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Factor Algebras of Signed Brauer's Algebras

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Abstract. In this paper we obtain a decomposition of certain factors of the signed Brauer algebra into a direct sum of simple algebras and we obtain the structure of the factor algebra.

Introduction

Brauer's algebras have a basis consisting of undirected graphs. Signed Brauer's algebra, denoted by $\vec{D}_f(x)$, introduced in [8], have a basis consisting of directed graphs. Young's theory of the decomposition of the group algebra into simple ideals is well known [13]. A corresponding decomposition of factor algebra $\vec{M}_r$ of signed Brauer's algebra $\vec{D}_f(n)$ is obtained as in [2]. We also realize in the semisimple case the algebra $\vec{M}_r$ contains an isomorphic copy of the group algebra of the symmetric group and a total matrix algebra and it is the direct product of them as in [2].

1. Preliminaries

In this section we recall the results needed for our purpose.

Definition 1.1([5]). Put $\mathbb{Z}_2^n = \{f | f : \{1, \cdots, n\} \to \mathbb{Z}_2\}$. Define

$$\mathbb{Z}_2 \wr S_n = \{(f, \pi) | f : \{1 \cdots n\} \to \mathbb{Z}_2, \pi \in S_n\}$$

where S_n is the symmetric group on n symbols. $\mathbb{Z}_2 \wr S_n$ is a group under the composition defined by

$$(f, \pi)(f\,', \pi\,') = (f f_\pi{}', \pi\pi\,'),$$

where $(ff\,')(i) = f(i) + f\,'(i), i \in \{1, \cdots n\}$ and $f_\pi = f \cdot \pi^{-1}$, for $\pi \in S_n$ and $f \in \mathbb{Z}_2^n$. This group is called the *wreath product* of $\mathbb{Z}_2$ by S_n. This hyperoctahedral group of type B_n is isomorphic to the Weyl group of type B_n.

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Definition 1.2([5], §2). Let α be a partition of n, denoted by $\alpha \vdash n$. Then the (i,j)-hook of α, denoted by $H^{\alpha}_{i,j}$ which is defined to be a Γ-shaped subset of diagram α which consists of the (i,j)-node called the *corner* of the hook and all the nodes to the right of it in the same row together with all the nodes lower down and in the same column as the corner.

The number h_{ij} of nodes of H^{α}_{ij} i.e.,

$$h_{ij} = \alpha_i - j + \alpha'_j - i + 1$$

where $\alpha_j{}'$ = number of nodes in the jth column of α, is called the *length* of $H^{\alpha}_{i,j}$, where $\alpha = [\alpha_1, \cdots \alpha_k]$. A hook of length q is called a *q-hook.* Then $H[\alpha] = (h_{ij})$ is called the *hook graph* of α.

Definition 1.3([5]). Let $\alpha \vdash n$. An (i,j)-node of α is said to be a rim node if there does not exist any $(i+1, j+1)$ node of α.

To H^{α}_{ij}, there corresponds a part of the rim of α which is of the same length. It consists of the nodes on the rim between the arm and leg of H^{α}_{ij} including the arm and the leg node where (i, α_i) node and $(\alpha_j{}', j)$-node are the *arm* and *leg* of the hook H^{α}_{ij} respectively. The associated part of the rim will be denoted by R^{α}_{ij} and will be called a *rim q-hook.*

Definition 1.4([5]). Diagrams α which do not contain any q-hook are called *q-cores.*

Definition 1.5([5]). For every $\alpha \vdash n$ there exists a uniquely determined q-core $\widetilde{\alpha}$ which is obtained from α by successive removal of rim q-hooks, called the q-core of α.

Notation 1.6. B^1_{2n+1} denotes the set of all partitions $\alpha \vdash 2n+1$ whose 2-core is [1].

Definition 1.7([12]). We shall call the (i,j) node of λ an $r-node$ if and only if $j - i \equiv r(\text{mod q})$. A hook of length $h_{ij} = mq$ whose head node (i, λ_i) in λ is of residue class r, is said to be a (q,r) hook. Such an (i,j) node is called a (q,r) node.

Theorem 1.8([12:4.46]). *If all the elements in the hook graph $H[\lambda]$, which are not divisible by q are deleted, then the remaining elements $h_{ij} = h^{(r)}_{ij}\ q$ $(r = 0, 1, \cdots, q-1)$ can be divided into disjoint sets whose (q,r) nodes constitute the right Young diagram, denoted by $\lambda^r_q (r = 0, 1, \cdots q-1)$ with hook graphs $(h^{(r)}_{ij})$. We shall call the skew diagram $\lambda_q = \lambda^0_q, \lambda^1_q, \cdots, \lambda^{q-1}_q$,* the q-quotient of λ.

