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LOCALLY PSEUDO-VALUATION DOMAINS OF THE FORM $D[X]_{N_v}$

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Abstract. Let D be an integral domain, X an indeterminate over D, $N_v = \{f \in D[X] | (A_f)_v = D\}$. Among other things, we introduce the concept of t-locally PVDs and prove that $D[X]_{N_v}$ is a locally PVD if and only if D is a t-locally PVD and a UMT-domain, if and only if $D[X]$ is a t-locally PVD, if and only if each overring of $D[X]_{N_v}$ is a locally PVD.

1. Introduction

Throughout this paper, D denotes an integral domain, $qf(D)$ is the quotient field of D, $\bar{D}$ is the integral closure of D in $qf(D)$, X is an indeterminate over D, and $D[X]$ is the polynomial ring. An *overring* of D is a ring between D and $qf(D)$. The *content* of a polynomial $f \in K[X]$, denoted by A_f, is the fractional ideal of D generated by the coefficients of f.

Let $K = qf(D)$ and let $\mathbf{F}(D)$ (resp., $f(D)$) be the set of nonzero (resp., nonzero finitely generated) fractional ideals of D. Obviously, $f(D) \subseteq \mathbf{F}(D)$ and equality holds if and only if D is a Noetherian domain. For any $A \in \mathbf{F}(D)$, define $A_v = (A^{-1})^{-1}$, where $A^{-1} = \{x \in K | xA \subseteq D\}$, and let $A_t = \cup\{I_v | I \subseteq A$ and $I \in f(D)\}$ and $A_w = \{x \in K | xJ \subseteq A$ for $J \in f(D)$ with $J^{-1} = D\}$. For $* = v, t$ or w, we say that A is a *$*$-ideal* if $A_* = A$, while A is a *maximal $*$-ideal* if A is maximal among proper integral $*$-ideals of D. Let $*$-Max(D) be the set of all maximal $*$-ideals of D. It is well known that if $* = t$ or w, then each maximal $*$-ideal is a prime ideal; each proper integral $*$-ideal is contained in a maximal $*$-ideal; and $*$-Max$(D) \neq \emptyset$ if D is not a field. Recall that D is a *Prüfer v-multiplication domain* (PvMD) if each $I \in f(D)$ is t-invertible, i.e., $(II^{-1})_t = D$. An upper to zero in $D[X]$ is a (height-one) prime ideal of $D[X]$ of the form $Q_f = fK[X] \cap D[X]$, where the polynomial $f \in D[X]$ is irreducible in $K[X]$. As in [20], we say that D is a *UMT-domain* if each upper to zero in $D[X]$ is a maximal t-ideal of $D[X]$. It is well known that D is a PvMD if and only if D_P is a valuation domain for each $P \in t$-Max(D) [17, Theorem 5], if and only if D is an integrally closed UMT-domain [20, Proposition 3.2].

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A prime ideal P of D is called *strongly prime* if $xy \in P$ and $x, y \in K$ imply $x \in P$ or $y \in P$. Following [18], we say that D is a *pseudo-valuation domain* (PVD) if every prime ideal of D is strongly prime; equivalently, D is a quasi-local domain whose maximal ideal is strongly prime. As a globalization of PVDs, D is called a *locally PVD* (an LPVD) if D_M is a PVD for each maximal ideal M of D. The concept of LPVDs was introduced by Dobbs and Fontana [11], and since a PVD is a generalization of valuation domains, an LPVD can be considered as a generalization of Prüfer domains. It is known that each integral overring of an LPVD is an LPVD [11, Proposition 2.6] and each overring of D is an LPVD if and only if D is an LPVD and $\bar{D}$ is a Prüfer domain [11, Theorem 2.9]. Let $\text{Int}(D) = \{f \in K[X] | f(D) \subseteq D\}$ be the ring of integer valued polynomials. It is also known that if D is a Prüfer domain, then $\text{Int}(D)$ is an LPVD if and only if $\text{Int}(D)$ is a Prüfer domain [7, Proposition 1.2] and that if D is a Noetherian domain, then $\text{Int}(D)$ is an LPVD if and only if D is an LPVD with finite residue fields [7, Theorem 2.4].

