International Journal of Fuzzy Logic and Intelligent Systems, vol. 8, no. 3, September 2008 pp. 170-174

C_{pk} Index Estimation under T_W (the weakest t-norm)-based Fuzzy Arithmetic Operations

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Abstract

The measurement of performance of a process considering both the location and the dispersion of information about the process is referred to as the process capacity indices (PCIs) of interest, C_{pk}. This information is presented by the mean and standard deviation of the producing process. Linguistic variables are used to express the evaluation of the quality of a product. Consequently, C_{pk} is defined with fuzzy numbers. Lee [Eur. J. Oper. Res. 129(2001) 683-688] constructed the definition of the C_{pk} index estimation presented by fuzzy numbers and approximated its membership function using the "min"-norm based Zadeh's extension principle of fuzzy sets . However, Lee's result was shown to be invalid by Hong [Eur. J. Oper. Res. 158(2004) 529-532]. It is well known that T_W (the weakest t-norm)-based addition and multiplication preserve the shape of L-R fuzzy numbers. In this paper, we allow that the fuzzy numbers are of L-R type. The object of the present study is to propose a new method to calculate the C_{pk} index under T_W-based fuzzy arithmetic operations.

Key words : Fuzzy sets; Process; Capacity; Index; Fuzzy arithmetic operations

1. Introduction

Process capacity indices (PCIs) were developed and have been successfully used by companies to compete in and dominate the high-profit markets by improving their quality and their productivity in the past two decades.

A common way to summarize this process performance is by using the process capacity indices (PCIs). These indices provide information with respect to the engineering specifications [12]. The measurement of performance of a process considering both the location and the dispersion of information about it is referred to, in the present study, the PCI of interest C_{pk}. There are two definitions which represent the same thing and get the same result for a specific sample. The first one is defined as follows

$$C_{pk} = (1-k) \times \frac{T}{6s},$$

where $k = |\bar{x} = \mu|/(T/2)$. The terms, $\bar{x}$ and s, are the mean and standard deviation calculated from a sample. The term, μ, is the central value of specifications. The term, T = USL − LSL, the difference in the upper and lower bounds of specifications is usually explained as the toleration of specifications. This index can be explained as the multiplication of the capability of accuracy that concerns the location information of the process and the capability of precision that concerns the dispersion of information of the process, respectively. Another definition for index C_{pk} is as follows:

$$C_{pk} = \min(C_{pk}(U), C_{pk}(L)).$$

where

$$C_{pk}(U) = \frac{\mathrm{USL} - \bar{x}}{3s}, \quad C_{pk}(L) = \frac{\bar{x} - \mathrm{LSL}}{3s}.$$

Then the index, C_{pk} is the shorter standardized distance from the center of the process to either USL or LSL. The PCIs are used in industry. There is an essential assumption, in the conventional application, wherein the output process measurements are precise and distributed as normal random variables. Since the assumption of a normal distribution is untenable, errors can occur if the C_{pk} index is computed using non-normal data. In the present study, we address the situation that the output of data from measurement of the quality of a product is insufficiently precise or not available. This situation can occur when the quality measurement involves to the decision-maker's subjective determination. In such a situation, using linguistic variables, that are easier to capture representing the decision-maker's subjective perception, are applied to construct the PCI C_{pk}. Fuzzy theory [2, 3, 4, 16-19] is applied

Manuscript received Apr. 8, 2007; revised Aug. 28, 2008.
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This work was supported by Grant 01-2008-000-10226-0 from the Korea Science and Engineering Foundation.

to construct the PCI. Since the index C_{pk} involves the location information and the dispersion information simultaneously, it can be applied to evaluate the capacity of a process where human judgments are involved in the evaluation of the process performance. This information is presented by the mean and standard deviation of the producing process. Recently, Lee [12] constructed the first definition of the C_{pk} index estimation presented by fuzzy numbers and approximated its membership function using the "min"-norm based extension principle of fuzzy sets [2, 3, 4]. However, Lee's result was shown to be invalid by Hong [10]. It is well known that T_W-based addition and multiplication preserve the shape of L-R fuzzy numbers [5, 6, 7, 8]. The analytic formula of T_W-based division was shown by Hong [9], recently. Using these T_W-based algebraic fuzzy operations, in this paper, we calculated the exact membership function of the second definition of the index C_{pk} index presented by L-R fuzzy numbers.

2. T_W-based algebraic operation of fuzzy numbers

A fuzzy number is a convex subset of the real line R with a normalized membership function.