Theorem 1.9([12:5.16]). *The partition $\lambda \vdash n$ is uniquely determined when its q-core λ and its q-quotient λ_q are given.*

Theorem 1.10([5], [7]). *A complete set of inequivalent irreducible representations* *of wreath product $\mathbb{Z}_2 \wr S_n$ is indexed by a pair of partitions $\lambda \vdash a, \mu \vdash b$ with $a+b=n$.*

Definition 1.11([8]). Signed Brauer's algebras Signed Brauer's algebra $\vec{D}_f(x)$ is defined over the field $k(x)$, where k is any arbitrary field and x an indeterminate.

A graph is said to be a signed diagram if every edge is labeled by a plus sign or a minus sign and edges of a signed diagram are called signed edges. An edge labeled by a plus (minus) sign will be called a *positive* (*negative*) edge. A positive vertical (horizontal) edge will be denoted by $\downarrow$ ($\rightarrow$) and a negative vertical (horizontal) edge will be denoted by $\uparrow$ ($\leftarrow$).

Let $\vec{V}_f$ be the set of all signed diagrams $\vec{b}$ with f signed edges and $2f$ vertices, arranged in two lines. The connected components of such graphs are single signed edges. The underlying diagram of any signed diagram is called Brauer diagram and any signed diagram whose edges are all positive is denoted by b.

Let $\vec{D}_f$ be the vector space spanned by $\vec{V}_f$ over $k(x)$. The multiplication in $\vec{D}_f$ is defined as follows: first, take the product of two undirected graphs a, b where $\vec{a}, \vec{b}$ are signed diagrams as in [14]; i.e., draw $\vec{b}$ below $\vec{a}$ and connect the ith upper vertex of $\vec{b}$ with the ith lower vertex of $\vec{a}$. Then $ab = x^d c$, where d is the number of loops in ab and c is a undirected graph. A new edge obtained in the product ab is labeled by a plus sign or a minus sign according as the number of negative edges obtained from $\vec{a}$ and $\vec{b}$ to form this edge is even or odd.

A loop β in ab is said to be positive (negative) if the number of negative edges obtained from $\vec{a}$ and $\vec{b}$ to form this loop is even (odd). A positive (negative) loop β in ab is replaced by the variable $x^2(x)$ in ab.

Now, $\vec{c}$ is the signed diagram where each edge is labeled as above and $ab = x^d c,\ d$ is the number of loops in c. Then,

$$\vec{a} \cdot \vec{b} = x^{2d_1+d_2}\vec{c},$$

where d_1 (d_2) is the number of positive (negative) loops in $\vec{ab}$.

Remark 1.12. A new edge in $\vec{ab}$ is positive or negative depending on whether the number of negative edges involved in $\vec{ab}$ to form this new edge is even or odd.
$e_i, g_i, \vec{h}_i$ denote the following graphs :

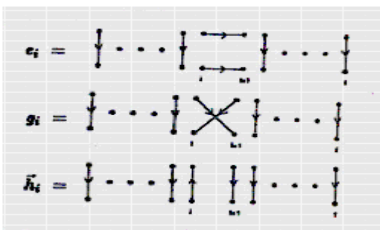

Fig 1.

As in the case of Brauer algebras, $\vec{D}_f(x)$ can be indentified with the sub-algebra of $\vec{D}_{f+1}(x)$ spanned linearly by all signed diagrams with a positive vertical edge on their right hand side. The linear span of signed diagrams of $\vec{V}_f$ having only positive edges forms an algebra isomorphic to the Brauer's centralizer algebra $D_f(x^2)$.

One can also define k-algebra $\vec{D}_f(n), n \neq 0$ similarly, its linear basis being the signed diagrams and in the multiplication $x^2(x)$ are replaced by $n^2(n)$ respectively.

Let $\vec{S}_f$ denote the set of signed diagrams having only vertical edges.

Proposition 1.13([5]). *$\vec{S}_f \cong \mathbb{Z}_2 \wr S_f$ where S_f is the symmetric group on f symbols and $\mathbb{Z}_2$ is the group consisting of two elements.*

Proposition 1.14([5]).

