

BEST PROXIMITY PAIRS AND NASH EQUILIBRIUM PAIRS

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ABSTRACT. Main purpose of this paper is to combine the optimal form of Fan's best approximation theorem and Nash's equilibrium existence theorem into a single existence theorem simultaneously. For this, we first prove a general best proximity pair theorem which includes a number of known best proximity theorems. Next, we will introduce a new equilibrium concept for a generalized Nash game with normal form, and as applications, we will prove new existence theorems of Nash equilibrium pairs for generalized Nash games with normal form.

1. Introduction

In 1969, Fan proved the well-known best approximation theorem which generalizes the Tychonoff fixed point theorem as follows:

Theorem A ([8]). *If K is a non-empty compact convex subset of a locally convex Hausdorff topological vector space E with a continuous seminorm p , and $f : K \rightarrow E$ is a single valued continuous function, then there exists an element $\bar{x} \in K$ such that*

$$p(f(\bar{x}) - \bar{x}) = d_p(f(\bar{x}), K) := \inf\{p(f(\bar{x}) - y) \mid y \in K\}.$$

Since then, a number of generalizations of this theorem have been obtained in various directions by several authors (e.g., see [2, 3, 5, 13, 18-23]). Indeed, Reich [18] has shown that even if K is a non-empty approximately p -compact convex subset of a locally convex Hausdorff topological vector space E with a relatively compact image $f(K)$, then the same conclusion holds. Later, Segal and Singh [21] have extended this result to convex valued continuous multifunctions. Even though a best approximation theorem guarantees the existence of an approximate solution, it is contemplated to find an approximate solution which is optimal. In this direction, Srinivasan and Veeramani [23] have proved the general forms of existence theorems for best proximity pairs, and Kim and Lee [12] prove two general existence theorems of best proximity pairs in a recent paper.

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On the other hand, in 1951, Nash established the pioneering result on the existence of equilibrium for generalized games as follows:

Theorem B ([15]). *Let I be a finite set of players. Assume that for all $i \in I$,*

(a) *the set $X_i \subset \mathbb{R}^{k_i}$ is non-empty compact and convex;*

(b) *the function $f_i : X := \prod_{i \in I} X_i \rightarrow \mathbb{R}$ is continuous on X ;*

(c) *the function $y_i \mapsto f_i(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)$ is concave on X_i .*

Then there exists a Nash equilibrium $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ such that for each $i \in I$,

$$f_i(\bar{x}_1, \dots, \bar{x}_i, \dots, \bar{x}_n) \geq f_i(\bar{x}_1, \dots, x_i, \dots, \bar{x}_n) \quad \text{for all } x_i \in X_i.$$

Next, in 1977, Friedman [9] established a generalization of Theorem B using the quasi-concavity assumption on every payoff function. Since then, the classical results of Nash [14, 15], Debreu [4] and Friedman [9] have served as basic references for the existence of Nash equilibrium for non-cooperative generalized games. In all of them, convexity of strategy spaces, continuity and concavity/quasi-concavity of the payoff functions were assumed. Till now, there have been a number of generalizations, and also many applications of those theorems have been found in several areas, e.g., see [1, 9] and references therein.

In this paper, we will combine the optimal form of Theorem A and Theorem B into a single existence theorem simultaneously. For this purpose, we first prove a general best proximity pair theorem which generalizes the theorems due to Srinivasan and Veeramani [23, 24], Kim and Lee [12], and others. Next, we will introduce a new concept of Nash equilibrium pair in a generalized Nash game with normal form, and as applications of the best proximity pair theorem, we will prove new existence theorems of Nash equilibrium pairs for generalized Nash games with normal form which includes Theorem B and its generalizations due to Nikaido and Isoda [16], Kim and Lee [11], and others.

2. Preliminaries

We begin with some notations and definitions. Let I be any (possibly infinite) index set. For each $i \in I$, let X_i be a non-empty topological space and denote $X_{\hat{i}} := \prod_{j \in I \setminus \{i\}} X_j$, and $X := \prod_{i \in I} X_i = X_i \times X_{\hat{i}}$. If $x = (x_1, \dots, x_n, \dots) \in X$, we shall write $x_{\hat{i}} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, \dots) \in X_{\hat{i}}$, and if $x_i \in X_i$ and $x_{\hat{i}} \in X_{\hat{i}}$, we shall use the notation $x := (x_i, x_{\hat{i}}) = (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n, \dots) \in X$.

Let I be a finite (or an infinite) index set. For each $i \in I$, let X_i and Y_i be non-empty subsets of a normed linear space E with a norm $\|\cdot\|$, and the metric $d(x, y)$ for E is induced by the norm, i.e., $d(x, y) = \|x - y\|$. Then we can use the following notations as in [12, 24]: for each $i, j \in I$,

$$d(X_i, Y_j) := \inf\{d(x, y) \mid x \in X_i, y \in Y_j\};$$

$$X_i^o := \{x \in X_i \mid \text{for each } j \in I, \exists y_j \in Y_j \text{ such that } d(x, y_j) = d(X_i, Y_j)\};$$

$$Y_i^o := \{y \in Y_i \mid \text{for each } j \in I, \exists x_j \in X_j \text{ such that } d(x_j, y) = d(X_j, Y_i)\}.$$

And we denote $X^\circ := \prod_{i \in I} X_i^\circ$ and $Y^\circ := \prod_{i \in I} Y_i^\circ$. If $X_i = \{x\}$, then $d(X_i, Y_j)$ is written as $d(x, Y_j)$. Also, if $X_i = \{x\}$ and $Y_j = \{y\}$, then $d(x, y)$ denotes $d(X_i, Y_j)$ which is precisely $\|x - y\|$.

