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Classification of Ruled Surfaces with Non-degenerate Second Fundamental Forms in Lorentz-Minkowski 3-Spaces

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Abstract. In this paper, we study some properties of ruled surfaces in a three-dimensional Lorentz-Minkowski space related to their Gaussian curvature, the second Gaussian curvature and the mean curvature. Furthermore, we examine the ruled surfaces in a three-dimensional Lorentz-Minkowski space satisfying the Jacobi condition formed with those curvatures, which are called the *II-W* and the *II-G* ruled surfaces and give a classification of such ruled surfaces in a three-dimensional Lorentz-Minkowski space.

1. Introduction

A surface M in a three-dimensional Euclidean space E^3 with positive Gaussian curvature K possesses a positive definite second fundamental form II if appropriately orientated. Therefore, the second fundamental form can be regarded as a Riemannian metric on M. In turn, it is possible to define its Gaussian curvature K_{II} formed with the second fundamental form viewed as a Riemannian metric. In other words, if a surface has non-zero Gaussian curvature everywhere, K_{II} can be defined formally and it is the curvature of the Riemannian or the pseudo-Riemannian manifold (M, II). Naturally, we can extend such a notion to that of surfaces in a three-dimensional Lorentz-Minkowski space L^3. Using a classical notation, we denote the component functions of the second fundamental form by e, f and g. Thus we define the *second Gaussian curvature* by (cf. [2],[6])

(1.1)
$$K_{II} = \frac{1}{(|eg| - f^2)^2} \left(\begin{vmatrix} -\frac{1}{2}e_{tt} + f_{st} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_t \\ f_t - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_t & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_t & \frac{1}{2}g_s \\ \frac{1}{2}e_t & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix} \right).$$

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It is well known that a minimal surface has vanishing second Gaussian curvature but that a surface with vanishing second Gaussian curvature need not be minimal. For the study of the second Gaussian curvature, D. Koutroufiotis ([8]) has shown that a closed ovaloid is a sphere if $K_{II} = cK$ for some constant c or if $K_{II} = \sqrt{K}$. Th. Koufogiorgos and T. Hasanis ([7]) proved that the sphere is the only closed ovaloid satisfying $K_{II} = H$, where H is the mean curvature. Also, W. Kühnel ([9]) studied surfaces of revolution satisfying $K_{II} = H$. One of the natural generalizations of surfaces of revolution is the helicoidal surfaces. In [1] C. Baikoussis and Th. Koufogiorgos proved that the helicoidal surfaces satisfying $K_{II} = H$ are locally characterized by constancy of the ratio of the principal curvatures. On the other hand, D. E. Blair and Th. Koufogiorgos ([2]) investigated a non-developable ruled surface in E^3 such that $aK_{II} + bH, 2a + b \neq 0$, is a constant along each ruling. Recently, in [6] two of the present authors investigated and classified a non-developable ruled surface in a three-dimensional Lorentz-Minkowski space L^3 satisfying the linear relations

$$aH + bK = \text{constant}, \quad a \neq 0, \tag{1.2}$$

$$aK_{II} + bH = \text{constant}, \quad 2a - b \neq 0, \tag{1.3}$$

$$aK_{II} + bK = \text{constant}, \quad a \neq 0 \tag{1.4}$$

along each ruling.

Now, we introduce the Jacobi function ϕ. Let f_1 and f_2 be smooth functions on M. The Jacobi function $\phi(f_1, f_2)$ formed with f_1, f_2 is defined by $\phi(f_1, f_2) = \det \begin{pmatrix} f_{1s} & f_{1t} \\ f_{2s} & f_{2t} \end{pmatrix}$, where $f_{1s} = \dfrac{\partial f_1}{\partial s}$, $f_{1t} = \dfrac{\partial f_1}{\partial t}$, $f_{2s} = \dfrac{\partial f_2}{\partial s}$ and $f_{2t} = \dfrac{\partial f_2}{\partial t}$. In particular, a surface satisfying the Jacobi condition $\phi(K, H) = 0$ is called a *Weingarten surface* or a *W-surface*. The present paper is to study non-developable ruled surfaces satisfying the Jacobi conditions extending (1.3) and (1.4), namely, the following equations hold:

$$\phi(K_{II}, H) = 0, \tag{1.5}$$

$$\phi(K_{II}, K) = 0. \tag{1.6}$$

If a surface satisfies the equation (1.5), a surface is said to be a *II-Weingarten surface* or simply a *II-W surface* ([10]) and we call a surface satisfying (1.6) a *II-Gauss surface* or simply a *II-G surface*. Needless to say, *II-W* and *II-G* surfaces in a three-dimensional Lorentz-Minkowski space are generalization of those surfaces satisfying (1.3) and (1.4).

For the study about W-surfaces, in [10] W. Kühnel studied the ruled W-surface and ruled II-W surface in E^3. Also, in [3] F. Dillen and W. Kühnel investigated some properties of the ruled W-surface in L^3.

In this paper, we study the *II-W* and *II-G* ruled surfaces in a three-dimensional Lorentz-Minkowski space L^3 and give a complete classification of those surfaces.

