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Intuitionistic Fuzzy Semigroups

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Abstract

We give some properties of intuitionistc fuzzy left, right, and two-sided ideals and bi-ideals of a semigroup. And we characterize a regular semigroup, a semigroup that is a lattice of left(right) simple semigroups, a semigroup that is a semilattice of left(right) groups and a semigroup that is a semilattice of groups in terms of intuitionistic fuzzy ideals and intuitionistic fuzzy bi-ideals.

Key words : intuitionstic fuzzy set, intuitionistic fuzzy subsemigroup, intuitionistic fuzzy left[resp. right]ideal, intuitionistic fuzzy bi-ideal, regular semigroup.

0. Introduction

As a generalization of fuzzy sets defined by Zadeh[26], the notion of intuitionistic fuzzy sets was introduced by Atanassov[2] 1986. After that time, Çoker et al.[6,7,8], Lee and Lee[22], and Hur et al.[13] applied the concept of intuitionistic fuzzy sets to topology. In particular, Hur et al.[12] applied the notion of intuitionistic fuzzy sets to topological group. Also, several researchers[1,3,4,9-11,14,15] applied one to algebra.

In this paper, we give some properties of intuitionistc fuzzy left, right, and two-sided ideals and bi-ideals of a semigroup. And we characterize a regular semigroup, a semigroup that is a lattice of left(right) simple semigroups, a semigroup that is a semilattice of left(right) groups and a semigroup that is a semilattice of groups in terms of intuitionistic fuzzy ideals and intuitionistic fuzzy bi-ideals. Another characterization of such semigroup can be seen in [14].

1. Preliminaries

We will list some concept and one result needed in the later sections.

For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0, 1]$ as I.

Definition 1.1[2,6]. Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set*(in short, IFS) in X if $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, where the mapping $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership(namely $\nu_A(x)$) of each $x \in X$ to A, respectively. In particular, $0_\sim$ and $1_\sim$ denote the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in a set X defined by $0_\sim(x) = (0, 1)$

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and $1_{\sim}(x) = (1, 0)$ for each $x \in X$, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definition 1.2[2]. Let X be a nonempty sets and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be an IFSs in X. Then

(1) $A \subset B$ if and only if $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

(2) $A = B$ if and only if $A \subset B$ and $B \subset A$.

(3) $A^c = (\nu_A, \mu_A)$.

(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

Definition 1.3[6]. Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then

(1) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$.

(2) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

Let S be a semigroup. By a *subsemigroup* of S we mean a non-empty subset A of S such that

$$A^2 \subset A$$

and by a *left*[resp. *right*]*ideal* of S we mean a non-empty subset A of S such that

$$SA \subset A\text{[resp. } AS \subset A].$$

By *two-sided ideal* or, simply, *ideal* we mean a subset A of S which is both a left and a right ideal of S we will denote the set of all *left ideals*[resp. *right ideals* and *ideals*] of S as LI(S)[resp. RI(S) and I(S)].

Definition 1.4[9]. Let S be a semigroup and let $A \in$ IFS(S). Then A is called an *intuitionistic fuzzy subsemigroup*(in short, *IFSG*) of S if for any $x, y \in S$,

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \text{ and } \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y).$$

We will denote the set of all IFSGs as IFSG(S).

Example 1.4. Let $S = \{a, b, c\}$ be the semigroup with the following operation on S:

$\cdot$	a	b	c
a	a	b	c
b	a	b	c
c	a	b	c

We define a complex mapping $A = (\mu_A, \nu_A) : S \to I \times I$ as follows : $A(a) = (\lambda_1, \mu_1), A(b) = (\lambda_2, \mu_2)$ and $A(c) = (\lambda_3, \mu_3)$ where $\lambda_i, \mu_i \in I$ such that $0 \leq \lambda_i + \mu_i \leq 1$ for $i = 1, 2, 3$. Then we can easily see that $A \in$ IFSG(S). ■

Definition 1.5[9]. Let S be a semigroup and let $A \in$ IFS(S). Then A is called an:

(1) *intuitionistic fuzzy left ideal*(in short, *IFLI*) of S if $\mu_A(xy) \geq \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(y)$ for any $x, y \in S$.

(2) *intuitionistic fuzzy right ideal*(in short, *IFRI*) of S if $\mu_A(xy) \geq \mu_A(x)$ and $\nu_A(xy) \leq \nu_A(x)$ for any $x, y \in S$.

(3) *intuitionistic fuzzy ideal*(in short, *IFI*) of S if it is both an IFLI and an IFRL of S.

We will denote the set of all IFRIs[resp. IFLIs, and IFIs] of S as IFRI(S)[resp. IFLI(S), and IFI(S)].

Example 1.5 Let A be the intuitionistic fuzzy semigroup of S defined in Example 1.4. Then we can easily see that $A \in$ IFLI(S). On the other hand, if $A(a) \neq A(b), A(a) \neq A(c)$ or $A(b) \neq A(c)$, then $A \notin$IFRI(S). So $A \notin$IFI(S). However, if $A(a) = A(b) = A(c)$, then clearly $A \in$ IFI(S). ■

Result 1.A[9, Proposition 3.8]. Let A be a non-empty subset of a semigroup S and let χ_A be the characteristic function of A.

(1) A is a subsemigroup of S if and only if $(\chi_A, \chi_{A^c}) \in$ IFSG(S).

(2) $A \in$ LI(S)[resp. RI(S) and I(S)] if and only if $(\chi_A, \chi_{A^c}) \in$ IFLI(S)[resp. IFRI(S) and IFI(S)].

A subsemigroup A of a semigroup S is called a *bi-ideal* of S if $ASA \subset A$. We will denote the set of all bi-ideal of S as BI(S).

Result 1.B[9, Proposition 3.2]. Let S be a semigroup and let $0_\sim \neq A \in \text{IFS}(S)$. Then $A \in \text{IFSG}(S)$ if and only if $A \circ A \subset A$.

Result 1.C[16, Lemmas 1.6 and 1.6$'$]. Let S be a semigroup and let $A \in \text{IFS}(S)$. Then $A \in \text{IFLI}(S)$ [resp. $\text{IFRI}(S)$] if and only if $1_\sim \circ A \subset A$ [resp. $A \circ 1_\sim \subset A$].

Result 1.D[16, Theorem 1.7]. Let S be a semigroup and let $A \in \text{IFS}(S)$. Then $A \in \text{IFI}(S)$ if and only if $1_\sim \circ A \subset A$ and $A \circ 1_\sim \subset A$.

2. Properties of intuitionistic fuzzy bi-ideals of a semigroup

For an intuitionistic fuzzy bi-ideal of a semigroup, the following characterization is well-known.

Result 2.A[14, Proposition 2.5]. Let A be a nonempty subset of a semigroup S. Then $A \in \text{BI}(S)$ if and only if $(\chi_A, \chi_{A^c}) \in \text{IFBI}(S)$.

Result 2.B[14, Proposition 2.7]. Every IFLI[resp. IFRI and IFI] of a semigroup S is an IFBI of S.

Definition 2.1[14]. Let S be a semigroup and let $A \in \text{IFSG}(S)$. Then A is called an *intuitionistic fuzzy bi-ideal*(in short, *IFBI*) of S if for any $x, y, z \in S$,

$\mu_A(xyz) \geq \mu_A(x) \wedge \mu_A(z)$ and $\nu_A(xyz) \leq \nu_A(x) \vee \nu_A(z)$.

We will denote the set of all IFBIs of S as $\text{IFBI}(S)$.

Example 2.1. Let A be the intuitionistic fuzzy semigroup of S defined in Example 1.4. Then we can easily see that $A \in \text{IFBI}(S)$. ■

Definition 2.2[9]. Let $(X, \cdot)$ be a groupoid and let $A, B \in \text{IFS}(X)$. Then the *intuitionistic fuzzy product* $A \circ B$ of A and B is defined as follows : For each $x \in X$,

$$\mathbb{F}(A \circ B)(x) = \begin{cases} \bigvee_{x=yz}[\mu_A(y) \wedge \mu_B(z)], \bigwedge_{x=yz}[\nu_A(y) \\ \vee \nu_B(z)] \ \text{ if } x \text{ is expressible as } x = yz, \\ (0,1) \quad \text{otherwise.} \end{cases}$$

From Proposition 2.3(1) in [9], it is clear that if S is a semigroup, then "0" is associative in $\text{IFS}(S)$.

Theorem 2.3. Let S be a semigroup and let $A \in \text{IFS}(S)$. Then $A \in \text{IFBI}(S)$ if and only if $A \circ A \subset A$ and $A \circ 1_\sim \circ A \subset A$.