A triangular fuzzy number, $\tilde{a}$, denoted by (a, α, β) is defined by

$$\tilde{a}(t) = \begin{cases} 1 - \dfrac{|a-t|}{\alpha} & \text{if } a - \alpha \le t \le a, \\ 1 - \dfrac{|a-\beta|}{\beta} & \text{if } a \le t \le a + \beta, \\ 0 & \text{otherwise,} \end{cases}$$

where $a \in R$ is the center, $\alpha > 0$ is the left spread, and $\beta > 0$ is the right spread of $\tilde{a}$.

If $\alpha = \beta$, then the triangular fuzzy number is called a symmetric triangular fuzzy number and denoted by (a, α).

A fuzzy number $\tilde{a} = (a, \alpha, \beta)_{LR}$ of type L-R is a function from the reals into the interval $[0, 1]$ satisfying

$$\tilde{a} = \begin{cases} R(\dfrac{t-a}{\beta}) & \text{for } a \le t \le a + \beta \\ L(\dfrac{a-t}{\alpha}) & \text{for } a - \alpha \le t \le a, \\ 0 & \text{else,} \end{cases}$$

where L and R are non-increasing and continuous functions from $[0, 1]$ to $[0, 1]$ satisfying $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$.

A binary operation T on the unit interval is said to be a triangular norm [13] (t-norm for short) if and only if T is associative, commutative, non-decreasing and $T(x, 1) = x$ for each $x \in [0, 1]$. Moreover, every t-norm satisfies the inequality

$$T_W(a, b) \le T(a, b) \le \min(a, b) = T_M$$

where

$$T_W(a, b) = \begin{cases} a & \text{if } b = 1, \\ b & a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The critical importance of $\min(a, b)$, $a \cdot b$, $\max(0, a + b - 1)$ and $T_W(a, b)$ is emphasized from a mathematical point of view in [13], as well as other publications.

The usual arithmetical operation of reals can be extended to arithmetical operations on fuzzy numbers by means of Zadeh's extension principle [19] based on a triangular norm T. Let $\tilde{A}, \tilde{B}$ be fuzzy numbers of the real line R. The fuzzy number arithmetic operations are summarized as follows:

Fuzzy number addition $\oplus$:

$$(\tilde{A} \oplus \tilde{B})(z) = \sup_{x+y=z} T(\tilde{A}(x), \tilde{B}(y)).$$

Fuzzy number multiplication $\otimes$:

$$(\tilde{A} \otimes \tilde{B})(z) = \sup_{x \cdot y=z} T(\tilde{A}(x), \tilde{B}(y)).$$

Fuzzy number division $\oslash$:

$$(\tilde{A} \oslash \tilde{B})(z) = \sup_{\frac{x}{y}=z} T(\tilde{A}(x), \tilde{B}(y)).$$

The addition (subtraction) rule for L-R fuzzy numbers is well known in the case of T_M-based addition. The resulting sum is again on L-R fuzzy numbers, i.e., the shape is preserved. It is also known that T_W-based addition and multiplication preserves the shape of L-R fuzzy numbers [5, 6, 7, 8, 11, 15]. Of course, we know that T_M-based multiplication does not preserve the shape of L-R fuzzy numbers. Recently, an analytic formula of T_W-based division was shown by Hong [9].

Let $T = T_W$ be the weakest t-norm and let $\tilde{A} = (a, \alpha_A, \beta_A)_{LR}$, $\tilde{B} = (b, \alpha_B, \beta_B)_{LR}$ be two L-R fuzzy numbers. By [5, 6, 7, 8, 11, 15],

$$\begin{aligned} \tilde{A} \oplus \tilde{B} &= (a, \alpha_A, \beta_A)_{LR} \oplus (b, \alpha_B, \beta_B)_{LR} \\ &= (a + b, \max(\alpha_A, \alpha_B), \max(\beta_A, \beta_B))_{LR}, \end{aligned} \tag{1}$$

$$\tilde{A} \otimes \tilde{B} = \begin{cases} (ab, \max(\alpha_A b, \alpha_B a), \max(\beta_A b, \beta_B a))_{LR} \\ \quad \text{for } a, b > 0, \\ (ab, \max(\beta_A b, \beta_B a), \max(\alpha_A b, \alpha_B a))_{RL} \\ \quad \text{for } a, b < 0, \\ (ab, \max(\alpha_A b - \beta_B a), \max(\beta_A b - \alpha_B a))_{RR} \\ \quad \text{for } a < 0, b > 0, L = R, \\ (0, \alpha_A b, \beta_A b)_{LR} \\ \quad \text{for } a = 0, b > 0, \\ (0, -\beta_A b, -\alpha_A b)_{RL} \\ \quad \text{for } a = 0, b < 0, \\ (0, 0, 0)_{LR} \\ \quad \text{for } a = 0, b = 0, \end{cases} \tag{2}$$