2. Preliminaries

Let L^3 be a three-dimensional Lorentz-Minkowski space with the scalar product of index 1 given by $\langle \cdot, \cdot \rangle = -dx_1^2 + dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) is a standard rectangular coordinate system of L^3. A vector x of L^3 is said to be *space-like* if $\langle x, x \rangle > 0$ or $x = 0$, *time-like* if $\langle x, x \rangle < 0$ and *light-like or null* if $\langle x, x \rangle = 0$ and $x \neq 0$. A time-like or light-like vector in L^3 is said to be *causal.* Now, we define a ruled surface M in a three-dimensional Lorentz-Minkowski space L^3. Let J_1 be an open interval in the real line R. Let $\alpha = \alpha(s)$ be a curve in L^3 defined on J_1 and $\beta = \beta(s)$ a transversal vector field along α. For an open interval J_2 of R we have the parametrization for M

$$x = x(s,t) = \alpha(s) + t\beta(s), \quad s \in J_1, \quad t \in J_2.$$

The curve $\alpha = \alpha(s)$ is called a *base curve* and $\beta = \beta(s)$ a *director vector field.* In particular, the ruled surface M is said to be *cylindrical* if the director vector field β is constant and *non-cylindrical* otherwise. First of all, we consider that the base curve α is space-like or time-like. In this case, the director vector field β can be naturally chosen so that it is orthogonal to α. Furthermore, we have ruled surfaces of five different kinds according to the character of the base curve α and the director vector field β as follows: If the base curve α is space-like or time-like, then the ruled surface M is said to be of type M_+ or type M_-, respectively. Also, the ruled surface of type M_+ can be divided into three types. In the case that β is space-like, it is said to be of type M_+^1 or M_+^2 if β' is non-null or light-like, respectively. When β is time-like, β' must be space-like by causal character. In this case, M is said to be of type M_+^3. On the other hand, for the ruled surface of type M_-, it is also said to be of type M_-^1 or M_-^2 if β' is non-null or light-like, respectively. Note that in the case of type M_- the director vector field β is always space-like (cf. [5]). The ruled surface of type M_+^1 or M_+^2 (resp. M_+^3, M_-^1 or M_-^2) is clearly space-like (resp. time-like). But, if the base curve α and the vector field β along α are both light-like, then the ruled surface M is called a *null scroll* ([4]). A non-null scroll ruled surface in L^3 is called a *ruled surface of polynomial kind* if the base curve and the director vector field are given by some polynomials and a *ruled surface of helical kind* if the base curve is given by a helix and the director vector field regarded as a curve is spherical. Throughout the paper, we assume the ruled surface M under consideration is connected unless stated otherwise.

Remark. Let M be a ruled surface in L^m defined by a null base curve α and a non-null director vector field β. In this case, passing to a curve defined by $\tilde{\alpha} = \alpha(s) + f(s)\beta(s)$ as a base curve for a certain function f, M can be determined by a non-null base curve $\tilde{\alpha}$ and a non-null director vector field β, i.e., M is reduced to one of $M_\pm^1, M_\pm^2$ or M_+^3-type. A ruled surface M with a non-null base curve α and a null director vector field β is turned out to be a null scroll by taking a null base curve $\tilde{\alpha} = \alpha(s) + f(s)\beta(s)$ for a suitable function f.

3. Main results

In this section we study ruled II-W and II-G surfaces M in a three-dimensional Lorentz-Minkowski space $\mathbb{L}^3$. Thus the ruled surface M under consideration must have the non-degenerate second fundamental form which automatically implies that M is non-developable. Let M be a ruled surface of one of three types M^1_+, M^3_+ or M^1_- in a three-dimensional Lorentz-Minkowski space $\mathbb{L}^3$ with non-degenerate second fundamental form. Then, M is assumed to be parametrized by

$$x = x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle \beta, \beta \rangle = \varepsilon_1 = \pm 1$, $\langle \beta', \beta' \rangle = \varepsilon_2 = \pm 1$ and $\langle \alpha', \beta' \rangle = 0$. In this case, α is the striction curve of x, and the parameter s is the arc-length of the pseudo-spherical curve β. The natural frame $x_s = \alpha' + t\beta'$ and $x_t = \beta$ give the first fundamental form with components $E = \langle x_s, x_s \rangle = \langle \alpha', \alpha' \rangle + \varepsilon_2 t^2, F = \langle x_s, x_t \rangle = \langle \alpha', \beta \rangle$ and $G = \langle x_t, x_t \rangle = \varepsilon_1$. For later use, we define the smooth functions Q, J and D as follows :

$$Q = \langle \alpha', \beta \times \beta' \rangle \neq 0, \quad J = \langle \beta'', \beta' \times \beta \rangle, \quad D = \sqrt{|EG - F^2|}.$$

On the other hand, the way of choosing the director curve β yields $\{\beta, \beta', \beta \times \beta'\}$ to be an orthonormal frame on M. Then,

$$\alpha' = \varepsilon_1 F\beta - \varepsilon_1\varepsilon_2 Q\beta \times \beta', \quad \beta'' = \varepsilon_1\varepsilon_2(-\beta + J\beta \times \beta'), \quad \alpha' \times \beta = \varepsilon_2 Q\beta' \tag{3.1}$$

and $EG - F^2 = -\varepsilon_2 Q^2 + \varepsilon_1\varepsilon_2 t^2$. Thus, a unit normal vector field N is obtained by

$$N = \frac{1}{D}(\varepsilon_2 Q\beta' - t\beta \times \beta'),$$

from which, the components e, f and g of the second fundamental form are expressed as

$$e = \frac{1}{D}(\varepsilon_1 Q(F - QJ) - Q't + Jt^2), \quad f = \frac{Q}{D}, \quad g = 0.$$

Hence, the function Q never vanishes everywhere on M.