If X_i and Y_j are non-empty compact and convex subsets of a normed linear space E , then when $|I| = 1$, we can see that X_i° and Y_i° are both non-empty compact and convex; on the other hand, when $|I| > 1$, it is easy to see that X_i° or Y_i° may be an empty set in general. In fact, when $E = \mathbb{R}^2$ with the Euclidean metric d and $X_1 = X_2 = \{x \in E \mid d(x, (0, 0)) \leq 1\}$;

$$Y_1 = \{x \in E \mid d(x, (3, 0)) \leq 1\}, \quad Y_2 = \{x \in E \mid d(x, (0, 3)) \leq 1\},$$

then each Y_i° is non-empty compact and convex but X_i° is an empty set.

For each $i \in I$, let X_i and Y_i be two non-empty subsets of a normed linear space E , and let $T_i : X = \prod_{i \in I} X_i \rightarrow 2^{Y_i}$ be a multifunction. Then the pair $(\bar{x}_i, T_i(\bar{x})) \in X_i \times Y_i$ is called the *best proximity pair* for T_i if $d(\bar{x}_i, T_i(\bar{x})) = d(X_i, Y_i)$ for each $i \in I$. Then the best proximity pair theorem seeks an appropriate solution $\bar{x} \in X$ which is optimal. In fact, the best proximity pair theorem analyzes the conditions which the problem of minimizing the real-valued function $x \mapsto d(x_i, T_i(x))$ has a simultaneous solution for each $i \in I$. In the best proximity pair, whenever $X_i = Y_i$ for each $i \in I$, then \bar{x} is actually the fixed point for the multifunction $T = \prod_{i \in I} T_i$, i.e., $\bar{x} \in T(\bar{x})$. Still there have been a number of generalized theorems on the existence of best proximity pairs which can be regarded as the optimal forms of Theorem A by several authors, e.g., see [2, 12, 19, 20, 23].

Let X be a non-empty subset of a normed linear space E , and consider the metric projection map $P_X : E \rightarrow 2^X$ defined by for each $z \in E$,

$$P_X(z) := \{x \in X \mid \|z - x\| = d(z, X) = d(z, X)\}.$$

Then, it is well-known that if X is non-empty compact and convex, then $P_X(x)$ is a non-empty compact and convex subset of X , and the multifunction P_X is upper semicontinuous in X . For the properties of the metric projection, see [18-20].

Next, we recall the following definition due to Kim and Lee which generalizes the concavity condition:

Definition 1 ([11]). Let X be a non-empty topological space, Y an arbitrary set. Then $f : X \times Y \rightarrow \mathbb{R}$ is called *C-concave on X* if for every $n \geq 2$, whenever n points $x_1, \dots, x_n \in X$ are arbitrarily given, there exists a continuous function $\phi_n : [0, 1]^n \rightarrow X$ such that

$$f(\phi_n(\lambda_1, \dots, \lambda_n), y) \geq \lambda_1 f(x_1, y) + \dots + \lambda_n f(x_n, y)$$

for all $\lambda_i \in [0, 1]$, $i = 1, \dots, n$, with $\sum_{i=1}^n \lambda_i = 1$, and for all $y \in Y$.

Remark. As remarked in [11], the concavity clearly implies the C-concavity by letting $\phi_n(\lambda_1, \dots, \lambda_n) := \lambda_1 x_1 + \dots + \lambda_n x_n$, whenever $x_1, \dots, x_n \in X$ are given; however, we do not know the implications between the quasi-concavity and the C-concavity.

Also recall that a topological space X is said to have the *fixed point property* (or is a *fixed point space*) [5] if every continuous mapping $f : X \rightarrow X$ has a fixed point in X . Clearly this property is topologically invariant, and note that the product of two fixed point spaces need not be a fixed point space. In contrast with finite products, an infinite product of non-empty compact fixed point spaces will be a fixed point space whenever every finite product of those spaces is a fixed point space, e.g. see [5, p.174].

A multifunction $T : X \rightarrow 2^Y$ from a topological space X to a Hausdorff topological vector space Y is said to be a *Kakutani multifunction* [13] if the following conditions are satisfied: (i) T is upper semicontinuous; (ii) either $T(x)$ is singleton for each $x \in X$ or for each $x \in X$, $T(x)$ is a non-empty compact convex subset of Y . A multifunction $S : X \rightarrow 2^Y$ is said to be a *Kakutani factorizable multifunction* [13] if it can be expressed as a composition of finitely many Kakutani multifunctions, i.e., $S = T_1 \circ T_2 \circ \cdots \circ T_n$, and each T_i is a Kakutani factorizable multifunction.