If $\tilde{A}$ and $\tilde{B}$ are symmetric fuzzy numbers, i.e., $L = R$ and $\alpha_A = \beta_A, \alpha_B = \beta_B$, the multiplication can be simplified as

$$\tilde{A} \otimes \tilde{B} = (ab, \max(\alpha_A|b|, \alpha_B|a|, \max(\alpha_A|b|, \alpha_B|a|)_{LL} \quad (3)$$

For the case of division, we have by [8], for $\tilde{A} = (a, \alpha_A, \beta_A)_{LR}$, $\tilde{B} = (b, \alpha_B, \beta_B)_{RL}$, and $a, b > 0$

$$(\tilde{A} \oslash \tilde{B})(z) = \begin{cases} L[(a/b - z)/((1/b)\max(\alpha_A, \beta_B z))] \\ \quad \text{if } \min\{(a-\alpha_A)/b, a/(\beta_B + b)\} \\ \qquad \leq z \leq a/b, \\ R[(z - a/b)/((1/b)\max(\beta_A, \alpha_B z))] \\ \quad \text{if } \max\{(a+\beta_A)/b, a/(b-\alpha_B)\} \\ \qquad \geq z \geq a/b, \\ 0 \quad \text{otherwise.} \end{cases} \quad (4)$$

Here, we note that $A \oslash B$ is not exactly a L-R fuzzy number.

3. The membership function for standard deviation of a fuzzy number

In order to obtain the membership function of C_{pk}, the membership functions of the two statistics, mean and standard deviation must be identified beforehand. For simplicity, we consider the case of symmetric triangular fuzzy numbers. (The non-symmetric LR type fuzzy number cases can be obtained similarly.)

We first note that by Zadeh's extension principle, for $\tilde{t} = (t, 0, 0)$

$$t\tilde{A} = \tilde{t}\tilde{A}, \quad \frac{t}{\tilde{A}} = \frac{\tilde{t}}{\tilde{A}}, \quad \frac{\tilde{A}}{t} = \frac{\tilde{A}}{\tilde{t}}.$$

But we know that $\tilde{A}+\tilde{A} \neq 2\tilde{A}$. For example, if $\tilde{A} = (a, \alpha)$, then $\tilde{A} + \tilde{A} = (2a, \alpha)$ by (2), but $2\tilde{A} = (2a, 2\alpha)$. In general, if $T \neq T_M$, then the n-tuple sum $\tilde{A} + \cdots + \tilde{A} \neq n\tilde{A}$ [see 5-8, 12, 15]. So for convenience we need to define a different type of constant multiplication with a new notation. We denote $\tilde{A} + \cdots + \tilde{A} = n \odot \tilde{A}$ and let $\tilde{A} = (a, \alpha)$. Then $n \odot \tilde{A} = \tilde{A} + \cdots + \tilde{A} = (na, \alpha)$. Hence, we can define for any non-zero real number t,

$$t \odot \tilde{A} = (ta, \alpha). \quad (5)$$

Similarly, suppose that $\tilde{A}\tilde{A} = \tilde{B}$, then $(a^2, |a|\alpha) = (b, \beta)$. Hence we can define

$$\sqrt{\tilde{B}} = \tilde{A} = (\sqrt{b}, \frac{\beta}{\sqrt{b}}). \quad (6)$$

Let us assume that $\tilde{x_j} = (x_j, \gamma_j)_L$, $j = 1, \cdots, n$, $\tilde{y_j} = (y_j, \delta_j)_L$, $j = 1, \cdots, n$, be samples of the random variables of symmetric fuzzy numbers and let

$$\tilde{\bar{x}} = \frac{1}{n} \odot \sum_{j=1}^{n} \tilde{x}_j \text{ and } \tilde{\bar{y}} = \frac{1}{n} \odot \sum_{j=1}^{n} \tilde{y}_j$$

be the average operation for fuzzy numbers $\tilde{x}$ and $\tilde{y}$, respectively. Then we have that by the fuzzy arithmetic operations in Section 2,

$$\tilde{\bar{x}} = (\frac{1}{n}\sum_{j=1}^{n} x_j, \max_{1\leq j\leq n} \gamma_j)_L$$

and

$$\tilde{x}_j - \tilde{\bar{x}} = (x_j - \frac{1}{n}\sum_{j=1}^{n} x_j, \max_{1\leq j\leq n} \gamma_j)_L.$$