We now define a new type of ruled surface.

Definition 3.1. A ruled surface M in a three-dimensional Lorentz-Minkowski space L^3 is said to be of homogeneous type if M is determined by a non-null director vector field β satisfying $J = C_1 Q$ and $F = C_2 Q^2$ for some constants C_1 and C_2 with $Q' \neq 0$ everywhere.

Example. Let β be a vector field in L^3 defined by $\beta(s) = (\cosh s, \sinh s, 0)$. Take a function $Q = \cosh s$ and $F = 0$ to get a homogeneous ruled surface defined by $x(s,t) = (t\cosh s, t\sinh s, \sinh s)$, where $-\infty < s, t < +\infty$ (See Fig 1).

First of all, let us consider $II - W$ and $II - G$ ruled surfaces of type $M^1_\pm$ or M^3_-. Making use of the data described above and (1.1), we obtain

$$\text{(3.2)}\quad \begin{aligned} K_{II} &= \frac{1}{f^4}\left\{ff_t(f_s - \frac{1}{2}e_t) - f^2(-\frac{1}{2}e_{tt} + f_{st})\right\} \\ &= \frac{1}{2Q^2D^3}\{Jt^4 + \varepsilon_1 Q(F - 2QJ)t^2 + 2\varepsilon_1 Q^2Q't + Q^3(F + QJ)\}. \end{aligned}$$

Furthermore, the mean curvature H is given by

$$\text{(3.3)}\quad \begin{aligned} H &= \frac{1}{2}\frac{Eg - 2Ff + Ge}{|EG - F^2|} \\ &= \frac{1}{2D^3}\{\varepsilon_1 Jt^2 - \varepsilon_1 Q't - Q(F + QJ)\}. \end{aligned}$$

Differentiating K_{II} and H with respect to s and t respectively, we get

$$\text{(3.4)}\quad \begin{cases} (K_{II})_s = \dfrac{1}{2Q^3D^5}\{(2\varepsilon_1 Q'J - \varepsilon_1 QJ')t^6 + (-Q^2F' + 3Q^3J' + QQ'F \\ \quad -5Q^2Q'J)t^4 - 2Q^3Q''t^3 + (4\varepsilon_1 Q^4Q'J - 3\varepsilon_1 Q^5J' \\ \quad -5\varepsilon_1 Q^3Q'F)t^2 + (2\varepsilon_1 Q^5Q'' - 6\varepsilon_1 Q^4Q'^2)t - 2Q^5Q'F \\ \quad -Q^6Q'J + Q^6F' + Q^7J'\}, \\ (K_{II})_t = \dfrac{1}{2Q^2D^5}\{-\varepsilon_1 Jt^5 + Q(F + 2QJ)t^3 + 4Q^2Q't^2 \\ \quad +\varepsilon_1 Q^3(5F - QJ)t + 2\varepsilon_1 Q^4Q'\}, \end{cases}$$

$$\text{(3.5)}\quad \begin{cases} H_s = \dfrac{1}{2D^5}\{-J't^4 + Q''t^3 + (\varepsilon_1 Q'F - \varepsilon_1 QQ'J + \varepsilon_1 QF' + 2\varepsilon_1 Q^2J')t^2 \\ \quad +(3\varepsilon_1 QQ'^2 - \varepsilon_1 Q^2Q'')t + (2Q^2Q'F + Q^3Q'J - Q^3F' - Q^4J')\}, \\ H_t = \dfrac{1}{2D^5}\{Jt^3 - 2Q't^2 - \varepsilon_1 Q(3F + QJ)t - \varepsilon_1 Q^2Q'\}. \end{cases}$$

Now, we assume that M is a II-W surface, that is M satisfies the Jacobi condition $\phi(K_{II}, H) = 0$. It follows

$$\text{(3.6)}\quad J(Q'J - QJ') = 0,$$

$$\text{(3.7)}\quad QQ''J - 4Q'^2J + 2QQ'J' = 0,$$

$$\text{(3.8)}\quad 2Q^3JJ' - QQ'JF - 2Q^2Q'J^2 + Q^2J'F = 0,$$

$$\text{(3.9)}\quad 5Q^3Q''J + Q^2Q''F - 2Q^2Q'F' + Q^3Q'J' + 2QQ'^2F - 11Q^2Q'^2J = 0,$$

(3.10) $$6Q^4Q'J^2-6Q^5JJ'+5Q^3Q'JF-3Q^4J'F+Q^3FF'-2Q^2Q'F^2-Q^4JF'=0,$$

(3.11) $$7Q^5Q''J-11Q^4{Q'}^2J-3Q^5Q'J'+2Q^3{Q'}^2F+2Q^4Q''F-3Q^4Q'F'=0,$$

(3.12) $$Q^6JF'+2Q^7JJ'-2Q^5Q'JF-2Q^6Q'J^2+2Q^4Q'F^2-Q^5FF'=0,$$

(3.13) $$Q^7Q'J'+2Q^5{Q'}^2F+5Q^6{Q'}^2J-Q^6Q''F-3Q^7Q''J=0,$$

(3.14) $$Q^7FF'+Q^8FJ'-2Q^6Q'F^2+Q^7Q'JF-Q^8JF'-Q^9JJ'+Q^8Q'J^2=0,$$

(3.15) $$Q^7Q'(Q^2J'+QF'-QQ'J-2Q'F)=0.$$

Consider an open subset $U_1=\{p\in M|Q'(p)\neq 0\}$. Suppose U_1 is not empty. If $(JQ'-QJ')(q)\neq 0$ for some $q\in U_1$, then $J(q)=0$ because of (3.6). For some open subset $U_2\subset U_1$, $J=0$ on U_2. So, $J'(q)=0$ on U_2 and hence, $JQ'-QJ'=0$ on U_2, which is a contradiction. Consequently, $JQ'-QJ'=0$ on U_1. Let $U_3=\{p\in U_1|J(p)\neq 0\}$. Suppose that $U_3\neq\phi$. On a component O of U_3, (3.6) implies