(i) *For each $\vec{b} \in \vec{D}_f$, there exists a unique $\vec{\epsilon}_{f-1}(\vec{b}) \in \vec{D}_{f-1}$ such that $e_f \vec{b} e_f = x^2 \vec{\epsilon}_{f-1}(\vec{b}) e_f$ and $\vec{\epsilon}_{f-1}(\vec{b}) = \vec{b}$ for all $\vec{b} \in \vec{D}_{f-1}$.*

(ii) *There exists a linear functional $\vec{\tau}$ on $\vec{D}_f$, defined inductively by $\vec{\tau}(1) = 1$ and $\vec{\tau}(\vec{b}) = \vec{\tau}(\vec{\epsilon}_{f-1}(\vec{b}))$ for $\vec{b} \in \vec{D}_f$.*

Proposition 1.15([5]). *The linear functional $\vec{\tau}$ defined on $\vec{D}_f$ is a nondegenerate trace form.*

Let $\vec{\Gamma}_f$ be the set of all partitions with $2(f-2k)+1$ nodes having $[1]$ as its 2-core, where k is an integer with $0 \leq k \leq \dfrac{f}{2}$. As in the case of Brauer's algebras the simple modules of $\vec{D}_f$ are also indexed by $\vec{\Gamma}_f$.

Theorem 1.16([5]). *The $k(x)$-algebra $\vec{D}_f$ is semisimple.*

$$\vec{D}_f = \bigoplus_{\lambda \in \vec{\Gamma}_f} \vec{D}_{f,\lambda},$$

where $\vec{D}_{f,\lambda}$ are full matrix algebras over $k(x)$. A simple $\vec{D}_{f,\lambda}$ module $\vec{V}_{f,\lambda}$ can be written as a direct sum of $\vec{D}_{f-1,\lambda}$ modules in the following way:

$$\vec{V}_{f,\lambda} = \bigoplus_{\mu} \vec{V}_{f-1,\mu},$$

where $\vec{V}_{f-1,\mu}$ is a simple $\vec{D}_{f-1,\mu}$ module and μ is obtained from λ either by removing or (if λ contains fewer than $2f+1$ nodes) adding two consecutive nodes of the form $\star\star$ or $\begin{smallmatrix}\star\\ \star\end{smallmatrix}$.

Theorem 1.17 ([9]). *There is a canonical embedding $i : \vec{D}_f(x) \to D_{2f}(x)$ where i is defined on the generators.*

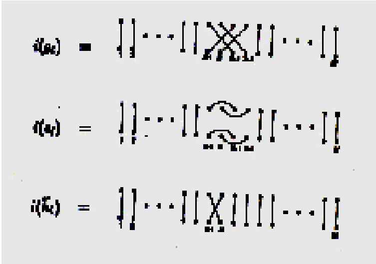

Fig 2.

1.18([9]).(Faithful representation of $\vec{D}_f(n)$) Let W be a vector space of dimension n over field k, $k = \mathbb{R}$ or $\mathbb{C}$, endowed with a nondegenerate symmetric bilinear form. Let $w_1, \cdots, w_n$ be an orthonormal basis of W. Put $V = W \otimes W$. Then $\{v_{ij} = w_i \otimes w_j\}_{1 \leq i,j \leq n}$ is an orthonormal basis for V.

By a result of Brauer [1], $D_{2f}(n)$ acts on $W^{\otimes 2f}$. Since we have an embedding $i : \vec{D}_f(n) \to D_{2f}(n)$ as in [9], it follows that $\vec{D}_f(n)$ acts on $W^{\otimes 2f}$. This action is given explicitly by

$$\begin{aligned}
(v_{i_1 j_1} \otimes \cdots \otimes v_{i_f j_f})\Phi(\sigma) &= v_{i_{\sigma(1)} j_{\sigma(1)}} \otimes \cdots \otimes v_{i_{\sigma(f)} j_{\sigma(f)}}, \quad \sigma \in S_f. \\
(v_{i_1 j_1} \otimes \cdots \otimes v_{i_f j_f})\Phi(e_l) &= \delta(i_l, i_{l+1})\delta(j_l, j_{l+1}) \\
&\quad \sum_{r,s} v_{i_1 j_1} \otimes \cdots \otimes v_{rs} \otimes v_{rs} \otimes \cdots \otimes v_{i_f j_f},
\end{aligned}$$

where v_{rs} are in the lth and $l+1$th position. It has already been proved that this defines a representation Φ of $D_f(n^2)$ into $End(V^{\otimes f})$ and its image is denoted by $B_f(n^2)$. This representation is faithful for $n^2 \geq f$. Let t denote the flip which sends $x \otimes y$ to $y \otimes x$, $\ x, y \in W$. Then

$$\Phi(\vec{h}_1) = t \otimes Id \otimes \cdots \otimes Id, \qquad \text{on } V^{\otimes f}.$$

Theorem 1.19([9]). $\Phi : \vec{D}_f(n) \to$ End $(V^{\otimes f})$ *is a faithful representation if* $n \geq 2f$.

