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Prior Estimates and Solutions of Boundary Value Problems for Higher-Order Nonlinear Finite Difference Equations

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Abstract. Sufficient conditions for the existence of at least one solution of boundary value problems for higher order nonlinear difference equations are established. We allow f to be at most linear, superlinear or sublinear in obtained results.

1. Introduction

Solvability of boundary value problems for finite difference equations were studied in many papers, one may see the text books [1], [2] and the papers [3], [4].

In [1], [2], the authors studied the solvability of problem

$$(1.1)\quad \begin{cases} \Delta^n x(k) = f(k, x(k), x(k+1), x(k+2), \cdots, x(k+n-1)), \ \ k \in [0, N], \\ \Delta^i x(0) = 0, \quad i = 0, \cdots, p, \\ \Delta^i x(N) = 0, \ \ i = p+1, \cdots, n-1, \end{cases}$$

where $n \geq 1$, $[0, N]$ denotes the integers set $\{0, 1, \cdots, N\}$. Under the assumption:
$(*)$ there are constants $a_i \geq 0$ such that

$$|f(t, x_0, x_1, \cdots, x_{n-1})| \leq \sum_{i=0}^{n-1} a_i |x_i| + a_n, \ \ (t, x_0, \cdots, x_{n-1}) \in [0, N] \times R^n.$$

and the other conditions imposed on a_i, it was prove that problem (1) has at least one solution. We call condition $(*)$ at most linear growth condition. When f is superlinear, problem (1) has not be solved till now.

The purposes of this paper are to establish sufficient conditions for the existence of at least one solutions of problems

$$(1.2)\quad \begin{cases} \Delta^n x(k) = f(k, x(k), \Delta x(k), \cdots, \Delta^{n-1} x(k)), \ \ k \in [0, N], \\ \Delta^i x(0) = 0, \quad i = 0, \cdots, p, \\ \Delta^i x(N+1) = 0, \ \ i = p+1, \cdots, n-1, \end{cases}$$

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and

$$(1.3)\quad \begin{cases} \Delta^n x(k) = f(k, x(k), \Delta x(k), \cdots, \Delta^{n-1}x(k), \Delta^{n-1}x(k+1)), & k \in [0, N], \\ \Delta^i x(0) = 0, \quad i = p+1, \cdots, n-1, \\ \Delta^i x(N+1) = 0, \quad i = 0, \cdots, p, \end{cases}$$

where f is continuous and N is a positive integer with $n \geq 2$, $0 \leq p \leq n-2$ and $p < N+1$. It is interesting that we allow that f to be sublinear, at most linear or superlinear.

Problem (2) is the discrete analogue of the well known (n, p) focal problem

$$\begin{cases} x^{(n)}(t) = f(t, x(t), x'(t), \cdots, x^{(n-1)}(t)), & t \in (0, 1), \\ x^{(i)}(0) = 0, \quad i = 0, \cdots, p, \\ x^{(j)}(1) = 0, \quad j = p+1, \cdots, n-1, \end{cases}$$

which was studied extensively in [1], [2] and the references therein.

This paper is organized as follows. In Section 2, we give the main results, and in Section 3, examples to illustrate the main results will be presented.

2. Main Results

To get existence results for solutions of BVP (2) and problem (3), we need the following fixed point theorem, which was used to solve multi-point boundary value problems for differential equations in many papers.

Let X and Y be Banach spaces, $L: \ D(L) \subset X \to Y$ be a Fredholm operator of index zero.

If Ω is an open bounded subset of X, $D(L) \cap \overline{\Omega} \neq \emptyset$, the map $N: \ X \to Y$ will be called $L-$compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N: \ \overline{\Omega} \to X$ is compact.