(3.16) $$J=C_1Q$$

for some non-zero constant C_1. Therefore, by continuity, O must be the whole surface M. Thus, we have one of the following: (1) Q is a non-zero constant. (2) $Q'\neq 0$ everywhere.

We now consider the cases (1) and (2).

`Case (1):` Q is a constant.

(3.6) implies that

$$QJJ'=0.$$

Thus, J is a constant, too. Consequently, $\phi(K_{II},H)$ gives

$$(K_{II})_sH_t-(K_{II})_tH_s=\frac{1}{2D^6}\{\varepsilon_1F'(F-QJ)\}t,$$

which vanishes if and only if F is constant. Thus, Q,J,F are constant. In this case, we can have by a straightforward computation

(3.17) $$\beta'''=\varepsilon_1(J^2-\varepsilon_2)\beta'.$$

We put $C=\varepsilon_1(J^2-\varepsilon_2)$. Without loss of generality, we may assume $C=1,0,-1$ according to its sign.

`Subcase (1.1)` : $C = 0$.

In this case, $J^2 = 1$, in other words, $\varepsilon_2 = 1$. If we compute the length of β'' by using the second equation of (3.1), we see that $\langle \beta'', \beta'' \rangle = 0$ because of $C = 0$. Therefore, we may put

$$\beta''(s) = (d_1, d_2, d_3)$$

for some constants d_1, d_2, d_3 satisfying $-d_1^2 + d_2^2 + d_3^2 = 0$ and so $\beta'(s) = (d_1 s + e_1, d_2 s + e_2, d_3 s + e_3)$ for some constants e_1, e_2 and e_3. Since $\langle \beta', \beta' \rangle = \varepsilon_2 = 1$, we may set $(e_1, e_2, e_3) = (0, 1, 0)$ up to an isometry and hence $\beta(s) = \left(\frac{d_1}{2} s^2 + c_1, \frac{d_2}{2} s^2 + s + c_2, \frac{d_3}{2} s^2 + c_3 \right)$ for some constants c_1, c_2 and c_3. But, $\langle \beta, \beta \rangle = \varepsilon_1$ implies $d_2 = c_2 = 0$ and $d_1^2 = d_3^2$, $-c_1^2 + c_3^2 = \varepsilon_1$, $-d_1 c_1 + d_3 c_3 + 1 = 0$. Thus, β takes the form $\beta(s) = \left(\frac{d_1}{2} s^2 + c_1, s, \frac{d_3}{2} s^2 + c_3 \right)$. Therefore, by using (3.1), up to a rigid motion, we can obtain the parametrization of M as

$$x(s,t) = (a_1 s^3 + b_1 s, a_2 s^2, a_3 s^3 + b_2 s) + t \left(\frac{d_1}{2} s^2 + c_1, s, \frac{d_3}{2} s^2 + c_3 \right) \tag{3.18}$$

for some constants a_1, a_2, a_3, b_1, b_2 satisfying $d_1^2 = d_3^2$, $-c_1^2 + c_3^2 = \varepsilon_1$, $-d_1 c_1 + d_3 c_3 + 1 = 0$ (See Fig 2). Thus, M is of polynomial kind.

`Subcase (1.2)` : $C = 1$.

First, suppose $(\varepsilon_1, \varepsilon_2) = (1, 1)$. Without loss of generality, we may assume $\beta'(0) = (0, 1, 0)$. Thus, $\beta'''(s) = \beta'(s)$ implies

$$\beta'(s) = (B_1 \sinh s, \cosh s + B_2 \sinh s, B_3 \sinh s)$$

for some constants B_1, B_2 and B_3. Since $\varepsilon_2 = 1$, we have $B_1^2 - B_3^2 = 1$ and $B_2 = 0$. From this, we can obtain

$$\beta(s) = (B_1 \cosh s + D_1, \sinh s, B_3 \cosh s + D_3) \tag{3.19}$$

for some constants D_1, D_3 satisfying $D_3^2 - D_1^2 = 2$, $B_1 D_1 = B_3 D_3$ and $B_1^2 - B_3^2 = 1$. We now change the coordinates by $\bar{x}, \bar{y}, \bar{z}$ such that $\bar{x} = B_1 x - B_3 z$, $\bar{y} = y$, $\bar{z} = -B_3 x + B_1 z$, that is,

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} B_1 & 0 & -B_3 \\ 0 & 1 & 0 \\ -B_3 & 0 & B_1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

With respect to the coordinates $(\bar{x}, \bar{y}, \bar{z})$, $\beta(s)$ turns into

$$\beta(s) = (\cosh s, \sinh s, D). \tag{3.20}$$

for a constant $D = B_1D_3 - B_3D_1$ with $D^2 = 2$. Thus, up to a rigid motion M has the parametrization of the form

$$x(s,t) = (a\sinh s, a\cosh s, bs) + t(\cosh s, \sinh s, D) \tag{3.21}$$

for some constants a, b and D with $D^2 = 2$ (See Fig 3).