3. Existence of best proximity pairs

The following Lassonde fixed point theorem for Kakutani factorizable multifunctions is a generalization of the well-known Fan-Glicksberg's fixed point theorem, and it is essential in proving the existence of best proximity pairs:

Theorem 1 ([13]). *If X is a non-empty compact and convex subset of a locally convex Hausdorff topological vector space, then any Kakutani factorizable multifunction $T : X \rightarrow 2^X$ has a fixed point, i.e., there exists a point $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$.*

Now we will prove a new existence theorem for the best proximity pairs as follow:

Theorem 2. *Let I be an (possibly infinite) index set and for each $i \in I$, let X_i and Y_i be non-empty compact and convex subsets of a normed linear space E , and let $A_i : X = \prod_{i \in I} X_i \rightarrow 2^{Y_i}$ be an upper semicontinuous multifunction in X such that $A_i(x)$ is a non-empty closed and convex subset of Y_i for each $x \in X$. Assume that for each $i \in I$,*

$$(*) \quad A_i(x) \cap Y_i^\circ \neq \emptyset \quad \text{for all } x \in X.$$

Then there exists a point $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ of best proximity pairs such that for each $i \in I$, $d(\bar{x}_i, A_i(\bar{x})) = d(X_i, Y_i)$.

Proof. For each $i \in I$, since X_i and Y_i are non-empty compact and convex, it is easy to prove that Y_i° is a non-empty compact and convex subset of Y_i . In fact, by the assumption (*), Y_i° is non-empty. Let $i \in I$ be arbitrarily fixed. In order to show Y_i° is convex, let $y_1, y_2 \in Y_i^\circ$ be given. Then, for each $j \in I$, there exist $x_j^1, x_j^2 \in X_j$ such that $d(x_j^k, y_k) = d(X_j, Y_i)$ for $k = 1, 2$. For any $\lambda \in (0, 1)$, we let $\hat{y} := \lambda y_1 + (1 - \lambda)y_2$, and $\hat{x}_j := \lambda x_j^1 + (1 - \lambda)x_j^2$. Since X_j

and Y_i are convex, $\hat{x}_j \in X_j$ and $\hat{y} \in Y_i$. Then for each $j \in I$, we have

$$\begin{aligned} d(\hat{x}_j, \hat{y}) &= \|(\lambda x_j^1 + (1 - \lambda)x_j^2) - (\lambda y_1 + (1 - \lambda)y_2)\| \\ &= \|\lambda(x_j^1 - y_1) + (1 - \lambda)(x_j^2 - y_2)\| \\ &\leq \lambda\|x_j^1 - y_1\| + (1 - \lambda)\|x_j^2 - y_2\| = d(X_j, Y_i) \end{aligned}$$

so that $\hat{y} \in Y_i^o$, and hence Y_i^o is convex. Similarly, the convexity for X_i^o can be proved. Next, in order to show that Y_i^o is a closed subset of Y_i , let $(y_n)_{n \in \mathbb{N}}$ be a sequence in Y_i^o which converges to $\tilde{y} \in Y_i$. For each $j \in I$, let $k_j := d(X_j, Y_i)$. Then, since $y_n \in Y_i^o$ for each $n \in \mathbb{N}$, there exists $x_j^n \in X_j$ such that $\|x_j^n - y_n\| = k_j$ for each $j \in I$. Since $\{x_j^n\} \subset X_j$ and X_j is compact, there exists a convergent subsequence $(x_j^{n_k})$ of (x_j^n) which converges to $\tilde{x}_j \in X_j$.

Then we can see that

$$\begin{aligned} k_j &\leq \|\tilde{x}_j - \tilde{y}\| = \|\tilde{x}_j - x_j^{n_k} + x_j^{n_k} - y_{n_k} + y_{n_k} - \tilde{y}\| \\ &\leq \|\tilde{x}_j - x_j^{n_k}\| + \|x_j^{n_k} - y_{n_k}\| + \|y_{n_k} - \tilde{y}\|. \end{aligned}$$

Since $\|\tilde{x}_j - x_j^{n_k}\| \rightarrow 0$ and $\|y_{n_k} - \tilde{y}\| \rightarrow 0$, and $\|x_j^{n_k} - y_{n_k}\| = k_j$, we have $\|\tilde{x}_j - \tilde{y}\| = k_j$ so that $\tilde{y} \in Y_i^o$. Therefore, Y_i^o is a closed subset of Y_i so that $Y^o := \prod_{i \in I} Y_i^o$ is a non-empty closed and convex subset of $Y = \prod_{i \in I} Y_i$.

Also, it is known that for each $i \in I$, the metric projection map $P_{X_i} : E \rightarrow 2^{X_i}$ is upper semicontinuous in E such that $P_{X_i}(z)$ is a non-empty compact and convex subset of X_i for each $z \in E$ (e.g., see [18, 24]).

For each $i \in I$, we now define a multifunction $A'_i : X \rightarrow 2^{Y_i^o}$ by

$$A'_i(x) := A_i(x) \cap Y_i^o \quad \text{for each } x \in X = \prod_{i \in I} X_i.$$

Then, by the assumptions, each $A'_i(x)$ is non-empty compact and convex in Y_i^o , and A'_i is upper semicontinuous in X .

Finally, we introduce two multifunctions $A' : X \rightarrow 2^{Y^o}$, defined by

$$A'(x) := \prod_{i \in I} A'_i(x) \quad \text{for each } x = (x_i)_{i \in I} \in \prod_{i \in I} X_i;$$

and $P'_X : \prod_{i \in I} Y_i \rightarrow 2^{\prod_{i \in I} X_i}$, defined by

$$P'_X(y) := \prod_{i \in I} P_{X_i}(y_i) \quad \text{for each } y = (y_i)_{i \in I} \in \prod_{i \in I} Y_i.$$