Theorem 2.1[5]. *Let X and Y be Banach spaces. Suppose $L: D(L) \subset X \to Y$ is a Fredholm operator of index zero with $KerL = \{0\}$, $N: X \to Y$ is $L-$compact on any open bounded subset of X. If $0 \in \Omega \subset X$ is a open bounded subset and $Lx \neq \lambda Nx$ for all $x \in D(L) \cap \partial\Omega$ and $\lambda \in [0, 1]$, then there is at least one $x \in \Omega$ so that $Lx = Nx$.*

Let $X = R^{N+n+1}$ be endowed with the norm $||x|| = \max_{n \in [0, N+n]} |x(n)|$, $Y = R^{N+1}$ be endowed with the norm $||y|| = \max_{n \in [0, N]} |y(n)|$. It is easy to see that X and Y are a Banach space. Choose

$$D(L) = \left\{x \in X: \ \Delta^i x(0) = 0, \ i \in [0, \cdots, p], \ \ \Delta^j x(N+1) = 0, \quad j \in [p+1, n-1]\right\}.$$

Set

$$L: D(L) \cap X \to Y, \quad L \bullet x(k) = \Delta^n x(k), \quad k \in [0, N],$$

and $N: X \to Y$ by

$$N \bullet x(k) = f(k, x(k), \Delta x(k), \cdots, \Delta^{n-1}x(k)), \ \ k \in [0, N],$$

for all $x \in X$.

It is easy to check the results.

(i) $\mathrm{Ker}L = \{x(k) \equiv 0,\ k \in [0, N+n-1]\}$.

(ii) L is a Fredholm operator of index zero.

(iii) Let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap D(L) \neq \emptyset$, then N is $L-$compact on $\overline{\Omega}$.

(iv) $x \in D(L)$ is a solution of $L \bullet x = N \bullet x$ implies that x is a solution of problem (2).

Theorem L1. Suppose that there is numbers $\beta > 0$, $\theta > 1$, nonnegative sequences $p_i(k), r(k) (i = 0, \cdots, n-1)$, functions $g(k, x_0, \cdots, x_{n-1})$, $h(k, x_0, \cdots, x_{n-1})$ such that

$$f(k, x_0, \cdots, x_{n-1}) = g(k, x_0, \cdots, x_{n-1}) + h(k, x_0, \cdots, x_{n-1})$$

$$g(k, x_0, x_1, \cdots, x_{n-1})x_{n-1} \geq \beta |x_{n-1}|^{\theta+1},$$

and

$$|h(k, x_0, \cdots, x_{n-1})| \leq \sum_{i=0}^{n-1} p_i(k)|x_i|^{\theta} + r(k),$$

for all $k \in \{0, \cdots, N\}$, $(x_0, x_1, \cdots, x_{n-1}) \in R^n$. Then problem (2) has at least one solution if

$$(N+1)^{1+\theta}\left[\left(\frac{(N+n-p-3)^{n-p-3}}{(n-p-3)!}\right)^{\theta} \sum_{i=0}^{p} ||p_i|| \left(\frac{(N+n-i-1)^{p-i}}{(p-i)!}\right)^{\theta} + \sum_{j=p+1}^{n-2} ||p_j|| \left(\frac{(N+n-j-2)^{n-j-2}}{(n-j-2)!}\right)^{\theta}\right] + ||p_{n-1}|| < \beta.$$

Proof. To apply Theorem 2.1, we should define an open bounded subset Ω of X so that conditions of Theorem 2.1 hold.

Let $\Omega_1 = \{x : Lx = \lambda Nx,\ (x, \lambda) \in [(D(L) \times (0,1)]\}$. For $x \in \Omega_1$, we have $L \bullet x = \lambda N \bullet x$, $\lambda \in (0,1)$, so

$$\Delta^n x(k) = \lambda f(k, x(k), \Delta x(k), \cdots, \Delta^{n-1}x(k)). \tag{2 1}$$

So

$$\sum_{k=0}^{N} [\Delta^n x(k)] \Delta^{n-1}x(k) = \lambda \sum_{k=0}^{N} f(k, x(k), \Delta x(k), \cdots, \Delta^{n-1}x(k)) \Delta^{n-1}x(k), \quad k \in [0, N].$$