Next, let $(\varepsilon_1, \varepsilon_2) = (-1, 1)$. In this case, we may also assume $\beta'(0) = (0, 1, 0)$. A similar argument implies $\beta(s) = (B_1\cosh s, \sinh s, B_3\cosh s)$ satisfying $B_1^2 - B_3^2 = 1$. If we change the original coordinates x, y, z as $\bar{x} = B_1x - B_3z$, $\bar{y} = y$, $\bar{z} = -B_3x + B_1z$, that is,

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} B_1 & 0 & -B_3 \\ 0 & 1 & 0 \\ -B_3 & 0 & B_1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Then, the director vector field β can be given by

$$\beta(s) = (\cosh s, \sinh s, 0). \tag{3.22}$$

Thus, with respect to $(\bar{x}, \bar{y}, \bar{z})$, M has the form up to a rigid motion

$$x(s,t) = (a\sinh s, a\cosh s, bs) + t(\cosh s, \sinh s, 0) \tag{3.23}$$

for some constants a and b.

We now suppose $(\varepsilon_1, \varepsilon_2) = (1, -1)$. Quite similarly as we did, we obtain

$$\beta(s) = (\sinh s, B_2\cosh s, B_3\cosh s)$$

satisfying $B_2^2 + B_3^2 = 1$. If we adopt the new coordinates $\bar{x}, \bar{y}, \bar{z}$ such that $\bar{x} = x$, $\bar{y} = B_2y + B_3z$, $\bar{z} = -B_3y + B_2z$, the director vector field β can be expressed as

$$\beta(s) = (\sinh s, \cosh s, 0)$$

and thus, M has parametrization up to a rigid motion

$$x(s,t) = (a\cosh s, a\sinh s, bs) + t(\sinh s, \cosh s, 0) \tag{3.24}$$

for some constants a and b. Therefore, the ruled surface M of Subcase (1.2) is of helical kind.

`Subcase (1.3)` : $C = -1$.

Let $(\varepsilon_1, \varepsilon_2) = (1, 1)$. We may give the initial condition by $\beta'(0) = (0, 1, 0)$ for the ordinary differential equation $\beta''' + \beta' = 0$. Under such initial condition, β is given by $\beta(s) = (B_1\cos s, \sin s, B_2\cos s)$, where B_1 and B_2 are some constant satisfying $B_2^2 - B_1^2 = 1$. If we take another coordinate system $(\bar{x}, \bar{y}, \bar{z})$ such that $\bar{x} = B_2x - B_1z$, $\bar{y} = y$, $\bar{z} = -B_1x + B - 2z$, β takes the form $\beta(s) = (0, \sin s, \cos s)$. So, using (3.1), we get the parametrization of M up to a rigid motion as

$$x(s,t) = (as, -b\cos s, b\sin s) + t(0, \sin s, \cos s) \tag{3.25}$$

for some constants a and b (See Fig 4).

Suppose $(\varepsilon_1, \varepsilon_2) = (-1, 1)$. Similarly to the previous case, we may take $\beta'(0) = (0, 1, 0)$. Then, we have $\beta(s) = (D_1 - B_1 \cos s, \sin s, D_2 - B_2 \cos s)$, where $B_2^2 - B_1^2 = 1, D_1^2 - D_2^2 = 2, B_1 D_1 = B_2 D_2$ and $(B_1 D_2 - B_2 D_1)^2 = 2$. If we choose another coordinate system $(\bar{x}, \bar{y}, \bar{z})$ such that $\bar{x} = -B_2 x + B_1 z, \bar{y} = y, \bar{z} = B_1 x - B_2 z$, the director vector field β turns into $\beta(s) = (\pm\sqrt{2}, \sin s, \cos s)$. Thus, the base curve α is derived as $\alpha(s) = (a_1 s, -a_2 \cos s, a_2 \sin s)$ for some constants a_1 and a_2 up to a rigid motion in L^3. Thus, we have a parametrization of M up to a rigid motion as

$$x(s,t) = (a_1 s, -a_2 \cos s, a_2 \sin s) + t(\pm\sqrt{2}, \sin s, \cos s) \tag{3.26}$$

for some constants a_1 and a_2.

One can easily verify that there do not exist *II-W* ruled surfaces for the cases of $(\varepsilon_1, \varepsilon_2) = (1, -1)$ by similarly examining the character of the director vector field β developed as above. Thus, the ruled surface M of Subcase (1.3) is of helical kind.

`Case (2)`: $Q' \neq 0$ everywhere.

In this case, $J = C_1 Q$ for some constant C_1. Thus, (3.7) implies

$$Q' = C_2 Q^2$$

for some non-zero constant C_2 if $C_1 \neq 0$. So, Q is given by $Q = \dfrac{1}{C_3 - C_2 s}$ and $J = \dfrac{C_1}{C_3 - C_2 s}$ for some constant C_3. By (3.8) and continuity, F is given by either $F = C_4 Q^2$ for some non-zero constant C_4 or $F \equiv 0$. So, we have a class of ruled surfaces of homogeneous type.

Conversely, if a ruled surface M with non-degenerate second fundamental form is given by polynomial kind or helical kind described as (3.18), (3.21), (3.23), (3.24), (3.25), (3.26) or one of homogeneous type ruled surfaces, it is easily seen $\phi(K_{II}, H) = 0$.