Notation 1.20. Let $\vec{B}_f(n^2)$ denote the image of $\vec{D}_f(n)$ under Φ.

Corollary 1.21([9]). $\vec{D}_f(n)$ *is semisimple for* $n \geq 2f$, *when* k *is the field of real numbers.*

Definition 1.22([9]). Let $V = W \otimes W, \dim W = n$ as in 2.18. Define

$$O_t(n^2) = \left\{A \in O(n^2) | tAt = A\right\}.$$

Theorem 1.23([9]). $\mathrm{End}_{O_t(n^2)}(V^{\otimes f}) = \vec{B}_f(n^2)$.

Remark 1.24([14],[8])**.** Let $\Gamma_f = \{\lambda \vdash f-2k \mid 0 \le k \le \lfloor f/2 \rfloor\}$. A partition $\lambda \in \Gamma_f$ is connected by an edge to a partition $\mu \in \Gamma_{f+1}$ if μ can be obtained from λ by adding a node to λ or by removing a node from λ. Let

$$\vec{\Gamma}_f = \{\lambda \vdash 2(f-2k)+1,\ 0 \le k \le \lfloor f/2 \rfloor \mid 2-\text{core of } \lambda \text{ is } [1]\}.$$

Denote by $\vec{B}$, the Bratteli diagram obtained by taking on the fth level, the elements of $\vec{\Gamma}_f$ as vertices and every vertex $\lambda \in \vec{\Gamma}_f$ is connected to $\mu \in \vec{\Gamma}_{f+1}$ if μ is obtained from λ by adding or removing one rim 2-hook of the form $\begin{smallmatrix}\star\\ \star\end{smallmatrix}$ or $\star\star$. Let

$$\Gamma_f^{(2)} = \bigcup_{\substack{p+q=f\\ 0\le p,q\le f}} \Gamma_p \times \Gamma_q.$$

Denote by $B^{(2)}$, the Bratteli diagram obtained by taking on the fth level, the elements of $\Gamma_f^{(2)}$ as vertices and every vertex $(\lambda^+,\lambda^-) \in \Gamma_f^{(2)}$ is connected to a vertex $(\mu^+,\mu^-) \in \Gamma_{f+1}^{(2)}$ if either μ^+ (or μ^-) is obtained from λ^+ (or λ^-) by adding or removing one node. Let $V = W \otimes W$, where $\dim W = n$ as in 2.18. Then $V = V^+ \oplus V^-$, where V^+ and V^- are the real vector spaces of dimensions $\dfrac{n^2+n}{2}$ and $\dfrac{n^2-n}{2}$ respectively such that $O\left(\dfrac{n^2+n}{2}\right), O\left(\dfrac{n^2-n}{2}\right)$ act on V^+, V^- naturally. It is easy to observe that $O_t(n^2) \cong O\left(\dfrac{n^2+n}{2}\right) \times O\left(\dfrac{n^2-n}{2}\right)$. Put $Z_f = \mathrm{End}_G\,(V^{\otimes f})$ where $G = O\left(\dfrac{n^2+n}{2}\right) \times O\left(\dfrac{n^2-n}{2}\right)$ and let $\hat{Z}_f$ be an indexing set for the finite dimensional irreducible representations of $\mathcal{Z}_f$. As in [6], $\hat{Z}_f$ can be identified with the subset of $\hat{G}$, where $\hat{G}$ stands for the finite dimensional irreducible representations of G.

Theorem 1.25([10]). *The elements of $\hat{\mathcal{Z}}_f$ are indexed by $\Gamma_f^{(2)}$ for $n \ge 2f$.*

Remark 1.26([10]). Let $V_{\lambda^+} \otimes V_{\lambda^-}$ be an irreducible summand in the decomposition of $V^{\otimes f}$ where $\lambda^+ \in \Gamma_p, \lambda^- \in \Gamma_q$, as $O\left(\dfrac{n^2+n}{2}\right) \times O\left(\dfrac{n^2-n}{2}\right)$ - modules. Now tensoring with V gives the following :