Since A'_i and P_{X_i} are upper semicontinuous with compact and convex values for each $i \in I$, by Lemma 3 in [6], A' and P'_X are both upper semicontinuous such that each $A'(x)$ is non-empty compact and convex in Y^o , and each $P'_X(y)$ is non-empty compact and convex in X . Hence, A' and P'_X are Kakutani multifunctions so that the composition map $P'_X \circ A' : X \rightarrow 2^X$ is a Kakutani factorizable multifunction. Therefore, by Theorem 1, there exists a fixed point $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ such that $\bar{x} \in (P'_X \circ A')(\bar{x})$. Then, $(\bar{x}_i)_{i \in I} \in P'_X(A'((\bar{x}_i)_{i \in I}))$ so that there exists an $(\bar{y}_i)_{i \in I} \in \prod_{i \in I} Y_i^o$ such that $(\bar{y}_i)_{i \in I} \in A'(\bar{x}) = \prod_{i \in I} (A_i(\bar{x}) \cap Y_i^o)$ and $\bar{x}_i \in P_{X_i}(\bar{y}_i)$ for each $i \in I$. Since each \bar{y}_i is an element in Y_i^o , there exists an $x'_i \in X_i$ such that $d(x'_i, \bar{y}_i) = d(X_i, Y_i)$

for each $i \in I$. Therefore, for each $i \in I$, we have

$$d(\bar{x}_i, A_i(\bar{x})) \leq d(\bar{x}_i, \bar{y}_i) = d(X_i, \bar{y}_i) \leq d(x'_i, \bar{y}_i) = d(X_i, Y_i)$$

so that $d(\bar{x}_i, A_i(\bar{x})) = d(X_i, Y_i)$ which completes the proof. \square

Remarks. (i) The assumptions of Theorem 2 is different from those of Theorem 1 and 2 in [12], and the conclusions have slight different forms. However, Theorem 2 can be regarded as an infinite generalization of Theorem 1 and 2 in [12].

(ii) If $X_i = Y_j$ and $A_i : X \rightarrow 2^{X_i}$ for each $i, j \in I$, then the assumption that " $A_i(x) \cap Y_i^\circ \neq \emptyset$ " is automatically satisfied. In fact, $Y_i^\circ = X_i$. Similarly, in case $Y_i = E$ for each $i \in I$, the assumption that " $A_i(x) \cap Y_i^\circ \neq \emptyset$ " is automatically satisfied and $Y_i^\circ = E$. In those cases, the conclusion means that \bar{x} is a fixed point for $\Pi_{i \in I} A_i$, which implies the Bohnenblust-Karlin fixed point theorem [5].

(iii) When $|I| = 1$, Theorem 2 generalizes Theorem 3.4 of Srinivasan-Veeramani [24] by relaxing the condition " $A(X^\circ) \subseteq Y^\circ$ " into the weaker condition " $A(x) \cap Y^\circ \neq \emptyset$ " for each $x \in X$.

Next we give a simple example which is suitable for Theorem 2, but the previous theorems due to Srinivasan-Veeramani [23, 24], Kim-Lee [12] and others can not be applied:

Example 1. Let $E = \mathbb{R}^2$ with the Euclidean metric d , and for each $i = 1, 2$, let

$$\begin{aligned} X_i &:= \{(x, 0) \mid 0 \leq x \leq 2\}; \\ Y_i &:= \{(x, y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}. \end{aligned}$$

And let $X := X_1 \times X_2$ and $Y := Y_1 \times Y_2$. Then it is easy to see that $X_i^\circ = X_i$ and $Y_i^\circ = \{(y, 1) \mid 0 \leq y \leq 2\}$, and $d(X_i, Y_i) = 1$ for each $i = 1, 2$. For each $i = 1, 2$, let $A_i : X \rightarrow 2^{Y_i}$ be a multifunction defined by for each $((x_1, 0), (x_2, 0)) \in X$,

$$\begin{aligned} A_1((x_1, 0), (x_2, 0)) &:= \{(2 - x_1, y) \mid 1 \leq y \leq |x_1 - 1| + 1\}; \\ A_2((x_1, 0), (x_2, 0)) &:= \{(x_2, y) \mid 1 \leq y \leq |x_2 - 1| + 1\}. \end{aligned}$$

Then, it is clear that each $A_i((x_1, 0), (x_2, 0))$ is a non-empty closed and convex subset of Y_i . Also, it is easy to see that A_i is upper semicontinuous in X . Then, for each $((x_1, 0), (x_2, 0)) \in X^\circ = X$, it is easy to see that $A_1((x_1, 0), (x_2, 0)) \cap Y_1^\circ = \{(2 - x_1, 1)\} \neq \emptyset$ and $A_2((x_1, 0), (x_2, 0)) \cap Y_2^\circ = \{(x_2, 1)\} \neq \emptyset$; however, $A_i((x_1, 0), (x_2, 0)) \not\subseteq Y_i^\circ$ for some $((x_1, 0), (x_2, 0)) \in X$, and $X = X_1 \times X_2$ is a product set so that the theorems due to Srinivasan-Veeramani [23, 24], Kim-Lee [12] can not be applied. However, all the hypotheses of Theorem 2 are satisfied so that there exists a point $\bar{x} = ((1, 0), (1, 0)) \in X$ of best proximity pairs such that $d((1, 0), A_i(\bar{x})) = d(X_i, Y_i) = 1$ for each $i = 1, 2$.