Thus, we have

Theorem 3.2. *Let M be a ruled surface of one of three types M_+^1, M_+^3 or M_-^1 in a three-dimensional Lorentz-Minkowski space L^3 with non-degenerate second fundamental form. Then, M is a II-W surface if and only if M is part of a ruled surface of polynomial kind, helical kind determined by* (3.18), (3.21), (3.23), (3.24), (3.25), (3.26) *or one of ruled surfaces of homogeneous type.*

Remark. Case (2) with $J = F = 0$ of Theorem 3.2 implies $K_{II} = -2H$. Furthermore, some of ruled surfaces satisfying $K_{II} = -2H$ are part of the conoids of the 1st, 2nd, 3rd kind (See [6]).

Theorem 3.3. *Let M be a ruled surface of the types M_+^1, M_+^3 or M_-^1 in a three-dimensional Lorentz-Minkowski space L^3 with non-degenerate second fundamental* *form. Then M is a II-G surface if and only if Q, J and F are constant, that is, M is part of a ruled surface of polynomial kind or helical kind of the form* (3.18), (3.21), (3.23), (3.24), (3.25) *or* (3.26).

Proof. As it is described in the proof of Theorem 3.2 we assume that the non-cylindrical ruled surface M of the three types M^1_+, M^3_+ or M^1_- is parametrized by

$$x = x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle\beta,\beta\rangle = \varepsilon_1(=\pm1), \langle\beta',\beta'\rangle = \varepsilon_2(=\pm1)$ and $\langle\alpha',\beta'\rangle = 0$. Following the same notations given in Theorem 3.2, the Gaussian curvature K is given by

$$K = \langle N,N\rangle\frac{eg-f^2}{EG-F^2} = \frac{Q^2}{D^4}.$$

It follows

$$\begin{cases} K_s = \dfrac{1}{D^6}(-2Q^3Q' - 2\varepsilon_1 QQ't^2), \\ K_t = \dfrac{1}{D^6}(4\varepsilon_1 Q^2 t). \end{cases} \tag{3.27}$$

Suppose that the surface M is a II-G surface, that is, M satisfies the Jacobi condition $\phi(K_{II}, K) = 0$. Then, by (3.4) and (3.27) we obtain

$$\begin{cases} 3Q^2Q'J - 2Q^3J' = 0, \\ 3Q^3Q'F - 2Q^4F' + 6Q^5J' - 9Q^4Q'J = 0, \\ Q^4Q'^2 - Q^5Q'' = 0, \\ 9Q^6Q'J - 6Q^7J' - 4Q^5Q'F = 0, \\ 2Q^7Q'' - 3Q^6Q'^2 = 0, \\ 2Q^8F' + 2Q^9J' + Q^7Q'F - 3Q^8Q'J = 0, \\ Q^8Q'^2 = 0, \end{cases}$$

from which, it is straightforward to get $J' = F' = Q' = 0$. By using Theorem 3.2, the ruled surface is determined by one of ruled surfaces of polynomial kind or helical kind of the form (3.18), (3.21), (3.23), (3.24), (3.25) or (3.26). The converse is straightforward. This completes the proof. □

Corollary 3.4. *Let M be a non-homogeneous type ruled surface of the type $M^1_\pm$ or M^3_+ in a three-dimensional Lorentz-Minkowski space L^3 with non-degenerate second fundamental form. Then M is a II-W surface if and only if M is a II-G surface.*

Next, we consider the II-W ruled surface M of type M^2_+ or M^2_-, that is, M satisfies the Jacobi condition $\phi(K_{II}, H) = 0$. In this case, the base curve α is space-like or time-like and the director vector field β is space-like but β' is light-like. So, we may take α and β satisfying $\langle\alpha',\beta\rangle = 0$, $\langle\beta,\beta\rangle = 1$, $\langle\beta',\beta'\rangle = 0$ and $\langle\alpha',\alpha'\rangle = \varepsilon_1(=\pm1)$. We have put the non-zero functions q and R as follows:

$$q = ||x_s||^2 = \varepsilon\langle x_s, x_s\rangle = \varepsilon(\varepsilon_1 + 2Rt), \quad R = \langle\alpha',\beta'\rangle$$

where ε denotes the sign of x_s. Since $\beta \times \beta'$ is a null vector field orthogonal to β', we can assume $\beta \times \beta' = \beta'$. Without loss of generality, we may assume that $\beta(0) = (0, 0, 1)$. Since β' is a null direction in the hyperboloid $\{\mathbf{x} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\}$, β can be chosen as a straight line. If we write $\beta(s) = (a, b, c)s + (0, 0, 1)$, we can see that $c = 0$ and $a^2 = b^2 \neq 0$. Without loss of generality, we may assume that $a = b$. Then, $\beta(s) = (as, as, 1)$ $(a \neq 0)$. Then, $\{\alpha', \beta, \alpha' \times \beta\}$ is a moving frame along M. Then, β' can be written as

$$\beta' = \varepsilon_1 R(\alpha' - \alpha' \times \beta). \tag{3.28}$$

It follows that the function R never vanishes everywhere on M. Since $\beta'' = 0$, (3.28) implies

$$\alpha'' = -R\beta + \frac{R'}{R}\alpha' \times \beta. \tag{3.29}$$

Let $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$. From (3.28), we have $a = \varepsilon_1 R(\alpha_1' + \alpha_2' - as\alpha_3')$ and $\alpha_3' = as(\alpha_1' - \alpha_2')$. Since $-(\alpha_1')^2 + (\alpha_2')^2 + (\alpha_3')^2 = \varepsilon_1$, it gives