Let $N_v = \{f \in D[X] | (A_f)_v = D\}$ and $S = \{f \in D[X] | A_f = D\}$. The rings $D[X]_{N_v}$ and $D[X]_S$ have many interesting ring-theoretic properties. For example, D is a PvMD if and only if $D[X]_{N_v}$ is a Prüfer domain, if and only if every ideal of $D[X]_{N_v}$ is extended from D [22, Theorems 3.1 and 3.7]. Also, D is a UMT-domain if and only if each prime ideal of $D[X]_{N_v}$ is extended from D [20, Theorem 3.1], if and only if $\bar{D}[X]_{N_v}$ is a Prüfer domain [14, Theorem 2.5]. It is also known that D is a Krull (resp., Dedekind) domain if and only if $D[X]_{N_v}$ (resp., $D[X]_S$) is a principal ideal domain [21, Theorem 22.7] (resp., [21, Theorem 18.6]). The purpose of this paper is to study when the rings $D[X]_{N_v}$ and $D[X]_S$ are PVDs or LPVDs.

Recall that if D is a PVD, then $\bar{D}$ is a valuation domain if and only if each overring of D is a PVD [19, Proposition 2.7] and that if $\bar{D}$ is a valuation domain, then D is a UMT-domain [14, Theorem 1.5]. But if $\text{Spec}(D)$ is linearly ordered under inclusion, then D is quasi-local and each prime ideal of D is a t-ideal [22, Theorem 3.19]; so D is a UMT-domain if and only if $\bar{D}$ is a Prüfer domain. This shows that if D is a PVD, then D is a UMT-domain if and only if each overring of D is a PVD (see Theorem 2.3). In Section 2, we show that if D is a PVD with maximal ideal M, then D is a UMT-domain if and only if D/P is a UMT-domain for each prime ideal P of D, if and only if $\bar{D}$ is a valuation domain and that if M is finitely generated, then D is a UMT-domain. Let D^w be the w-integral closure of D (see Section 3 for the definition of D^w). In Section 3, we first introduce the concept of t-locally PVDs; D is a t-locally PVD if D_P is a PVD for each maximal t-ideal P of D. Then we prove that if D is a t-locally PVD, then D is a UMT-domain if and only if D^w is a PvMD; and each t-linked overring R of D such that $R \subseteq D^w$ is a t-locally PVD. We also prove that D is a t-locally PVD and a UMT-domain if and only if $D[X]$ is a t-locally PVD, if and only if $D[X]_{N_v}$ is an LPVD, if and only if each overring of $D[X]_{N_v}$ is an LPVD.

The reader is referred to [16, Sections 32 and 34] and [28] for the t- and the w-operation; to [14, 20] for UMT-domains; to [1, 4, 10, 11, 18, 19] for PVDs; to [3], [8], [9], [15], [16, Section 33], or [21, Chapter IV] for the rings $D[X]_{N_v}$ and $D[X]_S$; and to [16, 23] for standard definitions and notations.

2. Pseudo-valuation domains and UMT-domains

In this section, we study PVDs that are also UMT-domains. One of the interesting and important characterizations of PVDs is that a (quasi-local) domain D is a PVD if and only if there exists a valuation overring V of D such that $\operatorname{Spec}(V) = \operatorname{Spec}(D)$ [18, Theorem 2.7]. Let D be a PVD with maximal ideal M. It is well known that $\operatorname{Spec}(D)$ is linearly ordered under inclusion [18, Corollary 1.3] and that if D is not a valuation domain, then $M^{-1} = \{x \in K | xM \subseteq D\}$ is a valuation domain such that $\operatorname{Spec}(M^{-1}) = \operatorname{Spec}(D)$ (in particular, M is the maximal ideal of M^{-1}) [18, Theorem 2.10]. We will say that (D, M) is a PVD if D is a PVD with maximal ideal M.

We begin by recalling some well-known properties of strongly prime ideals.

Lemma 2.1. *Let P be a strongly prime ideal of an integral domain D.*

(1) *Each ideal of D is comparable to P; so $P = PD_P$.*
(2) *D_P is a PVD with maximal ideal $P = PD_P$.*
(3) *P is a t-ideal of D.*
(4) *If Q is a prime ideal of D such that $Q \subsetneq P$, then Q is also a strongly prime ideal and D_Q is a valuation domain.*
(5) *The set of prime ideals of D contained in P is linearly ordered.*

Proof. (1) Let I be an ideal of D such that $I \not\subseteq P$. Choose $a \in I \setminus P$, and let $b \in P$. Then $\frac{b}{a} \in P$ since $\frac{b}{a}a = b \in P$ and P is strongly prime; hence $b \in aP \subseteq aD$. Thus $P \subsetneq aD \subseteq I$. This also implies that $P = PD_P$.

(2) Since $P = PD_P$ by (1), PD_P is a strongly prime ideal of D_P. Also, since PD_P is a maximal ideal of D_P, it follows from [18, Theorem 1.4] that D_P is a PVD.

(3) Note that $\operatorname{Spec}(D_P)$ is linearly ordered by (2); hence PD_P is a t-ideal of D_P, and thus $P = PD_P \cap D$ is a t-ideal of D [22, Theorem 3.19 and Lemma 3.17].

