\ 7.0187 & 4.5810\end{bmatrix}.$$

It is seen that as more iterations are performed, the bounds become tighter. By using the upper bound P_{U2} as the initial matrix P_0, three iterations of the algorithm stated in Remark 5 provides the following, tighter approximations to the actual solution of the CARE:

$$P_1=\begin{bmatrix}6.5056 & 6.1153\\ 6.1153 & 4.8165\end{bmatrix},$$
$$P_2=\begin{bmatrix}9.5227 & 3.4262\\ 3.4262 & 1.9381\end{bmatrix},$$
$$P_3=\begin{bmatrix}8.5563 & 2.3239\\ 2.3239 & 1.1939\end{bmatrix}.$$

Even though the upper bound P_{U2} is somewhat conservative in sense of tightness, the iterates above demonstrate that even with a conservative starting point, the actual solution can fast be approached.

Example 2: BB^T is nonsingular [19, Example]
Consider the CARE (1) with:

$$A=\begin{bmatrix}-3 & 0.5\\ 0.1 & 0.2\end{bmatrix},\ B=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix},\ Q=\begin{bmatrix}3 & 0.2\\ 0.2 & 3\end{bmatrix}.$$

Then the unique positive definite solution of the CARE (1) is:

$$P_{\text{exact}}=\begin{bmatrix}0.3967 & 0.0936\\ 0.0936 & 1.9603\end{bmatrix}.$$

With $K=\begin{bmatrix}-1 & -0.25\\ -0.1 & -2\end{bmatrix}$ and $M=I$, the upper bounds P_{U1} and P_{U2} are found by Theorems 1 and 2 to be:

$$P_{U1}=\begin{bmatrix}2.4579 & 0.0387\\ 0.0837 & 2.6609\end{bmatrix},$$

$$P_{U2} = \begin{bmatrix} 1.2701 & -0.0162 \\ -0.0162 & 2.0728 \end{bmatrix}.$$

Here, it is seen that P_{U2} is the tighter bound. Using 2 iterations of Algorithm 2, we can obtain the following tighter upper matrix bounds for the solution of the CARE (1):

$$S_1 = \begin{bmatrix} 0.9636 & 0.0921 \\ 0.0921 & 1.9708 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 0.6511 & 0.0949 \\ 0.0949 & 1.9625 \end{bmatrix}.$$

It can be seen that as more iterations of the algorithm are carried out, the bounds become tighter. By using the upper bound P_{U2} as the initial matrix P_0, three iterations of the algorithm stated in Remark 5 provides the following, tighter approximations to the actual solution of the CARE:

$$P_1 = \begin{bmatrix} 0.5859 & 0.0647 \\ 0.0647 & 1.9678 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.4102 & 0.0910 \\ 0.0910 & 1.9609 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0.3967 & 0.0909 \\ 0.0909 & 1.9601 \end{bmatrix}.$$

As in Example 1, the above iterates demonstrate that the algorithm stated in Remark 5 can deal with the possible conservativeness of the solution bounds proposed in this paper.

Since the upper bounds proposed in [20] are merely special cases of our upper bounds, we will not make a comparison with them. Instead, we shall compare our upper bounds only with the upper bound P_{U3} reported in [19]. Table 1 summarizes choices of the positive definite matrix U_1 for the bound P_{U3}.

Table 1. Choices of the matrix U_1 and the range of parameter ε.