Similarly we have that

$$\tilde{\bar{y}} = (\frac{1}{n}\sum_{j=1}^{n} y_j, \max_{1\leq j\leq n} \delta_j)_L$$

and

$$\tilde{y}_j - \tilde{\bar{y}} = (y_j - \frac{1}{n}\sum_{j=1}^{n} y_j, \max_{1\leq j\leq n} \delta_j)_L.$$

And hence we have

$$\begin{aligned} &(\tilde{x}_j - \tilde{\bar{x}})(\tilde{y}_j - \tilde{\bar{y}}) \\ = &\Bigg((x_j - \frac{1}{n}\sum_{j=1}^{n} x_j)(y_j - \frac{1}{n}\sum_{j=1}^{n} y_j), \\ &\max(|x_j - \frac{1}{n}\sum_{j=1}^{n} x_j| \max_{1\leq j\leq n} \delta_j, \\ &|y_j - \frac{1}{n}\sum_{j=1}^{n} y_j| \max_{1\leq j\leq n} \gamma_j)\Bigg)_L. \end{aligned} \quad (7)$$

Define $\mathrm{COV}(\tilde{x}, \tilde{y}) = 1/n \odot \sum_{j=1}^{n} (\tilde{x}_j - \tilde{\bar{x}})(\tilde{y}_j - \tilde{\bar{y}})$ as the covariance between the fuzzy numbers $\tilde{x}$ and $\tilde{y}$. Then we have the following result using (6) and (7).

Proposition 3.1. The membership for $\mathrm{COV}(\tilde{x}, \tilde{y})$ is obtained as

$$\begin{aligned} &\mathrm{COV}(\tilde{x}, \tilde{y}) \\ = &\Bigg(\frac{1}{n}\sum_{j=1}^{n}(x_j - \frac{1}{n}\sum_{k=1}^{n} x_k)(y_j - \frac{1}{n}\sum_{k=1}^{n} y_k), \\ &\max_{1\leq j\leq n}(|x_j - \frac{1}{n}\sum_{k=1}^{n} x_k| \max_{1\leq k\leq n} \delta_k, \\ &|y_j - \frac{1}{n}\sum_{k=1}^{n} y_k| \max_{1\leq k\leq n} \gamma_k)\Bigg)_L. \end{aligned}$$

From Proposition 1, the membership function for $SS_{\tilde{x}} = (1/n) \odot \sum_{j=1}^{n} (\tilde{x}_j - \tilde{\bar{x}})(\tilde{x} - \tilde{\bar{x}})$ as the sum of square error of the fuzzy number, $\tilde{x}$, is obtained as follows

$$\begin{aligned} SS_{\tilde{x}} &= COV(\tilde{x}, \tilde{x}) \\ &= (\frac{1}{n}\sum_{j=1}^{n}(x_j - \frac{1}{n}\sum_{k=1}^{n} x_k)^2, \\ &\quad \max_{1 \le j \le n} |x_j - \frac{1}{n}\sum_{k=1}^{n} x_k| \max_{1 \le k \le n} \gamma_k)_L. \end{aligned}$$

Hence by (6), the membership function for the standard deviation

$$\begin{aligned} S_{\tilde{x}} &= \sqrt{\frac{1}{n}\sum_{j=1}^{n}(\tilde{x}_j - \tilde{\bar{x}})(\tilde{x}_j - \tilde{\bar{x}})} \\ &= \left(\sqrt{\frac{1}{n}\sum_{j=1}^{n}(x_j - \frac{1}{n}\sum_{k=1}^{n} x_k)^2}, \frac{\max_{1 \le j \le n} |x_j - \frac{1}{n}\sum_{k=1}^{n} x_k| \max_{1 \le k \le n} \gamma_k}{\sqrt{\frac{1}{n}\sum_{j=1}^{n}(x_j - \frac{1}{n}\sum_{k=1}^{n} x_k)^2}} \right)_L . \quad (8) \end{aligned}$$

4. Membership function for the fuzzy index C_{pk}

Let the membership functions of the parameters, LSL and USL, be the symmetric L-fuzzy umbers $(l, m)_L$ and $(o, p)_L$, respectively. Then we have

$$USL - \tilde{\bar{x}} = (o - \frac{1}{n}\sum_{j=1}^{n} x_j, \max\{p, \max_{1 \le j \le n} \gamma_j\})_L$$

and

$$\tilde{\bar{x}} - LSL = (\frac{1}{n}\sum_{j=1}^{n} x_j - l, \max\{m, \max_{1 \le j \le n} \gamma_j\})_L.$$