4. Existence of Nash equilibrium pairs

We first recall the equilibrium concept in the game theory due to Nash [15]. Let $I = \{1, \dots, n\}$ be a set of players. A *non-cooperative Nash game* (or *n-person game*) $\Gamma = (X_i; f_i)_{i \in I}$ of *normal form* is an ordered $2n$ -tuple $(X_1, \dots, X_n; f_1, \dots, f_n)$, where for each player $i \in I$, the non-empty set X_i is the player's pure strategy space, and $f_i : X = \prod_{i=1}^n X_i \rightarrow \mathbb{R}$ is the player's payoff function. The set X , *joint strategy space*, is the Cartesian product of the individual strategy sets, and an element of X_i is called a *strategy*. A strategy n -tuple $(\bar{x}_1, \dots, \bar{x}_n) \in X$ is called a *Nash equilibrium* for Γ if the following system of inequalities holds: for each $i = 1, \dots, n$,

$$f_i(\bar{x}_i, \bar{x}_{\hat{i}}) \geq f_i(x_i, \bar{x}_{\hat{i}}) \quad \text{for all } x_i \in X_i.$$

This pioneering concept on the existence of equilibrium for a non-cooperative n -person game was established by Nash in 1951, and is regraded as the basic result in equilibrium theories. Here it should be noted that it is possible to generalize that the set of players I may be infinite in the model of a game.

In recent papers [23, 24], Srinivasan and Veeramani considered a general economic situation which is motivated to introduce the equilibrium pair concept in a constrained generalized games as follow. Suppose that goods are manufactured and sold in different locations. Each location (or player) can be a manufacturing as well as selling unit. It is agreed that the ultimate place where the goods get sold would be determining the payoff for the goods. Let there be n such locations. For each location, two strategies X_i and Y_i are associated, one to that of manufacturing unit and other to that of selling unit. Knowing the manufacturing strategy $x_{\hat{i}} \in X_{\hat{i}} := \prod_{j \in I \setminus \{i\}} X_j$ of all other locations, the choice of selling strategy at the i th location is restricted to $A_i(x) \subseteq Y_i$. Also, let the payoff function $f_i : Y = \prod_{i=1}^n Y_i \rightarrow \mathbb{R}$ at the i th location be given such that f_i is the real-valued function at the given strategy $y \in Y$ which can maximize the payoff associated with the i th location. In fact, the preferred strategies are strongly dependent to the consumer's behavior at the i th location. Under those settings, they investigated the system of multifunctions $A_i : X \rightarrow 2^{Y_i}$ and $f_i : Y \rightarrow \mathbb{R}$ for each $i \in I$. Moreover, the cost involved in the transportation of goods to different places should also be taken into account. In fact, the transporting cost from x_i to $A_i(x)$ can be considered as $d(x_i, A_i(x))$, and so it is very reasonable that its payoff function $f_i(y_i, y_{\hat{i}})$ on $A_i(x)$ can have a maximum whenever $d(x_i, A_i(x))$ has a minimum value $d(X_i, Y_i)$. In these situations, one can not expect an equilibrium for the Nash game of normal form by applying the previous results in [3, 4, 9, 11, 14-17] because the strategy sets X_i and Y_i may be different in general.

In [23, 24], as applications of their best proximity theorems, Srinivasan and Veeramani proved existence theorems of equilibrium pair for constrained generalized games by using the conditions (A) and (B), respectively. And, as remarked, the condition (c) of Definition 4.3 in [23] has less economic information

for constrained generalized games. This is the motivation of our equilibrium pair concept for a generalized Nash games, and also we shall prove an existence theorem of equilibrium pair for constrained generalized games without assuming the conditions (A) in [24] nor (B) in [23].

In this section, as an application of our best proximity pair theorem (Theorem 2), we shall prove an existence theorem of equilibrium pair for a generalized Nash games. Now we will introduce a new concept of equilibrium pairs in generalized Nash games as follows:

Definition 2. Let I be a finite or an infinite set of players or locations. For each $i \in I$, let X_i be a non-empty set of manufacturing commodities, and Y_i be a non-empty set of selling commodities. A *non-cooperative generalized Nash game* $\Gamma = (X_i, Y_i; A_i, f_i)_{i \in I}$ of normal form is defined as a family of ordered quadruples $(X_i, Y_i; A_i, f_i)$ where X_i and Y_i are non-empty subsets of a normed linear space E , and $A_i : X = \prod_{j \in I} X_j \rightarrow 2^{Y_i}$ is a constraint correspondence, and $f_i : Y = \prod_{i=1}^n Y_i \rightarrow \mathbb{R}$ is the player's payoff function. A *Nash equilibrium pair* for Γ is a pair of points $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i \in I}, (\bar{y}_i)_{i \in I}) \in X \times Y$ such that for each $i \in I$,

$$\bar{y}_i \in A_i(\bar{x}) \quad \text{with} \quad d(\bar{x}_i, \bar{y}_i) = d(\bar{x}_i, A_i(\bar{x})) = d(X_i, Y_i), \quad \text{and}$$

$$f_i(\bar{y}_i, \bar{y}_i) \geq f_i(y_i, \bar{y}_i) \quad \text{for all } y_i \in A_i(x_i).$$

In particular, when $I = \{1, \dots, n\}$, we may call Γ a *generalized n -person game of normal form*.