Proof.($\Rightarrow$): Suppose $A \in \text{IFBI}(S)$. Since $A \in \text{IFSG}(S)$, by Result 1.B, $A \circ A \subset A$. Let $a \in S$.

Case (i): Suppose $(A \circ 1_\sim \circ A)(a) = (0,1)$. Then it is clear that $A \circ 1_\sim \circ A \subset A$.

Case (ii): Suppose $(A \circ 1_\sim \circ A)(a) \neq (0,1)$. Then there exist $x, y, p, q \in S$ such that $a = xy$ and $x = pq$. Thus

$$\begin{aligned} \mu_{A \circ 1_\sim \circ A}(a) &= \mu_{(A \circ 1_\sim) \circ A}(a) \\ &= \bigvee_{a=xy}[\mu_{A \circ 1_\sim}(x) \wedge \mu_A(y)] \\ &= \bigvee_{a=xy}[(\bigvee_{x=pq}[\mu_A(p) \wedge \mu_{1_\sim}(q)]) \wedge \mu_A \\ &\quad \mu_A(y)] \\ &= \bigvee_{a=xy}[(\bigvee_{x=pq}(\mu_A(p) \wedge 1)) \wedge \mu_A(y)] \\ &= \bigvee_{a=xy}[\mu_A(p) \wedge \mu_A(y)] \\ &\leq \bigvee_{a=xy} \mu_A(pqy) \ (\text{Since } A \in \text{IFBI}(S)) \\ &= \bigvee_{a=xy} \mu_A(xy) \\ &= \mu_A(a) \end{aligned}$$

and

$$\begin{aligned} \nu_{A \circ 1_\sim \circ A}(a) &= \nu_{(A \circ 1_\sim) \circ A}(a) \\ &= \bigwedge_{a=xy}[\nu_{A \circ 1_\sim}(x) \vee \nu_A(y)] \\ &= \bigwedge_{a=xy}[(\bigwedge_{x=pq}[\nu_A(p) \vee \nu_{1_\sim}(q)]) \vee \nu_A \\ &\quad (y)] \\ &= \bigwedge_{a=xy}[(\bigwedge_{x=pq}(\nu_A(p) \vee 1)) \vee \nu_A(y)] \\ &= \bigwedge_{a=xy}[\nu_A(p) \vee \nu_A(y)] \\ &\geq \bigwedge_{a=xy} \nu_A(pqy) \\ &= \bigwedge_{a=xy} \nu_A(xy) \\ &= \nu_A(a). \end{aligned}$$

So $A \circ 1_\sim \circ A \subset A$. Hence, in all, $A \circ 1_\sim \circ A \subset A$.

($\Leftarrow$): Suppose $A \circ A \subset A$ and $A \circ 1_\sim \circ A \subset A$. Since $A \circ A \subset A$, by Result 1.B, it is clear that $A \in$ IFSG(S). Let $x, y, z \in S$ and let $a = xyz$. Then

$$\begin{aligned}\mu_A(xyz) &= \mu_A(a)\\ &\geq \mu_{(A\circ 1_\sim)\circ A}(a) \text{ (Since } A \circ 1_\sim \circ A \subset A)\\ &= \bigvee_{a=bc}[\mu_{A\circ 1_\sim}(b) \wedge \mu_A(c)]\\ &\geq \mu_{A\circ 1_\sim}(xy) \wedge \mu_A(z)\\ &= (\bigvee_{xy=pq}[\mu_A(p) \wedge \mu_{1_\sim(q)}] \wedge \mu_A(z)\\ &\geq \mu_A(x) \wedge \mu_{1_\sim}(y) \wedge \mu_A(z)\\ &= \mu_A(x) \wedge 1 \wedge \mu_A(z)\\ &= \mu_A(x) \wedge \mu_A(z)\end{aligned}$$

and

$$\begin{aligned}\nu_A(xyz) &= \nu_A(a)\\ &\geq \nu_{(A\circ 1_\sim)\circ A}(a)\\ &= \bigwedge_{a=bc}[\nu_{A\circ 1_\sim}(b) \vee \nu_A(c)]\\ &\leq \nu_{A\circ 1_\sim}(xy) \vee \nu_A(z)\\ &= (\bigwedge_{xy=pq}[\nu_A(p) \vee \nu_{1_\sim}(q)] \vee \nu_A(z)\\ &\leq \nu_A(x) \vee \nu_{1_\sim}(y) \vee \nu_A(z)\\ &= \nu_A(x) \vee 0 \vee \nu_A(z)\\ &= \nu_A(x) \vee \nu_A(z).\end{aligned}$$

Hence $A \in$ IFBI(S). This completes the proof. ■

Proposition 2.4. Let S be a semigroup, and let $A \in$ IFS(S) and let $B \in$ IFBI(S). Then $A \circ B, B \circ A \in$ IFBI(S).

Proof.
$$\begin{aligned}(A \circ B) \circ (A \circ B) &= A \circ [B \circ (A \circ B)]\\ &\subset A \circ (B \circ 1_\sim \circ B)\\ &\subset A \circ B. \text{ (By Theorem 2.3)}\end{aligned}$$

Thus it is clear that $A \circ B \in$ IFSG(S) from Result 1.B. On the other hand,

$$\begin{aligned}(A \circ B) \circ 1_\sim \circ (A \circ B) &= A \circ [B \circ (1_\sim \circ A) \circ B]\\ &\subset A \circ (B \circ 1_\sim \circ B)\\ &\subset A \circ B. \text{ (By Theorem 2.3)}\end{aligned}$$

Hence, by Theorem 2.3, $A \circ B \in$ IFBI(S). By the similar arguments, it can be seen that $B \circ A \in$ IFBI(S). This completes the proof. ■

3. Regular semigroups

A semigroup S is said to be *regular* if for each $a \in S$, there exists an $x \in S$ such that $a = axa$. As it is well-known (See Theorem 2.6 in [21]), a semigroup S is regular if and only if $B = BSB$ for each $B \in$ BI(S). Now we will give a characterization of a regular semigroup by intuitionistic fuzzy bi-ideals.

Theorem 3.1. Let S be a semigroup. Then S is regular if and only if $A = A \circ 1_\sim \circ A$ for each $A \in$ IFBI(S).

Proof.($\Rightarrow$): Suppose S is regular. Let $A \in$ IFBI(S) and let $a \in S$. Since S is regular, there exists an $x \in S$ such that $a = axa$. Thus

$$\begin{aligned}\mu_{A\circ 1_\sim \circ A}(a) &= \bigvee_{a=yz}[\mu_{A\circ 1_\sim}(y) \wedge \mu_A(z)]\\ &\geq \mu_{A\circ 1_\sim}(ax) \wedge \mu_A(a)\\ &= (\bigvee_{ax=pq}[\mu_A(p) \wedge \mu_{1_\sim}(q)] \wedge \mu_A(a)\\ &\geq \mu_A(a) \wedge \mu_{1_\sim}(x) \wedge \mu_A(a)\\ &= \mu_A(a) \wedge 1 \wedge \mu_A(a)\\ &= \mu_A(a)\end{aligned}$$

and

$$\begin{aligned}\nu_{A\circ 1_\sim \circ A}(a) &= \bigwedge_{a=yz}[\nu_{A\circ 1_\sim}(y) \vee \nu_A(z)]\\ &\leq \nu_{A\circ 1_\sim}(ax) \vee \nu_A(a)\\ &= (\bigwedge_{ax=pq}[\nu_A(p) \vee \nu_{1_\sim}(q)] \vee \nu_A(a)\\ &\leq \nu_A(a) \vee \nu_{1_\sim)}(x) \vee \nu_A(a)\\ &= \nu_A(a) \vee 0 \vee \nu_A(a)\\ &= \nu_A(a).\end{aligned}$$

So $A \subset A \circ 1_\sim \circ A$. Since $A \in$ IFBI(S), it is clear that $A \circ 1_\sim \circ A \subset A$ from Theorem 2.3. Hence $A = A \circ 1_\sim \circ A$.