(4) Note that $Q \subseteq QD_P = QD_P \cap PD_P = QD_P \cap P \subseteq QD_P \cap D = Q$ by (1); hence $QD_P = Q$. Since D_P is a PVD by (2), $Q = QD_P$ is a strongly prime ideal of D_P, and thus Q is a strongly prime ideal of D. Moreover, QD_P is a nonmaximal ideal of D_P, $D_Q = (D_P)_{QD_P}$ is a valuation domain [18, Proposition 2.6].

(5) This follows because $\operatorname{Spec}(D_P)$ is linearly ordered by (2). □

The concept of UMT-domains was introduced by Houston and Zafrullah [20] and has been studied by several authors (see, for example, [9, 12, 14, 20, 24]).

Lemma 2.2. *The following statements are equivalent for an integral domain D.*

(1) *D is a UMT-domain.*
(2) *D_P is a UMT-domain and PD_P is a t-ideal for each prime t-ideal P of D.*
(3) *D_P has Prüfer integral closure for each prime t-ideal P of D.*
(4) *D_P has Bezout integral closure for each prime t-ideal P of D.*

Proof. (1) $\Rightarrow$ (2) [14, Propositions 1.2 and 1.4]. (1) $\Leftrightarrow$ (3) [14, Theorem 1.5].

(2) $\Rightarrow$ (4) Replacing D with D_P and P with PD_P, we assume that D is quasi-local with maximal ideal P and P is a t-ideal. Let I be a nonzero finitely generated ideal of D, and let $f \in D[X]$ such that $A_f = I$. Since D is a UMT-domain and P is a t-ideal, there exists a polynomial $g \in qf(D)[X]$ such that $A_{fg} = D$ [14, Lemma 3.4]; hence $A_f A_g \subseteq (A_f \bar{D} A_g \bar{D})_v = (A_{fg}\bar{D})_v = \bar{D}$ [16, Proposition 34.8]. Let $R = D[A_f A_g]$. Then R is a finitely generated D-module, $R \subseteq \bar{D}$, and $(A_f R)(A_g R) = A_{fg} R = R$. Note that R has a finite number of maximal ideals [16, Ex. 11, p.131] since R is a finitely generated D-module and D is quasi-local. Hence $A_f R$ is principal [16, Proposition 7.4], and thus $I\bar{D} = A_f \bar{D}$ is principal.

Now, suppose that $J = (a_1, \dots, a_n)\bar{D}$ is a nonzero finitely generated ideal of $\bar{D}$. Since $\bar{D}$ is an overring of D, there exists a $0 \neq d \in D$ such that $da_i \in D$ for $i = 1, \dots, n$; so $(da_1, \dots, da_n)D$ is a nonzero finitely generated ideal of D. Hence $dJ = (da_1, \dots, da_n)\bar{D}$, and thus J is principal by the previous paragraph. Therefore, $\bar{D}$ is a Bezout domain.

(4) $\Rightarrow$ (3) Clear. □

We next give some characterizations of PVDs which are also UMT-domains.

Theorem 2.3. *The following statements are equivalent for a PVD (D, M).*

(1) *D is a UMT-domain.*
(2) *$\bar{D}$ is a valuation domain.*
(3) *$\bar{D} = (M : M)$.*
(4) *Each overring of D is a UMT-domain.*
(5) *There is an integral overring of D that is a UMT-domain.*
(6) *$\bar{D}$ is a UMT-domain.*
(7) *Each integrally closed overring of D is a valuation domain.*
(8) *Each overring of D is a PVD.*

Proof. Recall that $\bar{D}$ is a PVD with maximal ideal M and $V = (M : M)$ is a valuation domain with maximal ideal M [18, Theorems 1.7 and 2.10].

(1) $\Leftrightarrow$ (2) This is an immediate consequence of Lemma 2.2 since M is a t-ideal of D and $\bar{D}$ is quasi-local.

(2) $\Rightarrow$ (3) This follows directly from [16, Theorem 17.6] because $\bar{D} \subseteq V$ and M is a maximal ideal of both $\bar{D}$ and V.

(2) $\Rightarrow$ (4) and (7) Let R be an overring of D, and let $\bar{R}$ be the integral closure of R. Then $\bar{D} \subseteq \bar{R}$; so $\bar{R}$ is a valuation domain [16, Theorem 17.6].

Thus R is a UMT-domain by Lemma 2.2. Moreover, if R is integrally closed, then $R = \bar{R}$, and hence R is a valuation domain.

(3) $\Rightarrow$ (6); (4) $\Rightarrow$ (5); (6) $\Rightarrow$ (5); and (7) $\Rightarrow$ (2) Clear.