Remarks. (i) The transporting cost conditions $\|x - y_i\| = d(X, \hat{Y}_i)$ in Theorem 4.5 in [23] and $\|x - y\| = d(X, Y)$ in Definition 4.3 in [24] have reasonable economic meanings in their settings, and it should be noted that our condition is related with the Definition 4.3 in [24] as

$$\|x - y\| = d(X, Y) \iff \|x_i - y_i\| = d(X_i, Y_i) \quad \text{for each } i \in I.$$

(ii) When $X_i = Y_i$ and $A_i(x) = X_i$ for each $i \in I$ and $x \in X$ in Definition 2, then the Nash equilibrium pair can be reduced to the standard definitions of equilibrium in the game theory due to Nash [14, 15], Debreu [4] or Friedman [9], and so the Nash game $\Gamma' = (X_i, ; f_i)_{i \in I}$ can be considered as a subgame of the generalized Nash game $\Gamma = (X_i, Y_i; A_i, f_i)_{i \in I}$ of normal form. For more economic meanings and interpretations, see [3, 9, 14-17].

In [17], in order to obtain a Nash equilibrium for a game Γ , Nikaido and Isoda define the total sum of payoff functions $H : X \times X \rightarrow \mathbb{R}$ associated with the non-cooperative game $\Gamma = (X_i, ; f_i)_{i \in I}$, as follows:

$$H(x, y) := \sum_{i=1}^n f_i(y_1, \dots, y_{i-1}, x_i, y_{i+1}, \dots, y_n)$$

for every $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in X = \prod_{i=1}^n X_i$.

Here we note that if for each $i \in I$, the function $y_i \mapsto f_i(y_i, x_i)$ is concave as in Theorem B, then the function $x \mapsto H(x, y)$ is concave; and so it is \mathcal{C} -concave.

As mentioned before, we can generalize the total sum of payoff functions for the infinite generalized Nash game Γ . For this, we shall need the following. Let I be an infinite set of players, and let $\Gamma = (X_i, Y_i; A_i, f_i)_{i \in I}$ be a non-cooperative generalized Nash game of normal form. For each $i \in I$, $f_i : Y_i \times Y_{\hat{i}} \rightarrow \mathbb{R}$ is said to be *unconditionally summable* if for each $z_i \in Y_i$ and $y_{\hat{i}} \in Y_{\hat{i}}$, any rearrangement $\sum_{j \in I} f_j(z_j, y_j)$ of the infinite sum $\sum_{i \in I} f_i(z_i, y_{\hat{i}})$ converges to the same real value. Note that if $\Gamma = (X_i; f_i)_{i \in I}$ is a finite Nash game, then each f_i is automatically unconditionally summable.

Using the unconditionally summability, let us define the total sum of payoff functions $H : Y \times Y \rightarrow \mathbb{R}$ associated with the infinite generalized Nash game $\Gamma = (X_i, Y_i; A_i, f_i)_{i \in I}$ as follows:

$$H(z, y) := \sum_{i \in I} f_i(z_i, y_{\hat{i}}) \quad \text{for each } y = (y_i)_{i \in I}, z = (z_i)_{i \in I} \in Y.$$

The following is essential in proving the existence of generalized Nash equilibrium:

Lemma 1. *Let I be an infinite set of players and let $\Gamma = (X_i; f_i)_{i \in I}$ be a non-cooperative Nash game of normal form where f_i is unconditionally summable for each $i \in I$. If there exists a point $\bar{x} \in X = \prod_{i \in I} X_i$ with $H(\bar{x}, \bar{x}) \geq H(x, \bar{x})$ for any $x \in X$, then \bar{x} is a Nash equilibrium for Γ .*

Proof. For each $i \in I$ and for any $x = (x_i, \bar{x}_{\hat{i}}) \in X$, $x_i \in X_i$, by substitution, we can see that

$$f_i(\bar{x}_i, \bar{x}_{\hat{i}}) \geq f_i(x_i, \bar{x}_{\hat{i}}) \quad \text{for all } x_i \in X_i$$

so that \bar{x} is a Nash equilibrium. \square

We shall need the following which is an infinite generalization of Theorem 1 in [11] where their proof can be worked without any changes:

Lemma 2 ([11]). *Let I be an infinite set of players, and let $\Gamma = (X_i; f_i)_{i \in I}$ be a non-cooperative Nash game of normal form. For each $i \in I$, $f_i : X_i \times X_{\hat{i}} \rightarrow \mathbb{R}$ is unconditionally summable and satisfies the following:*

(i) *the set $X = \prod_{i \in I} X_i$ is non-empty compact and has the fixed-point property;*

(ii) *the function $H(x, y)$ is continuous on $X \times X$;*

(iii) *the function $x \mapsto H(x, y)$ is \mathcal{C} -concave on X .*

Then Γ has at least one Nash equilibrium.

Remark. As remarked in [11], Lemma 2 generalizes the previous equilibrium existence theorems due to Nash [17] and Nikaido and Isoda [16] in several aspects.

As an application of Theorem 2, we will prove the existence of Nash equilibrium pairs for a generalized Nash game with infinite set of players as follows:

Theorem 3. Let I be an infinite set of players and let $\Gamma = (X_i, Y_i; A_i, f_i)_{i \in I}$ be a generalized Nash game of normal form. For each $i \in I$, the payoff function $f_i : Y = Y_i \times Y_{-i} \rightarrow \mathbb{R}$ is unconditionally summable and assume the following conditions:

(1) X_i and Y_i are non-empty compact and convex subsets of a normed linear space E ;

(2) $A_i : X = \prod_{j \in I} X_j \rightarrow 2^{Y_i}$ is an upper semicontinuous correspondence such that each $A_i(x)$ is a non-empty closed and convex subset of Y_i , and satisfies the condition (*) in Theorem 2;

(3) the payoff function $f_i : Y_i \times Y_{-i} \rightarrow \mathbb{R}$ is continuous;

(4) the function $y_i \mapsto f_i(y_i, x_{-i})$ is concave.