($\Leftarrow$): Suppose $A = A \circ 1_\sim \circ A$ for each $A \in$ IFRI(S). Let $A \in$ BI(S) and let $a \in A$. Then, by Result 2.A, $(\chi_A, \chi_{A^c}) \in$ IFBI(S). By the hypothesis, $(\chi_A, \chi_{A^c}) \circ 1_\sim \circ (\chi_A, \chi_{A^c}) = (\chi_A, \chi_{A^c})$. Thus

$$\begin{aligned}[((\chi_A, \chi_{A^c}) \circ 1_\sim) \circ (\chi_A, \chi_{A^c})](a) &= (\chi_A, \chi_{A^c})(a)\\ &= (1, 0).\end{aligned}$$

On the other hand,

$$\begin{aligned}&[((\chi_A, \chi_{A^c}) \circ 1_\sim) \circ (\chi_A, \chi_{A^c})](a)\\ &= (\bigvee_{a=yz}[\mu_{(\chi_A,\chi_{A^c})\circ 1_\sim}(y) \wedge \chi_A(z)]\\ &\quad , \bigwedge_{a=yz}[\nu_{(\chi_A,\chi_{A^c})\circ 1_\sim}(y) \vee \chi_{A^c}(z)]).\end{aligned}$$

Then

$$\bigvee_{a=yz}[\mu_{(\chi_A,\chi_{A^c})\circ 1_\sim}(y) \wedge \chi_A(z)] = 1$$

and

$\bigwedge_{a=yz}[\nu_{(\chi_A,\chi_{A^c})\circ 1_\sim}(y) \vee \chi_{A^c}(z)] = 0.$

Thus there exist $b, c \in S$ with $a = bc$ such that

$$\mu_{(\chi_A,\chi_{A^c})\circ(\chi_S,\chi_{S^c})}(b) = 1,\ \nu_{(\chi_A,\chi_{A^c})\circ(\chi_S,\chi_{S^c})}(b) = 0$$

and

$$\chi_A(c) = 1\ ,\ \chi_{A^c}(c) = 0.$$

So $\bigvee_{b=pq}[\mu_A(p) \wedge \mu_{1_\sim}(q)] = 1$ and $\bigwedge_{b=pq}[\chi_{A^c}(p) \wedge \nu_{1_\sim}(q)] = 0$. Then there exist $d, e \in S$ with $b = de$ such that

$$\mu_A(d) = 1\ ,\ \chi_{A^c}(d) = 0 \text{ and } \mu_{1_\sim}(e) = 1,\ \nu_{1_\sim}(e) = 0.$$

Thus $d \in A$, $e \in S$ and $c \in A$, i.e., $a = bc = (de)c \in ASA$. So $A \subset ASA$. Since $A \in \mathrm{BI}(S)$, it is clear that $ASA \subset A$. Thus $A = ASA$. Hence S is regular. This completes the proof. ■

Theorem 3.2. Let S be a regular semigroup and let $A \in \mathrm{IFS}(S)$. Then $A \in \mathrm{IFBI}(S)$ if and only if there exist $B \in \mathrm{IFRI}(S)$ and $C \in \mathrm{IFLI}(S)$ such that $A = B \circ C$.

Proof.(⇒): Suppose $A \in \mathrm{IFBI}(S)$. Since S is regular, it is clear that $A = A \circ 1_\sim \circ A$ from Theorem 3.1. Then

$$\begin{aligned} A &= A \circ 1_\sim \circ A = A \circ 1_\sim \circ (A \circ 1_\sim \circ A) \\ &= (A \circ 1_\sim \circ A) \circ (1_\sim \circ A) \subset (A \circ 1_\sim \circ (1_\sim \circ A) \\ &= A \circ (1_\sim \circ 1_\sim) \circ A \subset A \circ 1_\sim \circ A \subset A. \end{aligned}$$

(By Theorem 2.3)

Thus $A = (A \circ 1_\sim) \circ (1_\sim \circ A)$. Let $A \circ 1_\sim = B$ and let $1_\sim \circ A = C$. Then, by Result 1.C, $B \in \mathrm{IFRI}(S)$ and $C \in \mathrm{IFLI}(S)$. Hence there exist $B \in \mathrm{IFRI}(S)$ and $C \in \mathrm{IFLI}(S)$ such that $A = BC$.

(⇐): Let $A \in \mathrm{IFS}(S)$. Suppose there exist $B \in \mathrm{IFRI}(S)$ and $C \in \mathrm{IFLI}(S)$ such that $A = BoC$. Then, by Result 2.B, $B, C \in \mathrm{IFBI}(S)$. By Proposition 2.4, $B \circ C \in \mathrm{IFBI}(S)$. Hence $A \in \mathrm{IFBI}(S)$. This completes the proof. ■

Result 3.A[18, Theorem 5]. Let S be a semigroup. Then S is regular if and only if $B \cap J = BJB$ for each $B \in \mathrm{BI}(S)$ and each $J \in \mathrm{I}(S)$.

We will give another characterization if such a semigroup.

Theorem 3.3. Let S be a semigroup. Then S is regular if and only if $A \cap B = A \circ B \circ A$ for each $A \in \mathrm{IFBI}(S)$ and each $B \in \mathrm{IFI}(S)$.

Proof.(⇒): Suppose S is regular. Let $A \in \mathrm{IFBI}(S)$ and let $B \in \mathrm{IFI}(S)$. Then, by Theorem 2.3, $A \circ B \circ A \subset A \circ 1_\sim \circ A \subset A$. By Result 1.D,

$$A \circ B \circ A \subset 1_\sim \circ B \circ 1_\sim \subset 1_\sim \circ B \subset B.$$

Thus $A \circ B \circ A \subset A \cap B$. Now let $a \in S$. Since S is regular, there exists an $x \in S$ such that $a = axa (= axaxa)$. Then, since $B \in \mathrm{IFI}(S)$,

$$\mu_B(xax) \geq \mu_B(ax) \geq \mu_B(a)$$

and

$$\nu_B(xax) \leq \nu_B(ax) \leq \nu_B(a).$$

Thus

$$\begin{aligned} \mu_{A\circ B\circ A}(a) &= \bigvee_{a=yz}[\mu_A(y) \wedge \mu_{B\circ A}(z)] \\ &\geq \mu_A(a) \wedge \mu_{B\circ A}(xaxa) \\ &= \mu_A(a) \wedge (\bigvee_{xaxa=pq}[\mu_B(p) \wedge \mu_A(q)]) \\ &\geq \mu_A(a) \wedge \mu_B(xax) \wedge \mu_A(a) \\ &\geq \mu_A(a) \wedge \mu_B(a) = \mu_{A\cap B}(a) \end{aligned}$$

and

$$\begin{aligned} \nu_{A\circ B\circ A}(a) &= \bigwedge_{a=yz}[\nu_A(y) \vee \nu_{B\circ A}(z)] \\ &\leq \nu_A(a) \vee \nu_{B\circ A}(xaxa) \\ &= \nu_A(a) \vee (\bigwedge_{xaxa=pq}[\nu_B(p) \vee \nu_A(q)]) \\ &\leq \nu_A(a) \vee \nu_B(xax) \vee \nu_A(a) \\ &\leq \nu_A(a) \vee \nu_B(a) = \nu_{A\cap B}(a). \end{aligned}$$

So $A \cap B \subset A \circ B \circ A$. Hence $A \circ B \circ A = A \cap B$.

(⇐): Suppose the necessary condition holds and let $A \in \mathrm{IFBI}(S)$. It is clear that $1_\sim \in \mathrm{IFI}(S)$. Then, by the hypothesis, $A = A \cap 1_\sim = A \circ 1_\sim \circ A$. Hence, by Theorem 3.1, S is regular. This completes the proof. ■

Result 3.B[17, Theorem 1]. Let S be a semigroup. Then S is regular if and only if $RL = R \cap L$ for each $R \in \mathrm{RI}(S)$ and each $L \in \mathrm{LI}(S)$.

We will given another characterization of such a semigroup.

Theorem 3.4. Let S be a semigroup. Then S is regular if and only if $A \circ B = A \cap B$ for each $A \in$ IFRI(S) and each $B \in$ IFLI(S).

Proof.($\Rightarrow$): It is clear from the proof of (1)$\Rightarrow$(2) in Theorem 3.1 in [1].