(5) $\Rightarrow$ (2) Let R be a UMT-domain such that $D \subseteq R \subseteq \bar{D}$. Then R is quasi-local since $\bar{D}$ is quasi-local and $\bar{D}$ is integral over R. Let Q be the maximal ideal of R. If P is a prime ideal of R with $P \subsetneq Q$, then $P \cap D \subsetneq M$, and hence $D_{P\cap D}$ is a valuation domain by Lemma 2.1. Since $D_{P\cap D} \subseteq R_P$, we have that R_P is a valuation domain [16, Theorem 17.6]. This also implies that Spec(R) is linearly ordered under inclusion; so Q is a t-ideal of R [22, Theorem 3.19]. Thus $\bar{D}$ is a valuation domain by Lemma 2.2.

(3) $\Leftrightarrow$ (8) [19, Proposition 2.7]. □

Corollary 2.4. *Let (D, M) be a PVD, and let $P \subsetneq M$ be a prime ideal of D.*

(1) *D is a UMT-domain if and only if D/P is a UMT-domain.*

(2) (cf. [25, Proposition 2.5]) *Each overring of D is a PVD if and only if each overring of D/P is a PVD.*

Proof. (1) Note that $P = PD_P$, $D_P = \bar{D}_{D\setminus P}$ is a valuation domain, P is a strongly prime ideal of both D and $\bar{D}$ (cf. Lemma 2.1), and D/P is a PVD [10, Lemma 4.5(v)]. Also, note that $(D/P)_{P/P} \cong D_P/PD_P = D_P/P$ [16, Proposition 5.8]; hence D_P/P (resp., $\bar{D}/P$) can be considered as the quotient field (resp., integral closure) of D/P. Moreover, since $\bar{D}_{D\setminus P}$ is a valuation domain, we have that $\bar{D}$ is a valuation domain if and only if $\bar{D}/P$ is a valuation domain [10, Lemma 4.5(v)]. Thus by Theorem 2.3, D is a UMT-domain if and only if $\bar{D}$ is a valuation domain, if and only if $\bar{D}/P$ is a valuation domain, if and only if D/P is a UMT-domain.

(2) Since D_P is a valuation domain and $P = PD_P$ by Lemma 2.1, we have that D is a PVD if and only if D/P is a PVD [10, Lemma 4.5(v)]. Thus the proof is completed by (1) and Theorem 2.3. □

Let (D, M) be a PVD, and let V be a valuation overring of D such that Spec(D) = Spec(V) [18, Theorem 2.10]. In [4, Theorem 14], Badawi showed that if D contains a nonzero finitely generated prime ideal, then $\bar{D} = (M : M)$ is a valuation domain; hence every overring of D is a PVD by Theorem 2.3. Let P be a nonzero finitely generated prime ideal D. Since Spec(D) = Spec(V), P is also a finitely generated ideal of V, and hence $P = aV$ for some $0 \neq a \in V$. Thus P is the maximal ideal of V, i.e., $P = M$. In [19], the authors also studied PVDs whose maximal ideal is finitely generated.

Corollary 2.5. *Let (D, M) be a PVD such that M is finitely generated, and let $\{P_\alpha\}$ be the set of prime ideals of D properly contained in M. Then*

(1) *$\cup P_\alpha \subsetneq M$.*

(2) *D is a UMT-domain, and hence each overring of D is a PVD.*

(3) *An overring R of D has a prime ideal lying over M (if and) only if R is integral over D.*

Proof. (1) Let $M = (a_1, \dots, a_n)$. Note that $\{P_\alpha\}$ is linearly ordered under inclusion (Lemmas 2.1(5)); so there exists an a_i such that $a_i \notin \cup P_\alpha$. Thus $\cup P_\alpha \subsetneq M$.

(2) Let $P = \cup P_\alpha$. Since $\{P_\alpha\}$ is linearly ordered, P is a prime ideal of D, and hence M/P is the unique nonzero prime ideal of D/P. Also, since M, and hence M/P, is finitely generated, D/P is a Noetherian domain by Cohen's theorem, and hence D/P is a UMT-domain [20, Theorem 3.7]. Thus D is a UMT-domain by Corollary 2.4(1) and each overring of D is a PVD by Theorem 2.3.