Since

$$
\begin{aligned}
& 2\sum_{k=0}^{N}[\Delta^n x(k)]\Delta^{n-1}x(k) \\
= \; & [\Delta^{n-1}x(N+1)]^2 - \sum_{k=0}^{N}[\Delta^{n-1}x(k+1)-\Delta^{n-1}x(k)]^2 - [\Delta^{n-1}x(0)]^2 \\
= \; & -\sum_{k=0}^{N}[\Delta^{n-1}x(k+1)-\Delta^{n-1}x(k)]^2 - [\Delta^{n-1}x(0)]^2 \le 0,
\end{aligned}
$$

we get

$$\sum_{k=0}^{N} f(k,x(k),\Delta x(k),\cdots,\Delta^{n-1}x(k))\Delta^{n-1}x(k) \le 0.$$

It follows that

$$
\begin{aligned}
\beta\sum_{k=0}^{N}|\Delta^{n-1}x(k)|^{\theta+1} \le \; & \sum_{k=0}^{N} g(k,x(k),\Delta x(k),\cdots,\Delta^{n-1}x(k))\Delta^{n-1}x(k) \\
\le \; & -\sum_{k=0}^{N} h(k,x(k),\Delta x(k),\cdots,\Delta^{n-1}x(k))\Delta^{n-1}x(k) \\
\le \; & \sum_{k=0}^{N} |h(k,x(k),\Delta x(k),\cdots,\Delta^{n-1}x(k))||\Delta^{n-1}x(k)| \\
\le \; & \sum_{i=0}^{n-1}\sum_{k=0}^{N} p_i(k)|\Delta^i x(k)|^{\theta}|\Delta^{n-1}x(k)| + \sum_{k=0}^{N}|r(k)||\Delta^{n-1}x(k)| \\
\le \; & \sum_{i=0}^{n-2}\|p_i\|\sum_{k=0}^{N}|\Delta^i x(k)|^{\theta}|\Delta^{n-1}x(k)| + \|r\|\sum_{k=0}^{N}|\Delta^{n-1}x(k)| \\
& +\|p_{n-1}\|\sum_{k=0}^{N}|\Delta^{n-1}x(k)|^{\theta+1}.
\end{aligned}
$$

For $x_i \ge 0, y_i \ge 0$, Holder inequality implies

$$\sum_{i=1}^{s} x_i y_i \le \left(\sum_{i=1}^{s} x_i^p\right)^{1/p}\left(\sum_{i=1}^{s} y_i^q\right)^{1/q}, \;\; 1/p+1/q=1, \;\; q>0, \; p>0.$$

It is easy to show, for $i = 0, \cdots, p$, that

$$\Delta^i x(k) = \sum_{s=0}^{k-1} d\frac{(k-1-s)^{p-i}}{(p-i)!}\Delta^{p+1}x(s), \quad k \in [0, N+n-i],$$

and for $j = p+1, \cdots, n-2$, that

$$\Delta^j x(k) = (-1)^{n-j-1} \sum_{s=k-n+j+2}^{N} \frac{(s-k+n-j-2)^{n-j-2}}{(n-j-2)!} \Delta^{n-1} x(s), \quad k \in [0, N+n-j].$$

Hence

$$|\Delta^j x(k)| = \frac{(N+n-j-2)^{n-j-2}}{(n-j-2)!} \sum_{k=0}^{N} |\Delta^{n-1} x(k)|, \quad j = p+1, \cdots, n-2.$$

and

$$|\Delta^i x(k)| = \frac{(N+n-i-1)^{p-i}}{(p-i)!} \frac{(N+n-p-3)^{n-p-3}}{(n-p-3)!} \sum_{k=0}^{N} |\Delta^{n-1} x(k)|, \quad i = 0, \cdots, p$$