$$\alpha_1' - \alpha_2' = -\frac{R}{a}, \tag{3.30}$$

$$\alpha_1' + \alpha_2' = \frac{a\varepsilon_1}{R} - as^2 R, \tag{3.31}$$

$$\alpha_3' = -sR, \tag{3.32}$$

from which, we can obtain from (3.30) and (3.31)

$$\alpha_1' = \frac{1}{2}\left(\frac{\varepsilon_1 a}{R} - as^2 R - \frac{R}{a}\right), \tag{3.33}$$

$$\alpha_2' = \frac{1}{2}\left(\frac{\varepsilon_1 a}{R} - as^2 R + \frac{R}{a}\right). \tag{3.34}$$

On the other hand, the unit normal vector field of M is given by

$$N = \frac{1}{\sqrt{q}}(\alpha' \times \beta - t\beta'),$$

from which the components of the second fundamental form e, f, and g are obtained as

$$e = -\frac{\varepsilon}{\sqrt{q}}\left\{tR' + \varepsilon_1 \frac{R'}{R}\right\}, \quad f = \frac{\varepsilon}{\sqrt{q}}R, \quad g = 0.$$

Thus, the mean curvature H and the second Gaussian curvature K_{II} are given respectively by

$$H = -\frac{\varepsilon}{2q^{3/2}}\left(tR' + \frac{\varepsilon_1 R'}{R}\right), \tag{3.35}$$

$$K_{II} = \frac{\varepsilon\varepsilon_1 R'}{2q^{3/2}R}. \tag{3.36}$$

Suppose that the ruled surface M is a II-W surface. Then M satisfies the Jacobi condition $\phi(K_{II}, H) = 0$. By straightforward computation from (3.35) and (3.36), we have

$$RR'R'' - (R')^3 = 0. \tag{3.37}$$

Let U $= \{p \in M | R'(p) \neq 0\}$ be an open subset of M. Suppose U is not empty. Then, on U, the function R can be solved as

$$R = be^{cs} \tag{3.38}$$

for some non-zero constants b and c. So, by continuity, R is either a non-zero constant or $R = be^{cs}$ along M.

`Case (1)`: R is a non-zero constant.

Up to a rigid motion, α is given by

$$\alpha(s) = (a_1 s + a_2 s^3, b_1 s + a_2 s^3, a_3 s^2) \tag{3.39}$$

for some non-zero constants a_1, a_2, a_3 and b_1. Thus, the II-W ruled surface M of type $M^2_\pm$ is given by

$$x(s,t) = (a_1 s + a_2 s^3, b_1 s + a_2 s^3, a_3 s^2) + t(as, as, 1) \tag{3.40}$$

for some non-zero constants a, a_1, a_2, a_3 and b_1 (See Fig 5).

`Case (2)`: $R = be^{cs}$ for some non-zero constants b and c.

Up to a rigid motion, we have α of the form

$$\begin{aligned} \alpha(s) = \Bigg(&\left(-\frac{ab}{2c}s^2 + \frac{ab}{c^2}s - \frac{ab}{c^3} - \frac{b}{2ac}\right)e^{cs} - \frac{\varepsilon_1 a}{2bc}e^{-cs}, \\ &\left(-\frac{ab}{2c}s^2 + \frac{ab}{c^2}s - \frac{ab}{c^3} + \frac{b}{2ac}\right)e^{cs} - \frac{\varepsilon_1 a}{2bc}e^{-cs}, \left(-\frac{b}{c}s + \frac{b}{c^2}\right)e^{cs}\Bigg). \end{aligned} \tag{3.41}$$

Thus, the II-W ruled surface M of type $M^2_\pm$ is given by

$$\begin{aligned} x(s,t) = \Bigg(&\left(-\frac{ab}{2c}s^2 + \frac{ab}{c^2}s - \frac{ab}{c^3} - \frac{b}{2ac}\right)e^{cs} - \frac{\varepsilon_1 a}{2bc}e^{-cs}, \\ &\left(-\frac{ab}{2c}s^2 + \frac{ab}{c^2}s - \frac{ab}{c^3} + \frac{b}{2ac}\right)e^{cs} - \frac{\varepsilon_1 a}{2bc}e^{-cs}, \left(-\frac{b}{c}s + \frac{b}{c^2}\right)e^{cs}\Bigg) \\ &+ t(as, as, 1), \end{aligned} \tag{3.42}$$

(See Fig 6).

Conversely, if a ruled surface M with non-degenerate second fundamental form is given by (3.40) or (3.42), one can see that $\phi(K_{II}, H) = 0$.

Thus, we have

Theorem 3.5. *Let M be a ruled surface of type $M^2_\pm$ with non-degenerate second fundamental form. Then, M is a II-W surface if and only if M is part of a ruled surface of polynomial kind of the form* (3.40) *or* (3.42).

Remark. In the proof of Theorem 3.2 and Theorem 3.5, the classes of ruled surfaces include minimal ones described in [5].