(3) Note that $\bar{D}$ is a valuation domain by (2) and Theorem 2.3. Let $\bar{R}$ be the integral closure of R, and let Q be a prime ideal of R such that $Q \cap D = M$. Since $\bar{D} \subseteq \bar{R}$, it follows that $\bar{R}$ is a valuation domain [16, Theorem 17.6]. Note that M is the maximal ideal of $\bar{D}$ [18, Theorem 1.7] and $M \subseteq Q \subseteq Q\bar{R} \subsetneq \bar{R}$. Hence $\bar{D} = \bar{R}$ [16, Theorem 17.6], and thus R is integral over D. □

Corollary 2.6. *Let (D, M) be a PVD, and let R an overring of D. If M is finitely generated, then each prime ideal Q of R with $Q \cap D = M$ is finitely generated.*

Proof. Note that $D \subseteq R \subseteq \bar{D} = (M : M)$ by Corollary 2.5(3) and Theorem 2.3 and that M is the maximal ideal of $(M : M)$. Hence $Q = M$, and thus Q is certainly finitely generated. □

3. *t*-locally pseudo-valuation domains

Let D be an integral domain with $qf(D) = K$ and $\bar{D}$ the integral closure D, and $N_v = \{f \in D[X] | (A_f)_v = D\}$. Following [11], we say that D is a *locally pseudo-valuation domain* (LPVD) if D_M is a PVD for each maximal ideal M of D. In this section, we study when $D[X]_{N_v}$ is an LPVD. To do this, we need the concept of t-locally PVD. We will call D a *t-locally PVD* (t-LPVD) if D_P is a PVD for all maximal t-ideals P of D.

Lemma 3.1. *An integral domain D is an LPVD if and only if D is a t-LPVD and each maximal ideal of D is a t-ideal.*

Proof. Let D be an LPVD, and let M be a maximal ideal of D. Then MD_M is a t-ideal by Lemma 2.1(3). Thus $M = MD_M \cap D$ is a t-ideal of D [22, Lemma 3.17]. The converse is clear. □

An overring R of D is said to be *t-linked* over D if for any nonzero finitely generated ideal I of D, $I^{-1} = D$ implies $(IR)^{-1} = R$; equivalently, if Q is a prime t-ideal of R, then $(Q \cap D)_t \subsetneq D$ [13, Proposition 2.1]. It is known that R is t-linked over D if and only if $R[X]_{N_v} \cap K = R$ [8, Lemma 3.2]. As in [27], we say that an element $u \in K$ is *w-integral* over D if $uI_w \subseteq I_w$ for some nonzero finitely generated ideal I of D. The *w-integral closure of D* is the set $D^w = \{x \in K | x$ is w-integral over $D\}$. We know that D^w is an integrally closed domain; $D \subseteq \bar{D} \subseteq D^w \subseteq K$; D^w is t-linked over D [9, Lemma 1.2];

$D^w = \bar{D}[X]_{N_v} \cap K$ [9, Theorem 1.3]; and if D is a UMT-domain, then D^w is a PVMD [9, Theorem 2.6]. For more on the w-integral closure of integral domains, see [9].

Lemma 3.2. *Let D be a t-LPVD and P a nonzero prime ideal of D with $P_t \subsetneq D$.*

(1) *P is a prime t-ideal of D.*
(2) *If P is not a maximal t-ideal, then D_P is a valuation domain.*
(3) *$\bar{D}_{D\setminus P} = (D^w)_{D\setminus P}$ and $\bar{D}_{D\setminus P}$ is a PVD.*

Proof. (1) and (2) Let Q be a maximal t-ideal of D such that $P_t \subseteq Q$; then D_Q is a PVD and PD_Q is a proper prime ideal of D_Q. Hence PD_Q, and thus $P = PD_Q \cap D$, is a t-ideal by Lemma 2.1(3) and [22, Lemma 3.17]. Moreover, if P is not a maximal t-ideal, then PD_Q is not a maximal ideal of D_Q, and hence $D_P = (D_Q)_{PD_Q}$ is a valuation domain by Lemma 2.1(4).

(3) Note that $\bar{D}_{D\setminus P}$ is an integrally closed t-linked overring of D [13, Proposition 2.9]; so $D^w \subseteq \bar{D}_{D\setminus P}$ (cf. [9, Theorem 1.3]). Thus $\bar{D}_{D\setminus P} = (D^w)_{D\setminus P}$. Moreover, since $\bar{D}_{D\setminus P}$ is the integral closure of D_P and D_P is a PVD, we have that $\bar{D}_{D\setminus P}$ is a PVD [18, Theorem 1.7]. □

Lemma 3.3. *Let D be a t-LPVD.*

(1) *D is a UMT-domain if and only if D^w is a PvMD.*
(2) *If D is an LPVD, then D is a UMT-domain if and only if $\bar{D}$ is a Prüfer domain.*

Proof. (1) Suppose that D^w is a PvMD. Let P be a maximal t-ideal of D, and let Q be a prime ideal of D^w such that $Q \cap D = P$ (cf. [9, Corollary 1.4(3)]). Then $(D^w)_{D\setminus P}$ is the integral closure of D_P and $(D^w)_{D\setminus P}$ is a PVD by Lemma 3.2(3). Note that $(D^w)_{D\setminus P} = (D^w)_Q$ and $Q_{D\setminus P}$ is a t-ideal of $(D^w)_{D\setminus P}$ by Lemma 2.1(3); so $Q = Q_{D\setminus P} \cap D^w$ is a t-ideal of D^w [22, Lemma 3.17]. Hence $(D^w)_{D\setminus P}$ is a valuation domain [17, Theorem 5]. Thus D is a UMT-domain by Lemma 2.2. The converse holds without the assumption that D is a t-LPVD (see [9, Theorem 2.6]).