$$
\begin{aligned}
V_{\lambda^+} \otimes V_{\lambda^-} \otimes V &= (V_{\lambda^+} \otimes V_{\lambda^-}) \otimes (V^+ \oplus V^-) \\
&= (V_{\lambda^+} \otimes V_{\lambda^-} \otimes V^+) \oplus (V_{\lambda^+} \otimes V_{\lambda^-} \otimes V^-) \\
&\cong (V_{\lambda^+} \otimes V^+) \otimes V_{\lambda^-} \oplus V_{\lambda^+} \otimes (V_{\lambda^-} \otimes V^-) \\
&\cong (\bigoplus_{\mu^+} V_{\mu^+}) \otimes V_{\lambda^-} \oplus V_{\lambda^+} \otimes (\bigoplus_{\mu^-} V_{\mu^-}) \\
&\cong \bigoplus_{\mu^+} (V_{\mu^+} \otimes V_{\lambda^-}) \bigoplus_{\mu^-} (V_{\lambda^+} \otimes V_{\mu^-}),
\end{aligned}
$$

where $\mu^+(\mu^-)$ is obtained from $\lambda^+(\lambda^-)$ by adding or removing one node. Hence the Branching rule for inclusion $\mathcal{Z}_f \subseteq \mathcal{Z}_{f+1}$ describes the decomposition as follows, as in [6] : Let $(\mu^+, \mu^-) \in \hat{\mathcal{Z}}_{f+1}$.

$$(\mu^+, \mu^-)|_{\hat{\mathcal{Z}}_f} = \bigoplus_{\lambda^+} (\lambda^+, \mu^-) \bigoplus_{\lambda^-} (\mu^+, \lambda^-),$$

where $\lambda^+(\lambda^-)$ is obtained from $\mu^+(\mu^-)$ by adding or removing one node. We now define a Bratteli diagram $\mathcal{Z}$ as follows : The elements of $\hat{Z}_f$, are the vertices on the fth level and every vertex (λ^+, λ^-) of $\hat{\mathcal{Z}}_f$ is connected to a vertex $(\mu^+, \mu^-) \in \hat{\mathcal{Z}}_{f+1}$ if either $\mu^+(\mu^-)$ is obtained from $\lambda^+(\lambda^-)$ by removing or adding one node for $n \geq 2f$.

1.27([10]). The roots of the polynomial $Q_f(x)$ Let $V = W \otimes W$, $\dim W = n$ with $n = 2m$ and m is odd. Let $\{E_{ij}\}_{1 \leq i,j \leq n}$ be a set of matrix units for End(W). Then one defines $G, E,\ H \in$ End$(V \otimes V)$ by

$$
\begin{aligned}
G &= \sum E_{ij} \otimes E_{rs} \otimes E_{ji} \otimes E_{sr}, \\
E &= \sum E_{ij} \otimes E_{rs} \otimes E_{ij} \otimes E_{rs}, \text{ and} \\
t &= \sum E_{ij} \otimes E_{ji} \otimes Id \otimes Id,
\end{aligned}
$$

where Id is the identity transformation. We embed these E, G, t into the linear maps on $V^{\otimes f}$ as G_i, E_i for $i = 1, 2, \cdots, f-1$ and H_i for $i = 1, 2, \cdots, f$ where E_i acts as E on the ith and $(i+1)$th factor of $V^{\otimes f}$ and as identity on the other ones. Put $E_t = \dfrac{1+t}{2}$. Denote VE_t by V^+ and $V(Id - E_t)$ by V^-. Then $\dim(V^+) = \dfrac{n^2+n}{2}$ and $\dim(V^-) = \dfrac{n^2-n}{2}$. It follows from Theorem 2.23 that the map $\Phi : \vec{D}_f(n) \to$ End$(V^{\otimes f})$ which maps g_i to G_i, e_i to E_i for $i = 1, 2, \cdots, f-1$ and h_i to H_i for $i = 1, 2, \cdots, f$ induces a homomorphism from $\vec{D}_f(n)$ onto $\text{End}_{O_t(n^2)}(V^{\otimes f})$, the centralizer of the fth tensor power of the standard representation of $O_t(n^2)$. It follows from [8] that $\dfrac{\vec{D}_f(n)}{\vec{I}_f(n)} \cong k\vec{S}_f$. For $n \geq 2f$, $\vec{D}_f(n)$ is semisimple by Corollary