Then there exists an equilibrium pair $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i \in I}, (\bar{y}_i)_{i \in I}) \in X \times Y$ for Γ , i.e., for each $i \in I$, $\bar{y}_i \in A_i(\bar{x})$ such that $d(\bar{x}_i, A_i(\bar{x})) = d(X_i, Y_i)$ and

$$f_i(\bar{y}_i, \bar{y}_{-i}) \geq f_i(y_i, \bar{y}_{-i}) \quad \text{for all } y_i \in A_i(\bar{x}).$$

Proof. For each $i \in I$, since A_i satisfies the whole assumptions of Theorem 2, there exists a point $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ of best proximity pairs, i.e., for each $i \in I$, $\{\bar{x}_i\} \times A_i(\bar{x}) \subseteq X_i \times Y_i$ such that $d(\bar{x}_i, A_i(\bar{x})) = d(X_i, Y_i)$. It remains to show the existence of Nash equilibrium on the constraint set $\prod_{i \in I} A_i(\bar{x})$. Since $A_i(\bar{x})$ is a non-empty closed convex subset of Y_i , we can now restrict the generalized Nash game $\Gamma = (X_i, Y_i; A_i, f_i)_{i \in I}$ to the Nash subgame $\Gamma' = (A_i(\bar{x}); f_i)_{i \in I}$, and also restrict the domain of the total sum of payoff functions H on $\mathcal{A} \times \mathcal{A}$ where $\mathcal{A} := \prod_{i \in I} A_i(\bar{x})$. Then the whole assumptions of Lemma 2 are satisfied so that there exists a Nash equilibrium $(\bar{y}_i)_{i \in I} \in \prod_{i \in I} A_i(\bar{x})$ for the Nash subgame $\Gamma' = (A_i(\bar{x}); f_i)_{i \in I}$, i.e., for each $i \in I$,

$$f_i(\bar{y}_i, \bar{y}_{-i}) \geq f_i(y_i, \bar{y}_{-i}) \quad \text{for all } y_i \in A_i(\bar{x}).$$

Therefore, we can obtain an equilibrium pair $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i \in I}, (\bar{y}_i)_{i \in I}) \in X \times \mathcal{A} \subseteq X \times Y$ for the generalized Nash game Γ . This completes the proof. \square

Remark. In Theorem 3, we can relax the assumptions (3) and (4) into the following weaker assumptions without affecting the conclusion:

(3') the total sum of payoff functions $H : Y \times Y \rightarrow \mathbb{R}$, defined by

$$H(z, y) := \sum_{i \in I} f_i(z_i, y_i) \quad \text{for each } y = (y_i)_{i \in I}, (z_i)_{i \in I} \in Y,$$

satisfies the following conditions

- (i) the function $(z, y) \mapsto H(z, y)$ is continuous on $Y \times Y$;
- (ii) the function $z \mapsto H(z, y)$ is C -concave on Y .

In Theorem 3, when $X_i = Y_i$ and $A_i(x) = X_i$ for each $i \in I$ and for all $x \in X$ where each X_i is assumed to be contained in the Euclidean space \mathbb{R}^n , we can obtain the infinite generalization of the existence of Nash equilibrium (i.e., Theorem B):

Corollary 1. *Let I be an infinite set of players and $\Gamma = (X_i; f_i)_{i \in I}$ be a non-cooperative Nash game where each f_i is unconditionally summable. Assume that for each $i \in I$,*

- (a) *the set $X_i \subset \mathbb{R}^{k_i}$ is non-empty compact and convex;*
- (b) *the function f_i is continuous;*
- (c) *the function $y_i \mapsto f_i(y_i, x_i)$ is concave.*

Then there exists at least one Nash equilibrium for Γ .

Here we note that the relations between the constraint multifunction A_i and the payoff function f_i on $A_i(x)$. In real market economies, whenever i th player can minimize his/her transporting cost from x_i to $A_i(x)$, it is very natural that its payoff function $f_i(y_i, y_i)$ can be maximizable. In fact, if $y_i \in A_i(x)$ with $d(x_i, y_i) > d(x_i, A_i(x)) = d(X_i, Y_i)$, then it is plausible that $f_i|_{A_i(x)}$ can not attain a maximum at y_i . Therefore, the following compatibility assumption on the relation between A_i and f_i is reasonable in the generalized Nash game with normal form: for each $i \in I$,

(†)

If $f_i|_{A_i(x)}$ attains a maximum at y_i in $A_i(x)$, then $d(x_i, y_i) = d(x_i, A_i(x))$.

Adding the above assumption (†) in Theorem 3, we can obtain the existence of Nash equilibrium pairs for a generalized Nash game Γ as follows:

Theorem 4. *Let I be an infinite set of players and let $\Gamma = (X_i, Y_i; A_i, f_i)_{i \in I}$ be a generalized Nash game with normal form. For each $i \in I$, the payoff function $f_i : Y_i \times Y_i \rightarrow \mathbb{R}$ is unconditionally summable and assume the following conditions:*