It follows that

$$\begin{aligned}
&\beta \sum_{k=0}^{N} |\Delta^{n-1} x(k)|^{\theta+1} \\
\leq\ & ||p_{n-1}|| \sum_{k=0}^{N} |\Delta^{n-1} x(k)|^{\theta+1} + ||r||(N+1)^{\frac{\theta}{\theta+1}} \left(\sum_{k=0}^{N} |\Delta^{n-1} x(k)|^{\theta+1} \right)^{\frac{1}{\theta+1}} \\
& + \sum_{i=0}^{p} ||p_i|| \sum_{k=0}^{N} |\Delta^i x(k)|^{\theta} |\Delta^{n-1} x(k)| + \sum_{i=p+1}^{n-2} ||p_i|| \sum_{k=0}^{N} |\Delta^i x(k)|^{\theta} |\Delta^{n-1} x(k)| \\
\leq\ & ||p_{n-1}|| \sum_{k=0}^{N} |\Delta^{n-1} x(k)|^{\theta+1} + ||r||(N+1)^{\frac{\theta}{\theta+1}} \left(\sum_{k=0}^{N} |\Delta^{n-1} x(k)|^{\theta+1} \right)^{\frac{1}{\theta+1}} \\
& + \sum_{i=0}^{p} ||p_i|| \left(\frac{(N+n-i-1)^{p-i}}{(p-i)!} \frac{(N+n-p-3)^{n-p-3}}{(n-p-3)!} \sum_{k=0}^{N} |\Delta^{n-1} x(k)| \right)^{\theta} \times \\
& \sum_{k=0}^{N} |\Delta^{n-1} x(k)| \\
& + \sum_{j=p+1}^{n-2} ||p_j|| \left(\frac{(N+n-j-2)^{n-j-2}}{(n-j-2)!} \sum_{k=0}^{N} |\Delta^{n-1} x(k)| \right)^{\theta} \sum_{k=0}^{N} |\Delta^{n-1} x(k)| \\
\leq\ & ||p_{n-1}|| \sum_{k=0}^{N} |\Delta^{n-1} x(k)|^{\theta+1} + ||r||(N+1)^{\frac{\theta}{\theta+1}} \left(\sum_{k=0}^{N} |\Delta^{n-1} x(k)|^{\theta+1} \right)^{\frac{1}{\theta+1}} \\
& + \left(\frac{(N+n-p-3)^{n-p-3}}{(n-p-3)!} \right)^{\theta} \sum_{i=0}^{p} \|p_i\| \left(\frac{(N+n-i-1)^{p-i}}{(p-i)!} \right)^{\theta} \times
\end{aligned}$$

$$
\begin{aligned}
&\left(\sum_{k=0}^{N}|\Delta^{n-1}x(k)|\right)^{\theta+1}\\
&+\sum_{j=p+1}^{n-2}\|p_j\|\left(\frac{(N+n-j-2)^{n-j-2}}{(n-j-2)!}\right)^{\theta}\left(\sum_{k=0}^{N}|\Delta^{n-1}x(k)|\right)^{\theta+1}\\
\leq\ & \|p_{n-1}\|\sum_{k=0}^{N}|\Delta^{n-1}x(k)|^{\theta+1}+||r||(N+1)^{\frac{\theta}{\theta+1}}\left(\sum_{k=0}^{N}|\Delta^{n-1}x(k)|^{\theta+1}\right)^{\frac{1}{\theta+1}}\\
&+(N+1)^{1+\theta}\left(\frac{(N+n-p-3)^{n-p-3}}{(n-p-3)!}\right)^{\theta}\sum_{i=0}^{p}\|p_i\|\left(\frac{(N+n-i-1)^{p-i}}{(p-i)!}\right)^{\theta}\times\\
&\sum_{k=0}^{N}|\Delta^{n-1}x(k)|^{\theta+1}\\
&+(N+1)^{\theta+1}\sum_{j=p+1}^{n-2}\|p_j\|\left(\frac{(N+n-j-2)^{n-j-2}}{(n-j-2)!}\right)^{\theta}\sum_{k=0}^{N}|\Delta^{n-1}x(k)|^{\theta+1}.
\end{aligned}
$$