1.21. The above quotient splits as a direct summand. As in [14], $z_f(n)$ will denote the central idempotent corresponding to the ideal $\vec{I}_f(n)$ and $p_{\lambda=(\lambda^+,\lambda^-)}$ will be any minimal idempotent in $k\vec{S}_f$ corresponding to $\lambda = (\lambda^+, \lambda^-) \in \Gamma_f^{(2)}$. Then $q(n) = (1-z_f(n))p_\lambda$ is a minimal idempotent of the quotient algebra $\dfrac{\vec{D}_f(n)}{\vec{I}_f(n)}$. Put $\rho^{\otimes f}(A) = A \otimes \cdots \otimes A$, f-times, for the $A \in O_t(n^2)$. Then $\rho^{\otimes f} : O_t(n^2) \to GL(V^{\otimes f})$ defines a representation. Therefore

$$A \to \Phi(q_\lambda(n))\rho^{\otimes f}(A)\Phi(q_\lambda(n))$$

is an irreducible representation of $O_t(n^2)$. Now $O_t(n^2) \cong O\left(\dfrac{n^2+n}{2}\right) \times O\left(\dfrac{n^2-n}{2}\right)$. Therefore it follows from Theorem 1.25 and Remark 1.26 that

$$(V^{\otimes f})\Phi(q_\lambda(n)) \cong V_{\lambda^+} \otimes V_{\lambda^-}$$

for $\lambda = (\lambda^+, \lambda^-) \in \Gamma_f^{(2)}$. Choose n such that $n = 2m$ and m is odd. Consider $V_{\lambda^+} \otimes V_{\lambda^-}$, where V_{λ^+} is a finite dimensional $O\left(\dfrac{n^2+n}{2}\right)$ - irreducible module occuring in the decomposition of tensor product representation of $O\left(\dfrac{n^2+n}{2}\right)$ and V_{λ^-} is a finite dimensional $O\left(\dfrac{n^2-n}{2}\right)$ - irreducible module occuring in the decomposition of tensor product representation of $O\left(\dfrac{n^2-n}{2}\right)$. By [4], there exists polynomials P_{λ^+} and P_{λ^-} derived from Weyl's dimension formulas such that the dimensions $\dim V_{\lambda^+}$ and $\dim V_{\lambda^-}$ are given by $\dim V_{\lambda^+} = P_{\lambda^+}\left(\dfrac{n^2+n}{2}\right)$ and $\dim V_{\lambda^-} = P_{\lambda^-}\left(\dfrac{n^2-n}{2}\right)$. Let $\lambda = (\lambda^+, \lambda^-) \in \Gamma_f^{(2)}$. Define $P_\lambda(n) = P_{\lambda^+}\left(\dfrac{n^2+n}{2}\right) P_{\lambda^-}\left(\dfrac{n^2-n}{2}\right)$. So that

$$\begin{aligned} \dim\,(V_{\lambda^+} \otimes V_{\lambda^-}) &= \dim\,(V_{\lambda_+})\dim(V_{\lambda^-}) \\ &= P_{\lambda^+}\left(\frac{n^2+n}{2}\right) P_{\lambda^-}\left(\frac{n^2-n}{2}\right) \\ &= P_\lambda(n). \end{aligned}$$

Let λ_i denote the length of the ith row and left λ'_j denote the length of the jth column. Define the hook length at a node $(i,j) \in \lambda$ to be

$$h(i,j) = \lambda_i - i + \lambda'_j - j + 1$$

and, for each node $(i,j) \in \lambda$, define

$$d(i,j) = \begin{cases} \lambda_i + \lambda_j - i - j + 1 & \text{if } i \le j \\ -\lambda'_i - \lambda'_j + i + j - 1 & \text{if } i > j \end{cases}. \tag{1.1}$$

Then it follows from [4] that the polynomials $P_{\lambda^+}\left(\frac{x^2+x}{2}\right)$ and $P_{\lambda^-}\left(\frac{x^2-x}{2}\right)$ can be written as

$$
\begin{aligned}
P_{\lambda^+}\left(\frac{x^2+x}{2}\right) &= \prod_{(i,j)\in\lambda^+} \frac{x^2+x-2+2d(i,j)}{2h(i,j)}, \\
P_{\lambda^-}\left(\frac{x^2-x}{2}\right) &= \prod_{(i,j)\in\lambda^-} \frac{x^2-x-2+2d(i,j)}{2h(i,j)}.
\end{aligned}
$$

We define $P_\lambda(x) = P_{\lambda^+}\left(\frac{x^2+x}{2}\right) P_{\lambda^-}\left(\frac{x^2-x}{2}\right)$. Then

$$
P_\lambda(x) = \left(\prod_{(i,j)\in\lambda^+} \frac{x^2+x-2+2d(i,j)}{2h(i,j)}\right)\left(\prod_{(i,j)\in\lambda^-} \frac{x^2-x-2+2d(i,j)}{2h(i,j)}\right).
$$