($\Leftarrow$): Suppose the necessary condition holds. Let $R \in$ RI(S) and let $L \in$ LI(S). Then it is clear that $RL \subset R \cap L$. On the other hand, by Result 1.A(2), $(\chi_R, \chi_{R^c}) \in$ IFRI(S) and $(\chi_L, \chi_{L^c}) \in$ IFLI(S). Then, by the hypothesis,

$$(\chi_R, \chi_{R^c}) \circ (\chi_L, \chi_{L^c}) = (\chi_R, \chi_{R^c}) \cap (\chi_L, \chi_{L^c}).$$

Let $a \in R \cap L$. Then $a \in R$ and $a \in L$. Thus

$$\begin{aligned}\bigvee_{a=yz}[\chi_R(y) \wedge \chi_L(z)] &= \mu_{(\chi_R,\chi_{R^c})\circ(\chi_L,\chi_{L^c})}(a) \\ &= \mu_{(\chi_R,\chi_{R^c})\cap(\chi_L,\chi_{L^c})}(a) \\ &= \chi_R(a) \wedge \chi_L(a) = 1\end{aligned}$$

and

$$\begin{aligned}\bigwedge_{a=yz}[\chi_{R^c}(y) \wedge \chi_{L^c}(z)] &= \nu_{(\chi_R,\chi_{R^c})\circ(\chi_L,\chi_{L^c})}(a) \\ &= \nu_{(\chi_R,\chi_{R^c})\cup(\chi_L,\chi_{L^c})}(a) \\ &= \chi_{R^c}(a) \vee \chi_{L^c}(a) = 0.\end{aligned}$$

So there exist $b, c \in S$ with $a = bc$ such that

$$\chi_R(b) = 1,\ \chi_{R^c}(b) = 0 \text{ and } \chi_L(c) = 1,\ \chi_{L^c}(c) = 0.$$

Then $b \in R$ and $c \in L$. Thus $a = bc \in RL$. So $R \cap L \subset RL$. Hence $RL = R \cap L$. Therefore, by Result 3.B, S is regular. This completes the proof. ■

Proposition 3.5. Let S be a regular semigroup. Then $A \circ A = A$ for each $A \in$ IFI(S).

Proof. Let $A \in$ IFI(S). Then, by Result 1.D, $A \circ A \subset A \circ 1_\sim \subset A$ and $A \circ 1_\sim \circ A \subset A \circ 1_\sim \subset A$. Thus, by Theorem 2.3, $A \in$ IFBI(S). Since S is regular, by Theorem 3.1, $A = A \circ 1_\sim \circ A \subset A \circ A$. Hence $A \circ A = A$. ■

4. Intraregular semigroups

A semigroup S is said to be *intraregular* if for each $a \in S$ there exist $x, y \in S$ such that $a = xa^2y$.

It is well-known(See Theorem 4.4 in [5] and Theorem II.4.5 in[25]) that a semigroup S is intraregular if and only if it is a semilattice of simple semigroups.

Result 4.A[21, Theorem 36]. Let S be a semigroup. Then S is intraregular if and only if $L \cap R \subset LR$ for each $L \in$ LI(S) and each $R \in$ RI(S).

We will give a characterization of an intraregular semigroup by intuitionistic fuzzy ideals (See Proposition 4.1 in [14]).

Theorem 4.1. Let S be a semigroup. Then S is intraregular if and only if $A \cap B \subset B \circ A$ for each $A \in$ IFRI(S) and each $B \in$ IFLI(S).

Proof.($\Rightarrow$): Suppose S is intraregular. Let $A \in$ IFRI(S) and let $B \in$ IFLI(S). Let $a \in S$. Since S is intraregular, there exist $x, y \in S$ such that $a = xa^2y$. Then,

$$\begin{aligned}\mu_{B\circ A}(a) &= \bigvee_{a=bc}[\mu_B(b) \wedge \mu_A(c)] \\ &\geq \mu_B(xa) \wedge \mu_A(ay) \\ &\geq \mu_B(a) \wedge \mu_A(a) (\text{Since } B \in \text{IFLI}(S) \text{ and } A \in \text{IFRI}(S)) \\ &= \mu_{A\cap B}(a)\end{aligned}$$

and

$$\begin{aligned}\nu_{B\circ A}(a) &= \bigwedge_{a=bc}[\nu_B(b) \vee \nu_A(c)] \\ &\leq \nu_B(xa) \vee \nu_A(ay) \\ &\leq \nu_B(a) \vee \nu_A(a) \\ &= \nu_{A\cap B}(a).\end{aligned}$$

Hence $A \cap B \subset B \circ A$.

($\Leftarrow$): Suppose the necessary condition holds. Let $R \in$ RI(S) and let $L \in$ LI(S). Let $a \in L \cap R$. Then $a \in L$ and $a \in R$. By Result 1.A(2), $(\chi_R, \chi_{R^c}) \in$ IFRI(S) and $(\chi_L, \chi_{L^c}) \in$ IFLI(S). By the hypothesis,

$$(\chi_R, \chi_{R^c}) \cap (\chi_L, \chi_{L^c}) \subset (\chi_L, \chi_{L^c}) \circ (\chi_R, \chi_{R^c}).$$

Thus

$$\begin{aligned}\bigvee_{a=pq}[\chi_L(p) \wedge \chi_R(q)] &= \mu_{(\chi_L,\chi_{L^c})\circ(\chi_R,\chi_{R^c})}(a) \\ &\geq \mu_{(\chi_L,\chi_{L^c})\cap(\chi_R,\chi_{R^c})}(a) \\ &= \chi_L(a) \wedge \chi_R(a) \\ &= 1\end{aligned}$$

and

$$\begin{aligned}\bigwedge_{a=pq}[\chi_L(p) \vee \chi_R(q)] &= \nu_{(\chi_L,\chi_{L^c})\circ(\chi_R,\chi_{R^c})}(a) \\ &\leq \nu_{(\chi_L,\chi_{L^c})\cup(\chi_R,\chi_{R^c})}(a) \\ &= \chi_L(a) \vee \chi_R(a) \\ &= 0.\end{aligned}$$

So there exist $b, c \in S$ with $a = bc$ such that

$$\chi_L(b) = 1,\ \chi_{L^c}(b) = 0 \text{ and } \chi_R(c) = 1,\ \chi_{R^c}(c) = 0.$$

Then $b \in L$ and $c \in R$. Thus $a = bc \in LR$. So $L \cap R \subset LR$. Hence, by Result 4.A, S is intraregular. This completes the proof. ■

Result 4.B[21, Theorems 37 and 38]. Let S be a semigroup. Then the following are equivalent :

(1) S is both regular and intraregular.

(2) $B^2 = B$ for each $B \in \mathrm{BI}(S)$

(3) $A \cap B \subset AB \cap BA$ for each $A, B \in \mathrm{BI}(S)$.

(4) $B \cap L \subset BL \cap LB$ for each $B \in \mathrm{BI}(S)$ and each $L \in \mathrm{LI}(S)$.

(5) $B \cap R \subset BR \cap RB$ for each $B \in \mathrm{BI}(S)$ and each $R \in \mathrm{RI}(S)$.

(6) $L \cap R \subset LR \cap RL$ for each $R \in \mathrm{RI}(S)$ and each $L \in \mathrm{LI}(S)$.

We will give a characterization of a semigroup that is both regular and intraregualr by intuitionistic fuzzy ideals.

Theorem 4.2. Let S be a semigroup. Then the following are equivalent :

(1) S is both regular and intraregualr.

(2) $A \circ A = A$ for each $A \in \mathrm{IFBI}(S)$.

(3) $A \cap B \subset (A \circ B) \cap (B \circ A)$ for any $A, B \in \mathrm{IFBI}(S)$.

(4) $A \cap B \subset (A \circ B) \cap (B \circ A)$ for each $A \in \mathrm{IFBI}(S)$ and each $B \in \mathrm{IFLI}(S)$.

(5) $A \cap B \subset (A \circ B) \cap (B \circ A)$ for each $A \in \mathrm{IFBI}(S)$ and each $B \in \mathrm{IFRI}(S)$.

(6) $A \cap B \subset (A \circ B) \cap (B \circ A)$ for each $A \in \mathrm{IFRI}(S)$ and each $B \in \mathrm{IFLI}(S)$.

Proof. It is clear that (3) $\Rightarrow$ (2), (3) $\Rightarrow$ (4) $\Rightarrow$ (6) and (3) $\Rightarrow$ (5) $\Rightarrow$ (6). We will prove that (1) $\Rightarrow$ (3), (2) $\Rightarrow$ (1) and (6) $\Rightarrow$ (1).

(1) $\Rightarrow$ (3): Suppose the condition (1) holds. Let $A, B \in \mathrm{IFBI}(S)$ and let $a \in S$. Since S is regular, there exists an $x \in S$ such that $a = axa (= axaxa)$. Since S is intraregualr, there exist $y, z \in S$ such that $a = ya^2z$. Then

$$a = (axya)(azxa).$$

Since $A, B \in \mathrm{IFBI}(S)$.

$$\begin{aligned}\mu_A(axya) &\geq \mu_A(a) \wedge \mu_A(a) \\ &= \mu_A(a),\ \nu_A(axya) \\ &\leq \nu_A(a) \vee \nu_A(a) \\ &= \nu_A(a)\end{aligned}$$

and

$$\begin{aligned}\mu_B(azxa) &\geq \mu_B(a) \wedge \mu_B(a) \\ &= \mu_B(a),\ \nu_B(azxa) \\ &\leq \nu_B(a) \vee \nu_B(a) \\ &= \nu_B(a).\end{aligned}$$

Thus

$$\begin{aligned}\mu_{A\circ B}(a) &= \bigvee_{a=pq}[\mu_A(p) \wedge \mu_B(q)] \\ &\geq \mu_A(axya) \wedge \mu_B(azxa) \\ &\geq \mu_A(a) \wedge \mu_B(a) \\ &= \mu_{A\cap B}(a)\end{aligned}$$

and

$$\begin{aligned}\nu_{A\circ B}(a) &= \bigwedge_{a=pq}[\nu_A(p) \vee \nu_B(q)] \\ &\leq \nu_A(axya) \vee \nu_B(azxa) \\ &\leq \nu_A(a) \vee \nu_B(a) \\ &= \nu_{A\cap B}(a).\end{aligned}$$

so $A \cap B \subset A \circ B$. By the similar arguments, we have $A \cap B \subset B \circ A$. Hence $A \cap B \subset (A \circ B) \cap (B \circ A)$.