Let M be a ruled surface of type $M^2_\pm$ with non-degenerate second fundamental form. So, we can take again the base curve α and the director vector field β as is given in Theorem 3.5 such that

$$\langle \alpha', \alpha' \rangle = \varepsilon_1 = \pm 1, \quad \langle \alpha', \beta \rangle = 0, \quad \langle \beta, \beta \rangle = 1, \quad \langle \beta', \beta' \rangle = 0.$$

Then, by definition, the Gauss curvature K is given by

$$K = \langle N, N \rangle \frac{eg - f^2}{EG - F^2} = \frac{R^2}{q^2}$$

where R and q are the same functions in the proof of Theorem 3.5 defined by $R = \langle \alpha', \beta' \rangle$ and $q = \varepsilon(\varepsilon_1 + 2Rt)$. We now suppose that the ruled surface M is II-G, that is, the Gaussian curvature K and the second Gaussian curvature K_{II} satisfy the Jacobi condition $\phi(K, K_{II}) = 0$. Then, by straightforward computation, we obtain

$$2RR'' - 5(R')^2 = 0. \tag{3.43}$$

Let $\mathrm{V} = \{p \in M | R'(p) \neq 0\}$. Suppose the open subset V of M is not empty. Then, (3.43) implies $2R''/R' = 5R'/R$. It gives that $R'/R^2 = C_1 R^{1/2}$ for some non-zero constant C_1. Therefore, $\frac{d}{ds}(-1/R) = C_1 R^{1/2}$. It yields $R = (C_1 s + C_2)^{-\frac{3}{2}}$ for some constant C_2. By the continuity of R, the function R is either a non-zero constant or $R = (C_1 s + C_2)^{-\frac{3}{2}}$ for some non-zero constant C_1 and a constant C_2 on M. Hence, if R is a constant, then M is part of ruled surface of polynomial kind given by (3.40) as it is given in the previous theorem . Suppose R is given by $R = (C_1 s + C_2)^{-\frac{3}{2}}$ for some non-zero constant C_1 and a constant C_2. Then, the parametrization of M is given by

$$x(s,t) = \int^s \alpha'(u)du + t(as, as, 1) \tag{3.44}$$

where α' is determined by (3.32), (3.33) and (3.34) with $R = (C_1 s + C_2)^{-\frac{3}{2}}$ for some non-zero constant C_1 and a constant C_2. Conversely, if M is given by a ruled surface of polynomial kind of the form (3.40) or (3.44), then it is easily seen that $\phi(K_{II}, K) = 0$, that is, M is a II-G surface.

Consequently, we have

Theorem 3.6. *Let M be a ruled surface of type $M^2_\pm$ with non-degenerate second fundamental form. Then, M is a II-G surface if and only if M is part of either a ruled surface of polynomial kind of the form given by* (3.40) *or* (3.44).

Remark. Let M be a null scroll in a three-dimensional Lorentz-Minkowski space with non-degenerate second fundamental form. If M is a $II-W$ surface or a $II-G$ surfacer, then M satisfies $K = Q^2$, $H = Q$ and $K_{II} = 1/Q$. (For details, see [6]).

Finally, we would like to propose the following :

Problem. Classify all non-trivial homogeneous $II - W$ ruled surfaces in L^3, i.e., they satisfy $J \neq 0$ and $F \neq 0$.

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Fig 1

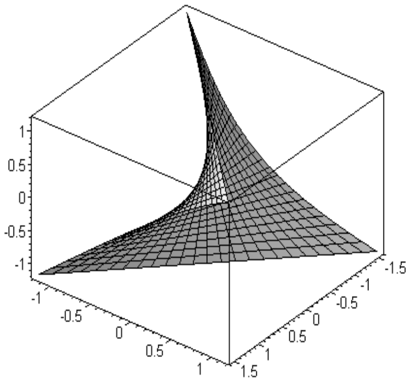

Fig 2

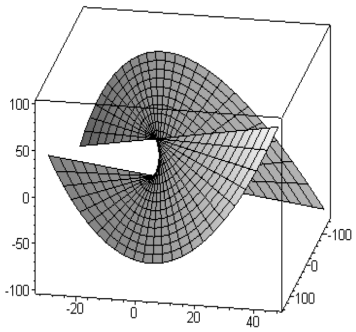

Fig 3

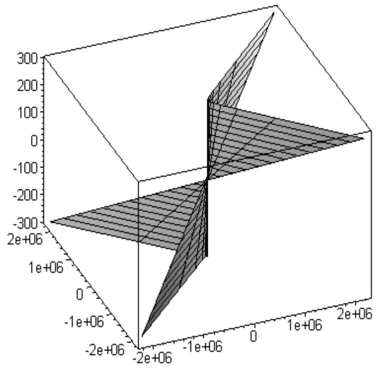

Fig 4

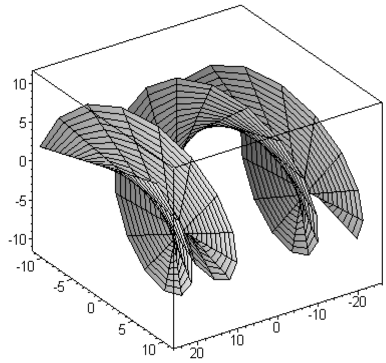

Fig 5

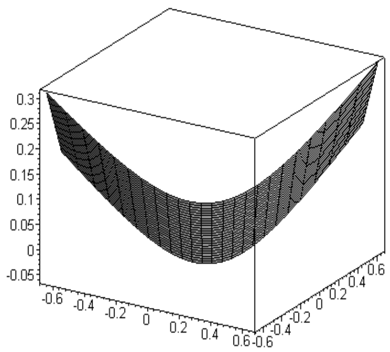

Fig 6