Let $\{b_i\}_{1\le i\le n(f)}$ be the linear basis consisting of signed diagrams of $\vec{D}_f$ and $\dim\vec{D}_f = n(f)$. Put $X = \left(\vec{\tau}(\vec{b}_i\vec{b}_j)\right)_{1\le i,j\le n(f)}$ and $Q_f(x) = \det(x^{2f-1}X)$. Then $Q_f(x)$ is a polynomial in x of degree $(2f-1)n(f)$. Now consider the matrix $X' = x^{2f}X$. Then $\det X' = x^{n(f)}Q_f(x)$, so that we have the following:

Theorem 1.28([10]).

$$
\begin{aligned}
x^{n(f)}Q_f(x) &= \prod_{\lambda=(\lambda^+,\lambda^-)\in\Gamma_f^{(2)}} P_\lambda(x) \\
&= \prod_{\substack{(i,j)\in\lambda^+ \\ (r,s)\in\lambda^-}} \left(\frac{x^2+x-2+2d(i,j)}{2h(i,j)}\right)\left(\frac{x^2-x-2+2d(r,s)}{2h(r,s)}\right),
\end{aligned}
$$

where the constants $d(i,j)$ and $h(i,j)$ are as given in (1.1).

Remark 1.29([10]). Let Λ_f be the set of all Young diagrams with f nodes. We say that a Young diagram μ is a subdiagram of the Young diagram λ, denoted by $\mu < \lambda$, if μ can be obtained from λ by taking away appropriate nodes. Let

$$
\Lambda_f^{(2)} = \bigcup_{\substack{p+q=f \\ 0\le p,q\le f}} (\Lambda_p \times \Lambda_q).
$$

Then (i) there is a one to one correspondence, denoted by ϕ, between the elements of B^1_{2f+1} and $\Lambda_f^{(2)}$. (ii) The elements of $\Gamma_f^{(2)}$ correspond bijectively to the elements of $\vec{\Gamma}_f$.

2. Subspaces of $\vec{D}_f(n)$

Denote by $\vec{W}_r$, the subspace of signed Brauer's algebra $\vec{D}_f(n)$ spanned by basis elements whose signed diagrams have exactly r-horizontal edges. $N(f,r)$ denotes the number of ways in which r -horizontal edges placed on a row of f vertices.

Theorem 2.1. *The algebra $\vec{D}_f(n)$ has the decomposition*

$$\vec{D}_f(n) = \bigoplus_{0 \leq r \leq \lfloor f/2 \rfloor} \vec{W}_r \ .$$

Furthermore $\dim(\vec{W}_r) = \left(\frac{f!}{r!}\right)^2 \frac{2^{f-2r}}{(f-2r)!}$.

Proof. Since $\vec{W}_r$ is the subspace of $\vec{D}_f(n)$ spanned by basis elements whose signed diagrams have exactly r - horizontal edges, $\vec{D}_f(n)$ is decomposed into $\lfloor f/2 \rfloor$ subspaces

$$\vec{D}_f(n) = \bigoplus_{0 \leq r \leq \lfloor f/2 \rfloor} \vec{W}_r \ .$$

The first horizontal edge in a signed diagram $\vec{d} \in \vec{W}_r$ may have one end placed in any one of f positions and the other end placed in any one of the remaining $(f-1)$ positions. Therefore the number of positions in which it may be placed is $f(f-1)$. The second horizontal edge in $\vec{d}$ have one end placed in any one of $(f-2)$ positions and the other end placed in any one of the remaining $(f-3)$ positions. Therefore the 2nd horizontal edge may be placed in $(f-2)(f-3)$ positions. Finally the r-th horizontal edge may be placed in $(f-(2r-2))(f-(2r-1))$ positions. But r-horizontal edges in $\vec{d}$ may be permuted in $r!$ ways. We have that

$$\begin{aligned} N(f,r) &= \frac{f(f-1)(f-2)...(f-(2r-2))(f-(2r-1))}{r!} \\ &= \frac{f!}{r!(f-2r)!}. \end{aligned}$$

Since there are $N(f,r)$ possible r-horizontal edges arrangements for each row and since the remaining $f-2r$ vertices in each row may be joined by vertical edges in $2^{f-2r}(f-2r)!$ ways, we see that

$$\begin{aligned} dim(\vec{W}_r) &= (N(f,r))^2 2^{f-2r}(f-2r)! \\ &= \left(\frac{f!}{r!}\right)^2 \frac{2^{f-2r}}{(f-2r)!} \ . \end{aligned}$$

□

Remark 2.2. The product of a basis element of $\vec{W}_r$ and any other basis element of the signed Brauer's algebra $\vec{D}_f(n)$ is an element of $\vec{W}_s, s \geq r$. It follows that the subspace

$$\vec{V}_r = \vec{W}_r \oplus \vec{W}_{r+1} \oplus \cdots \oplus \vec{W}_{\lfloor f/2 \rfloor}$$

is a two sided ideal of $\vec{D}_f(n)$. In particular $\vec{V}_{\lfloor f/2 \rfloor} = \vec{W}_{\lfloor f/2 \rfloor}$ is a two sided ideal.