(6) $\Rightarrow$ (1): Suppose the condition (6) holds. Let $A \in \mathrm{IFRI}(S)$ and let $B \in \mathrm{IFLI}(S)$. Then, by the hypothesis, $A \cap B \subset (A \circ B) \cap (B \circ A) \subset B \circ A$. Thus, by Theorem 4.1, S is intraregular. On the other hand, $A \cap B \subset (A \circ B) \cap (B \circ A) \subset A \circ B$. As it is stated in the proof of Theorem 3.4, we have $A \circ B \subset A \cap B$. Thus $A \cap B = A \circ B$. So, by Theorem 3.4, S is regular. Hence S is both regular and intraregualr.

(2) $\Rightarrow$ (1): Suppose the condition (2) holds. Let $B \in \mathrm{BI}(S)$ and let $a \in S$. Then, by Result 1.E, $(\chi_B, \chi_{B^c}) \in \mathrm{IFBI}(S)$. Thus, by the hypothesis,

$$(\chi_B, \chi_{B^c}) \circ (\chi_B, \chi_{B^c}) = (\chi_B, \chi_{B^c}).$$

So

$$\begin{aligned}(\bigvee_{a=pq}[\chi_B(p) \wedge \chi_B(q)], \bigwedge_{a=pq}[\chi_{B^c}(p) \wedge \chi_{B^c}(q)]) \\ = [(\chi_B, \chi_{B^c}) \circ (\chi_B, \chi_{B^c})](a) \\ = (\chi_B, \chi_{B^c})(a) \\ = (1, 0).\end{aligned}$$

Then there exist $b, c \in S$ with $a = bc$ such that

$$\chi_B(b) = \chi_B(c) = 1 \text{ and } \chi_{B^c}(b) = \chi_{B^c}(c) = 0.$$

Thus $b, c \in B$. So $a = bc \in BB$, i.e., $B \subset BB$. Since $B \in \mathrm{BI}(S)$, it is clear that $BB \subset B$. Thus $B^2 = B$. Hence, by Result 4.B, S is both regular and intraregular. This completes the proof. ■

5. Semilattice of left groups

A semigroup S is called a *left group* if it is regular and right cancellative, and is called a *semilattice of left groups* if it is the set-theoretical union of a family of left groups $G_i (i \in M)$:

$$S = \cup_{i \in M} G_i$$

such that for each $(i, j) \in M \times M$, $G_iG_j \subset G_k$ and $G_jG_i \subset G_k$ for some $k \in M$. A semigroup S is said to be *right*[resp. *left*] *regular* if for each $a \in S$ there exists an $x \in S$ such that $a = a^2x$[resp. $a = xa^2$].

Result 5.A[21, Theorem 80]. Let S be a semigroup. Then the following are equivalent:

(1) S is a semilattice of left groups.

(2) $BL = B \cap L$ for each $B \in \mathrm{BI}(S)$ and each $L \in \mathrm{LI}(S)$.

(3) $BJ = B \cap J$ for each $B \in \mathrm{BI}(S)$ and each $J \in \mathrm{I}(S)$.

(4) $XJ = X \cap J$ for each $X \in \mathrm{RI}(S)$[or $LI(S)$] and $J \in \mathrm{I}(S)$.

(5) S is regular, and every left ideal of S is an ideal of S.

(6) S is right regular, and every left ideal of S is an ideal of S.

Result 5.B[14, Propositions 3.1 and 3.1′]. Let S be a regular semigroup. Then every left[resp. right] ideal of S is an ideal of S if and only if every IFLI[resp. IFRI] of S is an IFI of S.

Result 5.C[14, Propositions 3.3 and 3.3′]. Let S be a regular semigroup. Then every bi-ideal of S is a left[resp. right] ideal of S if and only if every IFBI of S is an IFLI[resp. IFRI] of S.

Result 5.D[14, Proposition 6.1]. Let S be a left [resp. right] regular semigroup. Then every left[resp. right] ideal of S is an ideal of S if and only if every IFLI[resp. IFRI] of S is an IFI of S.

Now we will give a characterization of a semigroup that is a semilattice of left groups by intuitionistic fuzzy ideals.

Theorem 5.1. Let S be a semigroup. Then the following are equivalent :

(1) S is a semilattice of left groups.

(2) $A \circ B \subset A \cap B$ for each $A \in \mathrm{IFBI}(S)$ and $B \in \mathrm{IFLI}(S)$.

(3) $A \circ B \subset A \cap B$ for each $A \in \mathrm{IFBI}(S)$ and $B \in \mathrm{IFI}(S)$.

(4) $A \circ B \subset A \cap B$ for each $A \in \mathrm{IFLI}(S)$(or, $IFRI(S)$) and each $B \in \mathrm{IFI}(S)$.

(5) S is regular, and every IFLI of S is an IFI of S.

(6) S is right regular, and every IFLI of S is an IFI of S.

Proof. It is clear that (3) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4).

(1) $\Leftrightarrow$ (5): It is clear from Results 5.A and 5.B.

(1) $\Leftrightarrow$ (6): It is clear from Results 5.A and 5.B.

We will prove that (1) $\Rightarrow$ (3), (4) $\Rightarrow$ (1) and (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (3): Suppose the condition (1) holds. Let $A \in \mathrm{IFBI}(S)$ and let $B \in \mathrm{IFI}(S)$. Since S is an ideal of S, by Result 5.A, $B = B \cap S = BS$ for each $B \in \mathrm{BI}(S)$. Then $B \in \mathrm{RI}(S)$ for each $B \in \mathrm{BI}(S)$. Since S is regular, by Result 5.C, $A \in \mathrm{IFRI}(S)$. Hence, by

Theorem 3.4, $A \circ B = f \cap B$.

(4) $\Rightarrow$ (1): Suppose the condition (4) holds. Let $X \in$ RI(S)[or LI(S)] and let $J \in$ I(S). Then, by Result 1.A(2), $(\chi_X, \chi_{X^c}) \in$ IFRI(S)[or IFLI(S)] and $(\chi_J, \chi_{J^c}) \in$ IFI(S). Let $a \in X \cap J$. Then $a \in X$ and $a \in J$. Thus

$$\begin{aligned}(\textstyle\bigvee_{a=yz}[\chi_X(y) \wedge \chi_j(z)], \bigwedge_{a=yz}[\chi_{X^c}(y) \wedge \chi_{J^c}(z)]) \\ &= [(\chi_X, \chi_{X^c}) \circ (\chi_J, \chi_{J^c})](a) \\ &= [(\chi_X, \chi_{X^c}) \wedge (\chi_J, \chi_{J^c})](a) \\ &= (\chi_X(a) \wedge \chi_J(a), \chi_{X^c}(a) \wedge \chi_{J^c}(a)) \\ &= (1, 0).\end{aligned}$$

So there exist $b, c \in S$ with $a = bc$ such that

$\chi_X(b) = 1$, $\chi_{X^c}(b) = 0$ and $\chi_J(c) = 1$, $\chi_{J^c}(c) = 0$.