We get

$$
\begin{aligned}
&\left(\beta-(N+1)^{1+\theta}\left(\frac{N^{n-p-3}}{(n-p-3)!}\right)^{\theta}\sum_{i=0}^{p}\|p_i\|\left(\frac{(N-1)^{p-i}}{(p-i)!}\right)^{\theta}\right.\\
&\left.-(N+1)^{\theta+1}\sum_{i=p+1}^{n-2}\|p_i\|\left(\frac{(N)^{n-i-2}}{(n-i-2)!}\right)^{\theta}-\|p_{n-1}\|\right)\sum_{k=0}^{N}|\Delta^{n-1}x(k)|^{\theta+1}\\
\leq\ & \|r\|(N+1)^{\frac{\theta}{\theta+1}}\left(\sum_{k=0}^{N}|\Delta^{n-1}x(k)|^{\theta+1}\right)^{\frac{1}{\theta+1}}.
\end{aligned}
$$

It follows that there is $M>0$ such that $\sum_{k=0}^{N}|\Delta^{n-1}x(k)|^{\theta+1}\leq M$. Hence $|\Delta^{n-1}x(k)|\leq(M/(N{+}1))^{1/(\theta+1)}\}$ for all $k\in\{0,\cdots,N\}$. Thus for all $k\in[0,N{+}n]$,

$$
\begin{aligned}
|x(k)| &= \frac{(N+n-1)^p}{p!}\frac{(N+n-p-3)^{n-p-3}}{(n-p-3)!}\sum_{k=0}^{N}|\Delta^{n-1}x(k)|\\
&\leq (N+1)\frac{(N+n-1)^p}{p!}\frac{(N+n-p-3)^{n-p-3}}{(n-p-3)!}(M/(N+1))^{1/(\theta+1)}\}.
\end{aligned}
$$

So Ω_1 is bounded. Let Ω be a non-empty open bounded subset of X such that $\Omega\supset\Omega_1$ centered at zero. It is easy to see that L is a Fredholm operator of index zero and N is $L-$compact on $\overline{\Omega}$. Thus, from Lemma 2.1, that $Lx=Nx$ has at least one solution $x\in D(L)\cap\overline{\Omega}$, So x is a solution of problem (2). The proof is complete.

Now, consider problem (3). Choose

$$
D(L)=\left\{x\in X:\ \Delta^i x(N+1)=0,\ i\in[0,\cdots,p],\ \ \Delta^j x(0)=0,\quad j\in[p+1,n-1]\right\}.
$$

Set

$$L : D(L) \cap X \to Y, \quad L \bullet x(k) = \Delta^n x(k), \quad k \in [0, N],$$

and $N : X \to Y$ by

$$N \bullet x(k) = f(k, x(k), \Delta x(k), \cdots, \Delta^{n-1}x(k), \Delta^{n-1}x(k+1)), \ \ k \in [0, N],$$

for all $x \in X$. It is easy to show that $x \in D(L)$ is a solution of $L \bullet x = N \bullet x$ implies that x is a solution of problem (3). It is easy to check the results.

(i) $\mathrm{Ker}L = \{x(k) \equiv 0, \ k \in [0, N+n-1]\}$.

(ii) L is a Fredholm operator of index zero.

(iii) Let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap D(L) \neq \emptyset$, then N is $L-$compact on $\overline{\Omega}$.

□

Theorem L2. *Suppose that there is numbers $\beta > 0$, $\theta > 1$, nonnegative sequences $p(k), p_i(k), r(k) (i = 0, \cdots, n-1, n)$, functions $g(k, x_0, \cdots, x_{n-1}, x_n)$, $h(k, x_0, \cdots, x_{n-1}, x_n)$ such that*

$$f(k, x_0, \cdots, x_{n-1}, x_n) = g(k, x_0, \cdots, x_{n-1}, x_n) + h(k, x_0, \cdots, x_{n-1}, x_n)$$