- (1) *X_i and Y_i are non-empty compact and convex subsets of a normed linear space E ;*
- (2) *$A_i : X = \prod_{j \in I} X_j \rightarrow 2^{Y_i}$ is an upper semicontinuous correspondence such that each $A_i(x)$ is a non-empty closed and convex subset of Y_i , and satisfies the condition (*) in Theorem 2 and the condition (†);*
- (3) *the total sum of payoff functions $H : Y \times Y \rightarrow \mathbb{R}$ satisfies the following conditions*
 - (i) *the function $(z, y) \mapsto H(z, y)$ is continuous on $Y \times Y$;*
 - (ii) *the function $z \mapsto H(z, y)$ is C -concave on Y .*

Then there exists a Nash equilibrium pair $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i \in I}, (\bar{y}_i)_{i \in I}) \in X \times Y$ for Γ , i.e., for each $i \in I$, $\bar{y}_i \in A_i(\bar{x})$ such that $d(\bar{x}_i, \bar{y}_i) = d(\bar{x}_i, A_i(\bar{x})) = d(X_i, Y_i)$ and

$$f_i(\bar{y}_i, \bar{y}_i) \geq f_i(y_i, \bar{y}_i) \text{ for all } y_i \in A_i(\bar{x}).$$

Proof. By the condition (†), we can obtain that the equilibrium pair $(\bar{x}, \bar{y}) \in X \times Y$ in the conclusion of Theorem 3 is actually the Nash equilibrium pair for Γ . □

Remark. As already mentioned, the condition (†) can be reformulated as the contrapositive form:

(††) If $d(x_i, y_i) > d(x_i, A_i(x))$ for some $y_i \in A_i(x)$, then $f_i|_{A_i(x)}$ can not attain a maximum at y_i in $A_i(x)$.

Finally, we shall give an example of a generalized 2-person game which is suitable for Theorem 4, but the previous equilibrium existence theorems in [14-17, 23, 24] can not be applied:

Example 2. Let $\Gamma = (X_i, Y_i; A_i, f_i)_{i=1,2}$ be a generalized 2-person game such that $E = \mathbb{R}^2$ with the Euclidean metric d , and for each $i = 1, 2$,

$$X_i := \{(x, 0) \mid 0 \leq x \leq 2\};$$

$$Y_i := \{(x, y) \mid 0 \leq x \leq 2, 1 \leq y \leq 3\}.$$

And let $X := X_1 \times X_2$ and $Y := Y_1 \times Y_2$. Then it is easy to see that $X_i^\circ = X_i$ and $Y_i^\circ = \{(y, 1) \mid 0 \leq y \leq 2\}$, and $d(X_i, Y_i) = 1$ for each $i = 1, 2$.

Let $A_i : X \rightarrow 2^{Y_i}$ and $f_i : Y \rightarrow \mathbb{R}$ be correspondences defined by for each $x = ((x_1, 0), (x_2, 0)) \in X$ and $((y_1, y_2), (z_1, z_2)) \in Y$,

$$A_1((x_1, 0), (x_2, 0)) := \{(x_1, a) \mid 1 \leq a \leq (x_1 - 1)^2 + 2\};$$

$$A_2((x_1, 0), (x_2, 0)) := \{(2 - x_2, b) \mid 1 \leq b \leq (x_1 - 1)^2 + 2\};$$

$$f_1((y_1, y_2), (z_1, z_2)) := 1 - y_2^2; \quad f_2((y_1, y_2), (z_1, z_2)) := 1 - z_2^2.$$

Then, it is clear that each $A_i(x)$ is non-empty closed and convex subset of Y_i for each $x \in X$ and $i = 1, 2$. Also, it is easy to see that A_i is upper semicontinuous in X . And, we can see that $X^\circ = X$, $Y^\circ = \{(y_1, 1), (y_2, 1) \mid 0 \leq y_1, y_2 \leq 3\}$, and $d(X_i, Y_i) = 1$ for each $i = 1, 2$. Then, for each $x \in X^\circ = X$, it is easy to see that $A_1(x) \cap Y_1^\circ = \{(x_1, 1)\} \neq \emptyset$ and $A_2(x) \cap Y_2^\circ = \{(2 - x_2, 1)\} \neq \emptyset$; however, $A_i((1, 0), (1, 0)) \not\subseteq Y_i^\circ$ for each $i = 1, 2$, and $A_i^{-1}(a, 1)$ is not open in X for each $(a, 1) \in Y$ so that the previous theorems due to Debreu [4], Nash [14, 15], Kim-Lee [11, 12], Srinivasan-Veeramani [24] can not be applied in this model. It is easy to see that since each f_i is continuous and concave, the assumptions (3)(i) and (ii) of Theorem 4 are satisfied so that there exists a Nash equilibrium pair $(\bar{x}, \bar{y}) \in X \times Y$ for Γ , that is, we can obtain that $\bar{x} = ((1, 0), (1, 0))$, $\bar{y} = ((1, 1), (1, 1))$ is the desired Nash equilibrium pair for the generalized 2-person game Γ . In fact, for each $i = 1, 2$, we have

$$\bar{y}_i = (1, 1) \in A_i((1, 0), (1, 0)) = \{(1, a) \mid 1 \leq a \leq 2\};$$

$$1 = d(\bar{x}_i, A_i(\bar{x})) = d(X_i, Y_i); \text{ and}$$

$$0 = f_1((1, 1), (1, 1)) \geq f_1((y_1, y_2), (1, 1)) = 1 - y_2^2, \quad \forall (y_1, y_2) \in A_1((1, 0), (1, 0));$$

$$0 = f_2((1, 1), (1, 1)) \geq f_2((1, 1), (z_1, z_2)) = 1 - z_2^2, \quad \forall (z_1, z_2) \in A_2((1, 0), (1, 0)).$$

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