(2) Assume that $\bar{D}$ is a Prüfer domain, and let P be a maximal t-ideal of D. Then $\bar{D}_{D\setminus P}$, the integral closure of D_P, is a Prüfer domain. Thus D is a UMT-domain by Lemma 2.2. Conversely, assume that D is a UMT-domain. Let Q be a maximal ideal of $\bar{D}$, and put $P = Q \cap D$. Then P is a t-ideal of D by Lemma 3.1, and thus $\bar{D}_{D\setminus P}$ is a Prüfer domain by Lemma 2.2. So $\bar{D}_Q = (\bar{D}_{D\setminus P})_{Q_{D\setminus P}}$ is a valuation domain. Thus $\bar{D}$ is a Prüfer domain. □

Recall that each integral overring of an LPVD is an LPVD [11, Proposition 2.6]. Our next result is the t-LPVD analog.

Proposition 3.4. *Let D be a t-LPVD. If R is a t-linked overring of D such that $R \subseteq D^w$, then R is a t-LPVD.*

Proof. Let Q be a maximal t-ideal of R, and put $P = Q \cap D$. Then $P_t \subsetneq D$ since R is t-linked over D, and hence D_P is a PVD by Lemma 3.2(1). Note that $D_P \subseteq R_{D\setminus P} \subseteq (D^w)_{D\setminus P} = \bar{D}_{D\setminus P}$ by Lemma 3.2(3); so $R_{D\setminus P}$ is a PVD (cf. [18, Theorem 1.7] or [11, Proposition 2.6]). Thus $R_Q = R_{D\setminus P}$ is a PVD. □

Proposition 3.5. *Let $V = K + M$ be a valuation domain and $R = D + M$, where K is a field, M is the nonzero maximal ideal of V, and D is a proper subring of K.*

(1) *R is a PVD if and only if D is a PVD with quotient field K or D is a field.*
(2) *A prime ideal P of D is a maximal t-ideal if and only if $P + M$ is a maximal t-ideal of R.*
(3) *R is a t-LPVD if and only if either D is a t-LPVD with quotient field K or D is a field.*

Proof. (1) [10, Proposition 4.9].

(2) This follows from the fact that if I is a nonzero ideal of D, then $I + M$ is an ideal of R, I is finitely generated if and only if $I + M$ is finitely generated, and $(I + M)_v = I_v + M$ [2, Proposition 2.4].

(3) ($\Rightarrow$) Let R be a t-LPVD, and assume that D is not a field. Let P be a maximal t-ideal of D; then $P + M$ is a maximal t-ideal of R by (2). Hence $R_{P+M} = D_P + M$ [6, Theorem 2.1(g)] is a PVD; so D_P is a PVD with quotient field K by (1). Thus D is a t-LPVD with quotient field K. ($\Leftarrow$) If D is a field, then R is a PVD by (1), and hence R is a t-LPVD. So we assume that D is a t-LPVD with quotient field K. Let Q be a maximal t-ideal of R; then $Q = P + M$ [6, Theorem 2.1(c) and (d)], where P is a maximal t-ideal of D by (2). Note that $R_Q = D_P + M$ and D_P is a PVD with quotient field K. Thus R_Q is a PVD by (1). □

Recall that D is a *Mori domain* if D satisfies the ascending chain condition on integral divisorial ideals; equivalently, for each nonzero fractional ideal A of D, there is a finitely generated subideal I of A such that $A_v = I_v$. Clearly, a Noetherian domain is a Mori domain. It is well known that if D is a Noetherian PVD, then $\dim(D) \leq 1$ [18, Proposition 3.2] and that if D is a PVD, then D is a Mori domain if and only if M^{-1} is a rank one DVR [5, Theorem 3.2]. Thus if D is a Mori PVD, then $\dim(D) \leq 1$ (where $\dim(D)$ denotes the (Krull) dimension of D).