By (8), the membership function of $3S_{\tilde{x}}$ is as follows.

$$3S_{\tilde{x}} = \left(3\sqrt{\frac{1}{n}\sum_{j=1}^{n}(x_j - \frac{1}{n}\sum_{k=1}^{n} x_k)^2}, \frac{\max_{1 \le j \le n} |x_j - \frac{1}{n}\sum_{k=1}^{n} x_k| \max_{1 \le k \le n} \gamma_k}{\sqrt{\frac{1}{n}\sum_{j=1}^{n}(x_j - \frac{1}{n}\sum_{k=1}^{n} x_k)^2}} \right)_L .$$

Here, we can assume that $o - \frac{1}{n}\sum_{j=1}^{n} x_j > 0$ and $\frac{1}{n}\sum_{j=1}^{n} x_j - l > 0$. Now, by the division of fuzzy numbers in Section 2, we can easily find the membership functions of $(USL - \tilde{\bar{x}})/3S_{\tilde{x}}$ and $(\tilde{\bar{x}} - LSL)/3S_{\tilde{x}}$. Applying the following definition, we can now calculate the C_{pk} index completely.

Definition 4.1. Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers with $[\tilde{A}]^\alpha = \{\tilde{A} \ge \alpha\} = [a_1(\alpha), a_2(\alpha)]$ and $[\tilde{B}]^\alpha = [b_1(\alpha), b_2(\alpha)]$. We define

$$[\min\{\tilde{A}, \tilde{B}\}]^\alpha = [a_1(\alpha) \wedge b_1(\alpha), a_2(\alpha) \wedge b_2(\alpha)].$$

Finally, we consider the following example.

Example 4.2. Let $\tilde{x}_1 = (1, \frac{1}{2})$, $\tilde{x}_2 = (2, \frac{2}{3})$, $\tilde{x}_3 = (3, 1)$, let USL = (3, 1, 2), and let LSL = (0.9, 1, 1). Then we have $\tilde{\bar{x}} = (2, 1)$ and $S_{\tilde{x}} = (\frac{2}{3}, \frac{1}{2})$, $3S_{\tilde{x}} = (2, \frac{1}{2})$. We also have $USL - \tilde{\bar{x}} = (1, 1, 2)$ and $\tilde{\bar{x}} - LSL = (1.1, 1)$. Now applying by division of fuzzy numbers, we have

$$\frac{USL - \tilde{\bar{x}}}{3S_{\tilde{x}}} = \frac{(1,1,2)}{(2, \frac{1}{2})} = \begin{cases} 2z & \text{for } 0 \le z \le \frac{1}{2}, \\ \frac{3}{2} - z & \text{for } \frac{1}{2} \le z \le \frac{3}{2} \\ 0 & \text{otherwise,} \end{cases}$$

$$\frac{\tilde{\bar{x}} - LSL}{3S_{\tilde{x}}} = \frac{(1.1,1)}{(2, \frac{1}{2})} = \begin{cases} 2z - \frac{1}{10} & \text{for } \frac{1}{20} \le z \le \frac{11}{20}, \\ \frac{21}{10} - 2z & \text{for } \frac{11}{20} \le z \le 1, \\ \frac{1.1}{z} - 1 & \text{for } 1 \le z \le \frac{11}{10}, \\ 0 & \text{otherwise.} \end{cases}$$

By definition, we see the C_{pk} index in Fig.1.

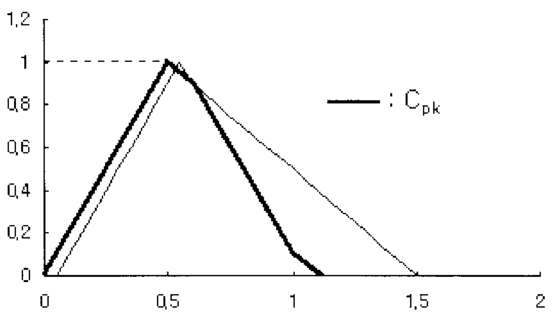

Fig. 1 C_{pk} index

5. Conclusion

Process capacity indices (PCIs) were developed and have been successfully used by companies to compete in and dominate high-profit markets by improving their quality and their productivity in the past two decades. There are two definitions that represent the same thing and get the same result for a specific sample. Recently, Lee [10] constructed the first definition for the C_{pk} index estimation represented by fuzzy numbers. Lee [10] approximated its membership function using the "min"-norm based extension principle of fuzzy sets. In this paper, using this T_W-based algebraic fuzzy operations, we calculated the exact membership function of the second definition of the index C_{pk} index presented by L-R fuzzy numbers.

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