Thus $b \in X$ and $c \in J$. Thus $a = bc \in XJ$. So $X \cap J \subset XJ$. Now let $a \in XJ$. Then there exist $b, c \in S$ such that $a = bc$. Thus

$$\begin{aligned}\chi_X(a) \wedge \chi_J(a) &= \mu_{(\chi_X,\chi_{X^c})\cap(\chi_J,\chi_{J^c})}(a) \\ &= \mu_{(\chi_X,\chi_{X^c})\circ(\chi_J,\chi_{J^c})}(a) \\ &= \textstyle\bigvee_{a=yz}[\chi_X(y) \wedge \chi_J(z)] \\ &\geq \chi_X(b) \wedge \chi_J(c) \\ &= 1\end{aligned}$$

and

$$\begin{aligned}\chi_{X^c}(a) \vee \chi_{J^c}(a) &= \nu_{(\chi_X,\chi_{X^c})\cap(\chi_J,\chi_{J^c})}(a) \\ &= \nu_{(\chi_X,\chi_{X^c})\circ(\chi_J,\chi_{J^c})}(a) \\ &= \textstyle\bigwedge_{a=yz}[\chi_{X^c}(y) \wedge \chi_{J^c}(z)] \\ &\leq \chi_{X^c}(b) \wedge \chi_{J^c}(c) \\ &= 0.\end{aligned}$$

So $\chi_X(a) = 1$, $\chi_{X^c}(a) = 0$ and $\chi_J(a) = 1$, $\chi_{J^c}(a) = 0$. Then $a \in X$ and $a \in J$. Thus $a \in X \cap J$. So $XJ \subset X \cap J$. Hence $XJ = X \cap J$. Therefore, by Result 5.A, S is a semilattice of left groups.

(2) $\Leftrightarrow$ (1): It can be seen in a similar way that (4) implies (1). This completes the proof. ■

Theorem 5.1′[The dual of Theorem 5.1]. Let S be a semigroup. Then the following are equivalent :

(1) S is a semilattice of right groups.

(2) $B \circ A \subset A \cap B$ for each $A \in$ IFBI(S) and $B \in$ IFRI(S).

(3) $B \circ A \subset A \cap B$ for each $A \in$ IFBI(S) and $B \in$ IFI(S).

(4) $B \circ A \subset A \cap B$ for each $A \in$ IFRI(S)(or, *IFLI*(S)) and each $B \in$ IFI(S).

(5) S is regular, and every *IFRI* of S is an *IFI* of S.

(6) S is left regular, and every IFRI of S is an IFI of S.

6. Semilattice of left simple semigroups

A semigroup S is called a *semilattice of left simple semigroups* if it is a set-theoretical union of a family of left simple semigroups $S_i (i \in M)$:

$$S = \bigcup_{i \in M} S_i$$

such that for each $(i, j) \in M \times M$, $S_iS_j \subset S_k$ and $S_jS_i \subset S_k$ for some $k \in M$.

Result 6.A[20, Theorem 8 ; 24, A Theorem. Let S be a semigroup. Then the following are equivalent:

(1) S is a semilattice of left simple semigroups.

(2) S is left regular, and every left ideal of S is an ideal.

(3) $AB = A \cap B$ for any $A, B \in$ LI(S).

(4) LI(S) is a semilattice under the multiplication of subset.

We will give a characterization of a semigroup that is a semilattice of left simple semigroups by intuitionistic fuzzy ideals.

Theorem 6.1. Let S be a semigroup. Then the following are equivalent:

(1) S is a semilattice of left simple semigroups.

(2) S is left regular, and every IFLI of S is an IFI of S.

(3) $A \circ B = A \cap B$ for any $A, B \in$ IFLI(S).

(4) IFLI(S) is a semilattice under the multiplication of intuitionistic fuzzy sets.

Proof. (1) $\Leftrightarrow$ (2): It is clear from Result 6.A and 5.D. (3) $\Rightarrow$ (4): It is clear. We will prove that (2) $\Rightarrow$

(3) and (4) $\Rightarrow$ (1).

(2) $\Rightarrow$ (3) : Suppose the condition (2) holds. Let $A, B \in$ IFLI(S) and let $a \in S$. Since S is left regular, there exists an $x \in S$ such that $a = xa^2$. Then $(A \circ B)(a) \neq (0,1)$. Thus

$$\begin{aligned}\mu_{A\circ B}(a) &= \bigvee_{a=yz}[\mu_A(y) \wedge \mu_B(z)]\\ &\geq \mu_A(xa) \wedge \mu_B(a) \quad (\text{Since } a = xa^2)\\ &\geq \mu_A(a) \wedge \mu_B(a) \quad (\text{Since } A \in \text{IFLI}(S))\\ &= \mu_{A\cap B}(a)\end{aligned}$$

and

$$\begin{aligned}\nu_{A\circ B}(a) &= \bigwedge_{a=yz}[\nu_A(y) \vee \nu_B(z)]\\ &\geq \nu_A(xa) \vee \nu_B(a) \geq \nu_A(a) \vee \nu_B(a)\\ &= \nu_{A\cap B}(a).\end{aligned}$$

So $A \cap B \subset A \circ B$. On the other hand,

$$\begin{aligned}\mu_{A\circ B}(a) &= \bigvee_{a=yz}[\mu_A(y) \wedge \mu_B(z)]\\ &\leq \bigvee_{a=yz}[\mu_A(yz) \wedge \mu_B(yz)] \quad (\text{Since } A \in \text{IFRI}(S) \text{ and } B \in \text{IFLI}(S))\\ &= \mu_A(a) \wedge \mu_B(a)\\ &= \mu_{A\cap B}(a)\end{aligned}$$

and

$$\begin{aligned}\nu_{A\circ B}(a) &= \bigwedge_{a=yz}[\nu_A(y) \vee \nu_B(z)]\\ &\geq \bigvee_{a=yz}[\nu_A(yz) \vee \nu_B(yz)]\\ &= \nu_A(a) \vee \nu_B(a)\\ &= \nu_{A\cap B}(a).\end{aligned}$$

Thus $A \circ B \subset A \cap B$. Hence $A \circ B = A \cap B$.

(4) $\Rightarrow$ (1): Suppose the condition (4) holds. Let $A, B \in$ LI(S) and let $a \in AB$. Then there exist $b \in A$ and $c \in B$ such that $a = bc$. By Result 1.A(2), $(\chi_A, \chi_{A^c}), (\chi_B, \chi_{B^c}) \in$ IFLI(S). Thus

$$\begin{aligned}(\bigvee_{a=yz}[\chi_B(y) \wedge \chi_A(z)] &= \mu_{(\chi_B,\chi_{B^c})\circ(\chi_A,\chi_{A^c})}(a)\\ &= \mu_{(\chi_A,\chi_{A^c})\circ(\chi_B,\chi_{B^c})}(a) \quad (\text{By the hypothesis})\\ &= \bigvee_{a=st}[\chi_A(s) \wedge \chi_B(t)\\ &\geq \chi_A(b) \wedge \chi_B(c)\\ &= 1\end{aligned}$$

and

$$\begin{aligned}(\bigwedge_{a=yz}[\chi_B(y) \vee \chi_A(z)] &= \nu_{(\chi_B,\chi_{B^c})\circ(\chi_A,\chi_{A^c})}(a)\\ &= \nu_{(\chi_A,\chi_{A^c})\circ(\chi_B,\chi_{B^c})}(a)\\ &= \bigwedge_{a=st}[\chi_A(s) \vee \chi_B(t)\\ &\leq \chi_A(b) \vee \chi_B(c)\\ &= 0\end{aligned}$$

So there exist $p, q \in S$ with $a = pq$ such that

$\chi_B(p) = 1,\ \chi_{B^c}(p) = 0$ and $\chi_A(q) = 1,$
$\chi_{A^c}(q) = 0.$

Then $p \in B$ and $q \in A$. Thus $a = pq \in BA$, i.e., $AB \subset BA$. By the similar arguments, we have $BA \subset AB$. So $AB = BA$. Now let $A \in$ LI(S) and let $a \in A$. By Result 1.A(2), $(\chi_A, \chi_{A^c}) \in$ IFLI(S). Then,

$$\begin{aligned}&(\bigvee_{a=yz}[\chi_A(y) \wedge \chi_A(z)], \bigwedge_{a=yz}[\chi_{A^c}(y) \vee \chi_{A^c}(z)])\\ &= [(\chi_A, \chi_{A^c}) \circ (\chi_A, \chi_{A^c})](a)\\ &= (\chi_A, \chi_{A^c})(a)\\ &= (1,0). \quad (\text{By the hypothesis})\end{aligned}$$

Thus there exist $b, c \in S$ with $a = bc$ such that

$\chi_A(b) = 1,\ \chi_{A^c}(b) = 0$ and $\chi_A(c) = 1,\ \chi_{A^c}(c) = 0.$

So $a = bc \in AA$. Then $A \subset AA$. since $A \in$ LI(S), it is clear that $AA \subset A$. Thus $AA = A$. So $LI(S)$ is a semilattice. Hence, by Result 6.A, S is a semilattice of left simple semigroups. This completes the proof. ■

Theorem 6.1$'$[The dual of Theorem 6.1]. Let S be a semigroup. Then the following are equivalent:

(1) S is a semilattice of right simple semigroups.

(2) S is right regular, and every IFRI of S is an IFI of S.

(3) $A \circ B = A \cap B$ for any $A, B \in$ IFRI(S).