U_1	Range of parameter ε
$(BB^T - \varepsilon I)^{-1}$	$0 < \varepsilon < \lambda_n(BB^T)$
$(BB^T - \varepsilon Q^{-2})^{-1}$	$0 < \varepsilon < \lambda_n(BB^T Q^2)$
$(BB^T - \varepsilon Q^2)^{-1}$	$0 < \varepsilon < \lambda_n(BB^T Q^{-2})$
$\frac{1}{\varepsilon} I$	$0 < \varepsilon < \lambda_n(BB^T)$
εQ	$\varepsilon > (Q^{-1}(BB^T)^{-1})$
εQ^{-1}	$\varepsilon > (Q(BB^T)^{-1})$
$\varepsilon(AA^T)^{-1}$	$\varepsilon > \lambda_1(A(BB^T)^{-1}A^T)$
$\varepsilon(AQA^T)^{-1}$	$\varepsilon > \lambda_1(QA(BB^T)^{-1}A^T)$

Using these choices, we shall now compare our bounds with the bound P_{U3} reported in [19]:

$$P_{U3} = \begin{bmatrix} 2.5444 & 0.0249 \\ 0.0249 & 2.0213 \end{bmatrix}$$

when $\varepsilon = 0.95$ and $U_1 = (BB^T - \varepsilon I)^{-1}$

$$P_{U3} = \begin{bmatrix} 15.8235 & 0.0575 \\ 0.0575 & 11.8727 \end{bmatrix}$$

when $\varepsilon = 5$ and $U_1 = (BB^T - \varepsilon Q^{-2})^{-1}$

$$P_{U3} = \begin{bmatrix} 0.1766 & -0.0150 \\ -0.0150 & 0.1766 \end{bmatrix}$$

when $\varepsilon = 0.05$ and $U_1 = (BB^T - \varepsilon Q^2)^{-1}$

$$P_{U3} = \begin{bmatrix} 2.4484 & -0.2801 \\ -0.2801 & 2.5712 \end{bmatrix}$$

when $\varepsilon = 0.5$ and $U_1 = \frac{1}{\varepsilon} I$

$$P_{U3} = \begin{bmatrix} 2.2249 & -0.2827 \\ -0.2827 & 3.1449 \end{bmatrix}$$

when $\varepsilon = 0.5$ and $U_1 = \varepsilon Q$

$$P_{U3} = \begin{bmatrix} 2.1683 & -0.4448 \\ -0.4448 & 3.6393 \end{bmatrix}$$

when $\varepsilon = 4$ and $U_1 = \varepsilon Q^{-1}$

$$P_{U3} = \begin{bmatrix} 2.5611 & -0.0465 \\ -0.0465 & 2.4729 \end{bmatrix}$$

when $\varepsilon = 3$ and $U_1 = \varepsilon(A^T A)^{-1}$

$$P_{U3} = \begin{bmatrix} 2.5611 & -0.0465 \\ -0.0465 & 2.4729 \end{bmatrix}$$

when $\varepsilon = 8$ and $U_1 = \varepsilon(A^T QA)^{-1}$.

In view of the above numerical experiments, it appears that our bounds give more precise solution estimates than the bound P_{U3} for this case.

4. CONCLUSIONS

The derivation of new upper matrix bounds for the solution of the CARE is the focus of this paper. Following the derivation of each bound, an iterative algorithm was proposed to derive sharper upper

matrix bounds. The bounds reported here are based on the results of [20], in which the results of [20] are extended and generalized. The restriction for validity (3) of [20] has been removed, and replaced with the less restrictive condition (7) of this note. The advantage of these new upper matrix bounds for the CARE over existing results is that they are always calculable if the stabilizing solution of the CARE exists, whereas existing upper matrix bounds might not be calculated, because they require other restrictions for validity in addition to the existence conditions for the CARE. When using the similarity transformation, the upper bounds for the solution of the CARE retain their validity. The numerical examples demonstrate that the presented bounds are tighter than existing ones for some cases. From (9) and the standard results of optimal control theory, it is known that the choice of K which gives the best upper solution bounds is $K = -B^T P$, but there remains the question as to which choice of the matrix D gives the best upper bounds. It is expected that future research will propose a method that can determine which choice of the matrices K and D give the best upper bounds for the CARE solution, which could also cope with the possible conservativeness of the upper solution bounds, as seen in the numerical examples. The problem of determining which choice of K and D result in the optimal upper matrix bounds could be considered as an optimization problem.

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