$$g(k, x_0, x_1, \cdots, x_{n-1}, x_n)x_n \leq -\beta |x_n|^{\theta+1},$$

and

$$|h(k, x_0, \cdots, x_{n-1}, x_n)| \leq \sum_{i=0}^{n} p_i(k)|x_i|^\theta + r(k),$$

for all $k \in \{0, \cdots, N\}$, $(x_0, x_1, \cdots, x_{n-1}, x_n) \in R^{n+1}$. Then problem (3) has at least one solution if

$$(N+1)^{1+\theta} \left[\left(\frac{(N+n-p-2)^{n-2}}{(n-2)!} \right)^\theta \sum_{j=0}^{p} \|p_j\| \left(\frac{(N+n-j)^{p-j-1}}{(p-j-1)!} \right)^\theta \right.$$
$$\left. + \sum_{i=p+1}^{n-2} \|p_i\| \left(\frac{(N+n-i-1)^{n-i-2}}{(n-i-2)!} \right)^\theta \right] + \|p_{n-1}\| + \|p_n\| < \beta.$$

Proof. Similar to the proof of Theorem L1, let $\Omega_1 = \{x : Lx = \lambda Nx, \ (x, \lambda) \in [(D(L) \times (0, 1)\}$. For $x \in \Omega_1$, we have $L \bullet x = \lambda N \bullet x$, $\lambda \in (0, 1)$, so

(2 2) $$\Delta^n x(k) = \lambda f(k, x(k), \Delta x(k), \cdots, \Delta^{n-1}x(k), \Delta^{n-1}x(k+1)).$$

So

$$\begin{aligned}&\sum_{k=0}^{N}[\Delta^n x(k)]\Delta^{n-1}x(k+1)\\=\;&\lambda\sum_{k=0}^{N}f(k,x(k),\Delta x(k),\cdots,\Delta^{n-1}x(k),\Delta^{n-1}x(k+1))\Delta^{n-1}x(k+1).\end{aligned}$$

Since

$$\begin{aligned}&2\sum_{k=0}^{N}[\Delta^n x(k)]\Delta^{n-1}x(k+1)\\=\;&[\Delta^{n-1}x(N+1)]^2+\sum_{k=0}^{N}[\Delta^{n-1}x(k+1)-\Delta^{n-1}x(k)]^2-[\Delta^{n-1}x(0)]^2\\=\;&[\Delta^{n-1}x(N+1)]^2+\sum_{k=0}^{N}[\Delta^{n-1}x(k+1)-\Delta^{n-1}x(k)]^2\geq 0,\end{aligned}$$

we get

$$\sum_{k=0}^{N}f(k,x(k),\Delta x(k),\cdots,\Delta^{n-1}x(k),\Delta^{n-1}x(k+1))\Delta^{n-1}x(k+1)\geq 0.$$

On the other hand, we have

$$\Delta^i x(k)=\sum_{s=0}^{k-1}\frac{(k-s-1)^{n-i-2}}{(n-i-2)!}\Delta^{n-1}x(s),\;\; k\in[0,N+n-i],\; i=p+1,\cdots,n-1,$$

and

$$\begin{aligned}\Delta^j x(k)\;&=\;(-1)^{p+1-j}\sum_{s=k-n+j}^{N}\frac{(s-k+n-j)^{p-j-1}}{(p-j-1)!}\Delta^{p+1}x(s),\\&\quad k\in[0,N+n-j],\; j=0,\cdots,p.\end{aligned}$$

The remainder of the proof is just similar to that of the proof of Theorem L1 and is omitted. □

3. Examples

In this section, we present examples, which can not be solved by known results, to illustrate the main results in section 2.