(4) IFRI(S) is a semilattice under the multiplication of intuitionistic fuzzy sets.

7. Semisimple semigroups

A semigroup S is said to be *semisimple* if $J^2 = J$ for each $J \in$ I(S).

Result 7.A[19, Lemma 7.1]. Let S be a semigroup. Then the following are equivalent:

(1) S is semisimple.

(2) $a \in SaSaS$ for each $a \in S$.

(3) $A \cap B = AB$ for any $A, B \in$ I(S).

The equivalence of (1) and (2) of the above Result 7.A is due to Theorem 3 in [25].

We will give a characterization of a semisimple semigroup by intuitionistic fuzzy ideals.

Theorem 7.1. Let S be a semigroup. Then the following are equivalent:

(1) S is semisimple.

(2) $A \circ A = A$ for each $A \in \mathrm{IFI}(S)$.

(3) $A \circ B = A \cap B$ for any $A, B \in \mathrm{IFI}(S)$.

proof. (1)⇒(3): Suppose S is semisimple. let $A, B \in \mathrm{IFI}(S)$ and let $a \in S$. By Result 7.A, there exist $x, y, z \in S$ such that $a = xayaz$. Then $(A \circ B)(a) \neq (0,1)$. Thus

$$\begin{aligned}\mu_{A\circ B}(a) &= \bigvee_{a=bc}[\mu_A(b) \wedge \mu_B(c)]\\ &\geq \mu_A(xa) \wedge \mu_B(yaz)\ (a = xayaz)\\ &\geq \mu_A(a) \wedge \mu_B(a)\ (\text{Since } A, B \in \mathrm{IFI}(S))\\ &= \mu_{A\cap B}(a)\end{aligned}$$

and

$$\begin{aligned}\nu_{A\circ B}(a) &= \bigwedge_{a=bc}[\nu_A(b) \vee \nu_B(c)]\\ &\leq \nu_A(xa) \vee \nu_B(yaz)\\ &\leq \nu_A(a) \vee \nu_B(a)\\ &= \nu_{A\cap B}(a).\end{aligned}$$

So $A \cap B \subset A \circ B$. On the other hand,

$$\begin{aligned}\mu_{A\circ B}(a) &= \bigvee_{a=bc}[\mu_A(b) \wedge \mu_B(c)]\\ &\leq \bigvee_{a=bc}[\mu_A(bc) \wedge \mu_B(bc)\ (\text{Since } A, B \in \mathrm{IFI}(S))\\ &= \mu_A(a) \wedge \mu_B(a)\\ &= \mu_{A\cap B}(a)\end{aligned}$$

and

$$\begin{aligned}\nu_{A\circ B}(a) &= \bigwedge_{a=bc}[\nu_A(b) \vee \nu_B(c)]\\ &\geq \bigwedge_{a=bc}[\nu_A(bc) \vee \nu_B(bc)\\ &= \nu_A(a) \vee \nu_B(a)\\ &= \nu_{A\cap B}(a).\end{aligned}$$

Then $A \circ B \subset A \cap B$. Hence $A \circ B = A \cap B$.

(3) ⇒ (2) It is clear.

(2) ⇒ (1): Suppose the condition (2) holds. Let $J \in \mathrm{I}(S)$ and let $a \in J$. By Result 1.A(2) $(\chi_J, \chi_{J^c}) \in \mathrm{IFI}(S)$. Then

$$(\chi_J, \chi_{J^c})\circ(\chi_J, \chi_{J^c})](a) = (\chi_J, \chi_{J^c})(a) = (1, 0).$$

Thus $[(\chi_J, \chi_{J^c}) \circ (\chi_J, \chi_{J^c})](a) \neq (0,1)$. So

$$(\bigvee_{a=bc}[\chi_J(b) \wedge \chi_J(c)], \bigwedge_{a=bc}[\chi_{J^c}(b) \vee \chi_{J^c}(c)]) = (1, 0).$$

Then there exist $p, q \in S$ such that $a = pq$ such that

$$\chi_J(b) = 1,\ \chi_{J^c}(p) = 0 \text{ and } \chi_J(q) = 1,\ \chi_{J^c}(q) = 0.$$

Thus $a = pr \in JJ$. So $J \subset JJ$. Since $J \in \mathrm{I}(S)$, it is clear that $JJ \subset J$. Hence $JJ = J$. Therefore S is semisimple. This completes the proof. ■

8. Semilettice of groups

A semigroup S is called a *semilattice of groups* if it is the set-theoretical union of a family of mutually disjoint subgroups $G_i (i \in M)$ such that for each $(i, j) \in M \times M$, $G_iG_j \subset G_k$ and $G_jG_i \subset G_k$ for some $k \in M$.

Result 8.A[20, Theorem 3]. Let S be a semigroup. Then the following are equivalent:

(1) S is semilattice of groups.

(2) $LR = L \cap R$ for each $L \in \mathrm{LI}(S)$ and each $R \in \mathrm{LI}(S)$.

(3) $LB = L \cap B$ for each $L \in \mathrm{LI}(S)$ and each $B \in \mathrm{BI}(S)$.

(4) $BR = B \cap R$ for each $B \in \mathrm{BI}(S)$ and each $R \in \mathrm{RI}(S)$.

(5) S is regular, and every one-sided ideal of S is an ideal.

Result 8.B[14, Corollary 3.4]. Let S be a semigroup which is a semilattice of groups. Then every IFBI of S is an IFI of S.

Now we will give a characterization of a semigroup which is a semilattice of groups by intuitionistic fuzzy ideals.

Theorem 8.1. Let S be a semigroup. Then the following are equivalent:

(1) S is a semilattice of groups.

(2) $B \circ A = B \cap A$ for each $A \in \mathrm{IFRI}(S)$ and each

$B \in$ IFLI(S).

(3) $B \circ C = B \cap C$ for each $C \in$ IFBI(S) and each $B \in$ IFLI(S).

(4) $C \circ A = C \cap A$ for each $C \in$ IFBI(S) and each $A \in$ IFRI(S).

(5) $C_1 \circ C_2 = C_1 \cap C_2$ for any $C_1, C_2 \in$ IFBI(S).

Proof. Since any IFRI[resp. IFLI] of S is an IFBI of S, it follows that (5) implies (3), (3) implies (2), (5) implies (4), and (4) implies (2). We will prove that (2)⇒(1) and (1)⇒(5).

(2)⇒(1): Suppose the condition (2). Let $L \in$ LI(S) and let $R \in$ RI(S). Let $a \in L \cap R$. Then $a \in L$ and $a \in R$. By Result 1.A(2), $(\chi_L, \chi_{L^c}) \in$ IFLI(S) and $(\chi_R, \chi_{R^c}) \in$ IFRI(S). Thus

$$\begin{aligned}[(\chi_L, \chi_{L^c}) \circ (\chi_R, \chi_{R^c})](a) &= [(\chi_L, \chi_{L^c}) \cap (\chi_R, \chi_{R^c})](a) \text{ (By the hypothesis)} \\ &= (\chi_L(a) \wedge \chi_R(a), \chi_{L^c}(a) \vee \chi_{R^c}(a)) \\ &= (1, 0). \text{ (Since } a \in L \text{ and } a \in R)\end{aligned}$$

So $[(\chi_L, \chi_{L^c}) \circ (\chi_R, \chi_{R^c})](a) \neq (0, 1)$. Moreover,

$$(\bigvee_{a=xy}[\chi_L(x) \wedge \chi_R(y)], \bigwedge[\chi_{L^c}(x) \vee \chi_{R^c}(z)] = (1, 0)).$$

Then there exist $b, c \in S$ with $a = bc$ such that

$$\chi_L(b) = 1,\ \chi_{L^c}(b) = 0 \text{ and } \chi_R(c) = 1,\ \chi_{R^c}(c) = 0.$$

Thus $b \in L$ and $c \in R$. So $a = bc \in LR$, i.e., $L \cap R \subset LR$. Now let $a \in LR$. Then there exist $b \in L$ and $c \in R$ such that $a = bc$. Thus

$$\begin{aligned}\chi_L(a) \wedge \chi_R &= \mu_{(\chi_L, \chi_{L^c}) \wedge (\chi_R, \chi_{R^c})}(a) \\ &= \mu_{(\chi_L, \chi_{L^c}) \circ (\chi_R, \chi_{R^c})}(a) \text{ (By the hypothesis)} \\ &= \bigvee_{a=xy}[\chi_L(x) \wedge \chi_R(y)] \\ &\geq \chi_L \wedge \chi_R(c) \text{ (Since } a = bc) \\ &= 1 \text{ (Since } b \in L \text{ and } c \in R)\end{aligned}$$

and

$$\begin{aligned}\chi_{L^c}(a) \wedge \chi_{R^c} &= \nu_{(\chi_L, \chi_{L^c}) \wedge (\chi_R, \chi_{R^c})}(a) \\ &= \nu_{(\chi_L, \chi_{L^c}) \circ (\chi_R, \chi_{R^c})}(a) \\ &= \bigwedge_{a=xy}[\chi_{L^c}(x) \vee \chi_{R^c}(y)] \\ &\leq \chi_{L^c} \vee \chi_{R^c}(c) \\ &= 0.\end{aligned}$$

So $\chi_L(a) = 1$, $\chi_{L^c}(a) = 0$ and $\chi_R(a) = 1$, $\chi_{R^c}(a) = 0$. Then $a \in L \cap R$. Thus $LR \subset L \cap R$. Hence $LR = L \cap R$. Therefore, by Result 8.A, S is a semilattice of groups.