Proposition 3.6. *Let D be a t-LPVD which is not a field. If D is a Mori domain, then each maximal t-ideal of D has height-one.*

Proof. Let P be maximal t-ideal of D; then D_P is a Mori domain [26, Corollary 3] and a PVD. Hence $\mathrm{ht}P = \dim(D_P) \leq 1$ by the preceding statement, and thus $\mathrm{ht}P = 1$. □

Let $S = \{f \in D[X] | A_f = D\}$; then S is a saturated multiplicative subset of $D[X]$. The quotient ring $D[X]_S$, denoted by $D(X)$, is called the *Nagata* *ring of D*. Let $N_v = \{f \in D[X] | (A_f)_v = D\}$; then N_v is also a saturated multiplicative subset of $D[X]$ such that $S \subseteq N_v$. Thus $D[X]_{N_v}$ is an overring of $D(X)$. It is not difficult to show that D is a PvMD (resp., Prüfer domain) if and only if $D[X]_{N_v}$ (resp., $D(X)$) is a Prüfer domain [22, Theorem 3.7] (resp., [3, Theorem 4]).

The rest of this section is devoted to the study of the following question: *when does the ring* $D[X]_{N_v}$ *become an LPVD ?*

Lemma 3.7. *Let D be a quasi-local domain and $D(X)$ the Nagata ring of D. Then D is a PVD and a UMT-domain if and only if $D(X)$ is a PVD.*

Proof. Let M be the maximal ideal of D and $K = qf(D)$. Note that $D(X)$ is quasi-local with maximal ideal $M(X) = MD(X)$ [16, Proposition 33.1].

($\Rightarrow$) Let D be a PVD and a UMT-domain. Then $\bar{D}$ is a valuation domain with maximal ideal M by Theorem 2.3, and hence $\bar{D}(X)$ is a valuation domain with maximal ideal $M(X)$ [16, Proposition 18.7]. Thus $D(X)$ is a PVD [18, Theorem 2.7]. ($\Leftarrow$) Suppose that $D(X)$ is a PVD. Note that $M(X)$ is a strongly prime ideal of $D(X)$ and $M(X) \cap K = M$ (cf. [16, Proposition 33.1(4)]); so M is a strongly prime ideal of D. Thus D is a PVD. Next, we show that D is a UMT-domain. Assume to the contrary that D is not a UMT-domain. Note that M, and hence $M[X]$, is a t-ideal [22, Corollary 2.3]; so there is a polynomial $f \in K[X] \setminus D[X]$ such that $Q_f = fK[X] \cap D[X] \subsetneq M[X]$ and Q_f is a prime ideal of $D[X]$. It is clear that $(Q_f)_S \subsetneq M[X]_S = M(X)$; hence $(Q_f)_S$ is a strongly prime ideal of $D(X)$ by Lemma 2.1(4). Let $0 \neq a \in (A_f)^{-1}$; then $af \in Q_f \subseteq (Q_f)_S$. Since $f \notin D[X]$, we have $f \notin D(X)$; hence $a \in (Q_f)_S \cap K \subseteq D$. Thus $a \in Q_f \cap D = (0)$, a contradiction. □

We next give the main result of this paper.

Theorem 3.8. *The following statements are equivalent for an integral domain D.*

(1) *D is a t-LPVD and D is a UMT-domain.*
(2) *D is a t-LPVD and D^w is a PvMD.*
(3) *For each maximal t-ideal P of D, each overring of D_P is a PVD.*
(4) *For each maximal t-ideal P of D, each overring of D_P is an LPVD.*
(5) *Each t-linked overring R of D is a t-LPVD and if Q is a prime ideal of R with $(Q \cap D)_t \subsetneq D$, then Q is a t-ideal.*
(6) *$D[X]$ is a t-LPVD.*
(7) *$D[X]_{N_v}$ is an LPVD, where $N_v = \{f \in D[X] | (A_f)_v = D\}$.*
(8) *$D[X]_{N_v}$ is a t-LPVD.*
(9) *Each overring of $D[X]_{N_v}$ is an LPVD.*

Proof. (1) $\Leftrightarrow$ (2) Lemma 3.3.

(1) $\Rightarrow$ (3) Let P be a maximal t-ideal of D. Then D_P is a PVD and a UMT-domain by Lemma 2.2, and thus each overring of D_P is a PVD by Theorem 2.3.

(3) $\Rightarrow$ (5) Let Q be a prime ideal of R such that $(Q \cap D)_t \subsetneq D$, and let P be a maximal t-ideal of D containing $(Q \cap D)_t$. Then $D_P \subseteq D_{Q \cap D} \subseteq R_Q$; so R_Q

is a PVD. Hence QR_Q is a t-ideal by Lemma 2.1(3), and thus $Q = QR_Q \cap R$ is a t-ideal of R [22, Lemma 3.17]. Moreover, if Q' is a maximal t-ideal of R, then since R is t-linked over D, we have $(Q' \cap D)_t \subsetneq D$. Thus $R_{Q'}$ is a PVD.