Example 3.1. Consider the following equation

(3.1)
$$\begin{cases} \Delta^n x(k) = \beta[\Delta^{n-1}x(k)]^{2m+1} + \sum_{i=0}^{n-1} p_i(k)[\Delta^i x(k)]^{2m+1} + r(k), \quad n \in [0, N], \\ \Delta^i x(0) = 0, \;\; i = 0, \cdots, p, \\ \Delta^j x(N+1) = 0, \;\; j = p+1, \cdots, n-1, \end{cases}$$

where m, N, $0 \le p \le n-2$ and $n \ge 2$ are a positive integer, $\beta > 0$, $p_i(n), r(n)$ are sequences. Corresponding to the assumptions of Theorem L1, we set $g(k, x_0, \cdots, x_{n-1}) = \beta[x_{n-1}]^{2m+1}$, $h(k, x_0, \cdots, x_{n-1}) = \sum_{i=0}^{n-1} p_i(k)x_i^{2m+1} + r(k)$ with $\theta = 2m+1$. It is easy to see that conditions of Theorem L1 hold. It follows from Theorem L1 that (6) has at least one solution if

$$(N+1)^{2m+2}\left[\left(\frac{(N+n-p-3)^{n-p-3}}{(n-p-3)!}\right)^{2m+1}\sum_{i=0}^{p}\|p_i\|\left(\frac{(N+n-i-1)^{p-i}}{(p-i)!}\right)^{2m+1}\right.$$
$$\left.+\sum_{j=p+1}^{n-2}\|p_j\|\left(\frac{(N+n-j-2)^{n-j-2}}{(n-j-2)!}\right)^{2m+1}\right] + \|p_{n-1}\| < \beta.$$

Example 3.2. Consider the following equation

(3.2)
$$\begin{cases} \Delta^n x(k) = -\beta[\Delta^{n-1}x(k+1)]^{2m+1} + \sum_{i=0}^{n-1} p_i(k)[\Delta^i x(k)]^{2m+1} + r(k), \quad n \in [0, N], \\ \Delta^i x(N+1) = 0, \;\; i = 0, \cdots, p, \\ \Delta^j x(0) = 0, \;\; j = p+1, \cdots, n-1, \end{cases}$$

where m, N, $0 \le p \le n-2$ and $n \ge 2$ are a positive integer, $\beta > 0$, $p_i(n), r(n)$ are sequences. Corresponding to the assumptions of Theorem L2, we set $g(k, x_0, \cdots, x_{n-1}, x_n) = -\beta[x_n]^{2m+1}$, $h(k, x_0, \cdots, x_{n-1}, x_n) = \sum_{i=0}^{n-1} p_i(k)x_i^{2m+1} + r(k)$ with $\theta = 2m+1$. It is easy to see that conditions of Theorem L2 hold. It follows from Theorem L2 that (7) has at least one solution if

$$(N+1)^{2m+2}\left[\left(\frac{(N+n-p-2)^{n-2}}{(n-2)!}\right)^{2m+1}\sum_{j=0}^{p}\|p_j\|\left(\frac{(N+n-j)^{p-j-1}}{(p-j-1)!}\right)^{2m+1}\right.$$
$$\left.+\sum_{i=p+1}^{n-2}\|p_i\|\left(\frac{(N+n-i-1)^{n-i-2}}{(n-i-2)!}\right)^{2m+1}\right] + \|p_{n-1}\| < \beta.$$

References

[1] R. P. Agarwal, *Focal Boundary Value Problems for Differential and Difference Equations*, Kluwer, Dordrecht, 1998.

[2] R. P. Agarwal, D. O'Regan and P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht, 1999.

[3] L. Kong, Q. Kong and B. Zhang, *Positive solutins of boundary value problems for third order functional difference equations*, Comput. Math. with Applications, **44**(2002), 481-489.

[4] R. P. Agarwal and J. Henderson, *Positive solutions and nonlinear eigenvalue problems for third order difference equations*, Comput. Math. with Applications, **36**(1998), 347-355.

[5] J. Mawhin, *Topological degree and boundary value problems for nonlinear differential equations, in: P. M. Fitzpertrick, M. Martelli, J. Mawhin, R. Nussbanm(Eds.), Topological Methods for Ordinary Differential Equations*, Lecture Notes in Math. Springer-Verlag, New York/Berlin, **1537**(1991).