(1)⇒ (5): Suppose the condition (1) holds. Let $C_1, C_2 \in$ IFBI(S). Then, by Result 8.B, $C_1, C_2 \in$ IFI(S). By Result 8.A, S is regular. Hence, by Theorem 3.4, $C_1 \circ C_2 = C_1 \cap C_2$. This completes the proof. ■

Result 8.C[20, Theorem 1]. Let S be a semigroup. Then S is a semilattice of groups if and only if $BI(S)$ is a semilattice under the multiplication of subsets.

Theorem 8.2. Let S be a semigroup. Then S is a semilattice of groups if and only if IFBI(S) is a semilattice under the multiplication of intuitionistic fuzzy sets.

Proof. (⇒): It is clear From Theorem 8.1.

(⇐) : Suppose the necessary condition holds. Let $A, B \in$ BI(S) and let $a \in AB$. Then there exist $b \in A$ and $c \in B$ such that $a = bc$. By Result 2.A, $(\chi_A, \chi_{A^c}), (\chi_B, \chi_{B^c}) \in$ IFBI(S). Thus

$$\begin{aligned}\bigvee_{a=yz}[\chi_B(y) \wedge \chi_A(z)] &= \mu_{(\chi_B, \chi_{B^c}) \circ (\chi_A, \chi_{A^c})}(a) \\ &= \mu_{(\chi_A, \chi_{A^c}) \circ (\chi_B, \chi_{B^c})}(a) \text{ (By the hypothesis)} \\ &= \bigvee_{a=st}[\chi_A(s) \wedge \chi_B(t)] \\ &\geq \chi_A(b) \wedge \chi_B(c) \\ &= 1\end{aligned}$$

and

$$\begin{aligned}\bigwedge_{a=yz}[\chi_{B^c}(y) \wedge \chi_{A^c}(z)] &= \nu_{(\chi_B, \chi_{B^c}) \circ (\chi_A, \chi_{A^c})}(a) \\ &= \nu_{(\chi_A, \chi_{A^c}) \circ (\chi_B, \chi_{B^c})}(a) \\ &= \bigwedge_{a=st}[\chi_{A^c}(s) \vee \chi_{B^c}(t)] \\ &\leq \chi_A(b) \wedge \chi_B(c) \\ &= 0.\end{aligned}$$

So there exist $p, q \in S$ with $a = pq$ such that

$$\chi_B(p) = 1,\ \chi_{B^c}(p) = 0 \text{ and } \chi_A(q) = 1,\ \chi_{A^c}(q) = 0.$$

Then $p \in B$ and $q \in A$. Thus $a = pq \in BA$, i.e., $AB \subset BA$. By the similar arguments, we have $BA \subset AB$. So $AB = BA$. Now let $A \in$ BI(S) and let $a \in A$. By Result 2.A, $(\chi_A, \chi_{A^c}) \in$ IFBI(S). Then, by the hypothesis,

$$(\chi_A, \chi_{A^c}) \circ (\chi_A, \chi_{A^c})](a) = (\chi_A, \chi_{A^c})(a) = (1, 0).$$

Thus $[(\chi_A, \chi_{A^c}) \circ (\chi_A, \chi_{A^c})](a) \neq (0, 1)$. Moreover,

$$(\bigvee_{a=yz}[\chi_A(y) \wedge \chi_A(z)], \bigwedge_{a=yz}[\chi_{A^c}(y) \vee \chi_{A^c}(a)])$$
$$= (1, 0).$$

So there exist $b, c \in S$ with $a = bc$ such that

$$\chi_A(b) = 1,\ \chi_{A^c}(b) = 0 \text{ and } \chi_A(c) = 1,\ \chi_{A^c}(c) = 0.$$

Then $a = bc \in AA$. Thus $A \subset AA$. It is clear that $AA \subset A$. So $A = AA$. Hence BI(S) is a semilattice under the multiplication of subsets. Therefore, by Result 8.C, S is a semilattice of groups. This completes the proof. ■

References

1. Y.S.Ahn, K.Hur and D.S.Kim, Intuitionistic fuzzy quasi-ideals of a semigroup, International J. Pure and Applied Math, 23(2)(2005), 207-227.
2. K.Atanassove, Intuitionistic fuzzy sets, Fuzzy Set and Systems, 20(1986) 87-96.
3. Baldev Banerjee and Dhiren Kr. Basnet, Intuitionistic fuzzy subrings and ideals, J.Fuzzy Math 11(1)(2003), 139-155.
4. R.Biswas,Intuitionistic fuzzy subrings, Mathematical Forum x(1989), 37-46.
5. A.H.Clifford and G.B. Preston, The Algebraic Theory of Semigroups, Vol I, AMS, Providence(1961).
6. D.Çoker, An introduction to intuitionistic fuzzy topological spaces, Fuzzy Set and Systems, 88(1997), 81-89.
7. D.Çoker and A.Haydar Es, On fuzzy compactness in intuitionistic fuzzy topological spaces, J.Fuzzy Math. 3(1995), 899-909.
8. H.Gurcay, D.Çoker and A.Haydar Es, On fuzzy continuity in intuitionistic fuzzy topological spaces, J.Fuzzy Math. 5(1997), 365-378.
9. K.Hur, S.Y.Jang and H.W.Kang, Intuitionistic fuzzy subgroupoids, International Journal of Fuzzy Logic and Intelligent Systems 3(1)(2003), 72-77.
10. K.Hur, H.W.Kang and H.K.Song, Intuitionistic fuzzy subgroups and subrings, Honam Math.J.25(1)(2003), 19-41.
11. K.Hur, S.Y.Jang and H.W.Kang, Intuitionistic fuzzy subgroups and cosets, Honam Math.J.26(1)(2004), 17-41.
12. K.Hur, Y.B.Jun and J.H.Ryou, Intuitionistic fuzzy topological groups, Honam Math.J.26(2)(2004), 163-192.
13. K.Hur, J.H.Kim and J.H.Ryou, Intuitionistic fuzzy topological spaces, J.Korea Soc. Math. Edu. Ser. B : Pure Appl. Math. 11(3)(2004), 243-265.
14. K.Hur, K.J.Kim and H.K.Song, Intuitionistic fuzzy ideals and bi-ideals, Honam Math. J.26(3)(2004), 309-330.
15. K.Hur, S.Y.Jang and H.W.Kang, Intuitionistic fuzzy normal subgroups and intuitionistic fuzzy cosets, Honam Math. J. 26(4)(2004), 559-587.
16. K.Hur, H.W.Kang and J.H.Ryou, Intuitionistic fuzzy generalized bi-ideals of a semigroup, International Mathematical Forum 1(6)(2006), 257-274.
17. K.Iseki, A characterization of regular semigroups, Proc. Japan Acad. 32(1965), 676-677.
18. N.Kuroki, Two-sided purity on regular semigroups. Comment. Math. Univ. St.Paul. 24(1975), 21-26
19. __________, On fuzzy semigroups, Inform. Sci. 63(1991), 203-236.
20. S.Lajos, A note on semilattices of groups, Acta Sci. Math.(Szeged) 31(1970), 179-265.
21. ______, Theorems on (1, 1)-ideals in semigroup II, Depart. of Mathematics, Karl Mark Univ. of Economics. Budapest(1975).
22. S.J.Lee and E.P.Lee, The category of intuitionistic fuzzy topological spaces, Bull. Korean Math. Soc. 37(1)(2000), 63-76.
23. M.Petrich, Introduction to Semigroup, Charles E.Merrill(1973)
24. T. Saito, On semigroups which are semilattice of left simple semigroups, Math. Japan.(18)(1973),95-97.
25. G. Szász, Semigroups with idempotent ideals, Publ. Math. Debrecen 21(1974),115-117.
26. L.A.Zadeh, Fuzzy sets, Inoform. and Centrol 8(1965), 338-353.

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