(5) $\Rightarrow$ (4) Let P be a maximal t-ideal of D, and let R be an overring of D_P. First, note that D_P is t-linked over D and PD_P is a t-ideal of D_P by (5); hence D_P is a PVD. Also, D_P being t-linked over D implies that R is t-linked over D. Hence R is a t-LPVD. Let Q be a maximal ideal of R. Then $Q \cap D_P$ is a t-ideal of D_P (cf. Lemma 2.1(3)), and hence $Q \cap D$ is a t-ideal of D [22, Lemma 3.17]. Hence Q is a t-ideal of R by (5), and thus R_Q is a PVD.

(4) $\Rightarrow$ (1) Clearly, D is a t-LPVD. Let P be a prime t-ideal of D. If P is not a maximal t-ideal, then D_P is a valuation domain by Lemma 3.2(2). Next, if P is a maximal t-ideal, then the integral closure of D_P is a Prüfer domain by (4) and [11, Theorem 2.9]. Thus D is a UMT-domain by Lemma 2.2.

(1) $\Rightarrow$ (6) Assume that D is a t-LPVD and a UMT-domain. Let Q be a maximal t-ideal of $D[X]$; then either $Q \cap D = (0)$ or $Q = (Q \cap D)[X]$ with $Q \cap D$ maximal t-ideal of D [14, Proposition 2.2]. If $Q \cap D = (0)$, then $D[X]_Q$ is a valuation domain. Next, assume that $Q = (Q \cap D)[X]$, and note that $D_{Q \cap D}$ is a PVD and a UMT-domain by Lemma 2.2. Thus $D[X]_Q = (D_{Q \cap D}[X])_{Q_{Q \cap D}} = D_{Q \cap D}(X)$, the Nagata ring of $D_{Q \cap D}$, is an PVD by Lemma 3.7.

(6) $\Rightarrow$ (7) Suppose that $D[X]$ is a t-LPVD. Let Q be a maximal ideal of $D[X]_{N_v}$; then $Q = P[X]_{N_v}$, where P is a maximal t-ideal of D [22, Proposition 2.1]. Note that $P[X]$ is a maximal t-ideal of $D[X]$ [14, Lemma 2.1(4)]. Thus $(D[X]_{N_v})_Q = (D[X]_{N_v})_{P[X]_{N_v}} = D[X]_{P[X]}$ is a PVD.

(7) $\Rightarrow$ (1) Let P be a maximal t-ideal of D. Then $P[X]_{N_v}$ is a maximal ideal of $D[X]_{N_v}$ [22, Proposition 2.1]. Hence $(D[X]_{N_v})_{P[X]_{N_v}} = D[X]_{P[X]} = D_P(X)$, the Nagata ring of D_P, is a PVD. Thus D_P is a PVD and a UMT-domain by Lemma 3.7. Moreover, since PD_P is a t-ideal (Lemma 2.1(3)), D is a UMT-domain by Lemma 2.2.

(7) $\Leftrightarrow$ (8) This follows from Lemma 3.1 because each maximal ideal of $D[X]_{N_v}$ is a t-ideal (cf. [22, Propositions 2.1 and 2.2]).

(7) $\Rightarrow$ (9) By the equivalence of (1) and (7), D is a UMT-domain, and hence the integral closure of $D[X]_{N_v}$ is a Prüfer domain [14, Theorem 2.5]. Thus each overring of $D[X]_{N_v}$ is an LPVD [11, Theorem 2.9].

(9) $\Rightarrow$ (7) Clear. □

Note that D is an LPVD if and only if D is a t-LPVD and each maximal ideal of D is a t-ideal by Lemma 3.1. Thus by Lemma 3.3(2) and Theorem 3.8, we have

Corollary 3.9. *The following statements are equivalent for an integral domain D.*

(1) *D is an LPVD and D is a UMT-domain.*
(2) *D is an LPVD and $\bar{D}$ is a Prüfer domain.*
(3) *The Nagata ring $D(X)$ of D is an LPVD.*

(4) *Each overring of $D(X)$ is an LPVD.*

It is known that if D is an LPVD, then $\bar{D}$ is a Prüfer domain if and only if each overring of D is an LPVD [11, Theorem 2.9]. As the t-operation analog, we proved that if the w-integral closure of a t-LPVD D is a PvMD, then each t-linked overring of D is a t-LPVD (see the equivalence of (2) and (5) of Theorem 3.8). But, we do not know whether the converse holds. This is equivalent to whether the second half of (5) in Theorem 3.8 is superfluous.

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