By the decomposition in Theorem 1.1, we have chain of ideals

$$\vec{D}_f(n) = \vec{V}_0 \supset \vec{V}_1 \supset \cdots \supset \vec{V}_r \supset \cdots \supset \vec{V}_{\lfloor f/2 \rfloor} \tag{2.1}$$

2.3. A notation for basis elements of $\vec{W}_r$

In Theorem 2.1 we have seen that r-horizontal edges may be arranged on f vertices in $N(f, r)$ ways. We may therefore assign indices $1, 2, \cdots, N(f, r)$ one to each possible arrangement. Such a scheme of indices will be referred to as an r-horizontal edge scheme. The edge structure of signed diagram corresponding to a basis element of $\vec{W}_r$ is therefore specified by a pair of indices $\binom{i}{j}$ which indicate the edge structure of the upper row has the index i and that of lower row has index j. We may also say that corresponding basis elements has edge structure $\binom{i}{j}$.

Consider the construction of vertical lines. Suppose that in each row of an incomplete signed diagram there is a set of t unoccupied vertices. In each row we number these vertices $1, 2, \cdots, t$ from left to right. A signed diagram for a permutation $\vec{a}$ of $\vec{S}_t$ may then be constructed upon these vertices by joining vertex k of the lower set to vertex $\vec{a}(k)$ of the upper set, for $1 \leq k \leq t$.

We then say that the signed diagram possesses the permutation $\vec{a}$ of $\vec{S}_t$ on the two sets of t vertices and when there is no fear of confusion that the signed diagram or the corresponding basis elements possesses the permutation $\vec{a}$ of $\vec{S}_t$.

The element whose edge structure is $\binom{i}{j}$ and which possesses the permutation $\vec{a}$ of $\vec{S}_{(f-2r)}$ is an element of $\vec{W}_r$. A complete basis for $\vec{W}_r$ is obtained by allowing i and j to range through r-horizontal edge scheme and $\vec{a}$ to range $\vec{S}_{f-2r}$. Element possessing identity permutation will be called special elements.

We will denote by $\vec{w}_{i,j}$ the basis element of $\vec{W}_r$ whose edge structure is $\binom{i}{j}$ and possesses the identity permutation of $\vec{S}_{f-2r}$. The elements $\vec{w}_{i,j}$ of $\vec{W}_r$ do not provide a complete basis when i and j range through the r-horizontal edge scheme. We now develop a notation for the remaining elements.

If i is an index of the r-horizontal edge scheme and if $\vec{a} \in \vec{S}_{f-2r}$ then we define $\vec{a}_i$ to be the element of $\vec{W}_0$ whose signed diagram is constructed as follows:

(i) The identity permutation of $\vec{S}_{2r}$ is constructed upon those vertices which are the end points of horizontal edges in the signed diagram for $\vec{w}_{i,i}$.

(ii) The permutation of $\vec{a}$ of $\vec{S}_{f-2r}$ is constructed upon those vertices which are the end points of vertical edges in the signed diagram for $\vec{w}_{i,i}$.

It is now seen that $\vec{a}_i\vec{w}_{i,j}$ is the basis element of $\vec{W}_r$ whose edge structure is $\binom{i}{j}$ whose permutation is $\vec{a}$ of $\vec{S}_{f-2r}$. Since $\vec{w}_{i,j}$ possesses the identity permutation of $\vec{S}_{f-2r}$ and since the identity commutes with every other permutation, it follows that $\vec{a}_i\vec{w}_{i,j} = \vec{w}_{i,j}\vec{a}_j$. The set of elements $\vec{a}_i\vec{w}_{i,j}$ forms a complete basis for $\vec{W}_r$ when $\vec{a}$ ranges through $\vec{S}_{f-2r}$ and i,j take all values in $1, 2, \cdots, N(f,r)$.

The number of elements $\vec{a}_i\vec{w}_{i,j} \in \vec{W}_r$

$$\begin{aligned}
&= (N(f,r))^2 \times \text{ number of elements } \vec{a} \text{ in } \vec{S}_{f-2r} \\
&= (N(f,r))^2 \times 2^{f-2r}(f-2r)! \\
&= \left(\frac{f!}{r!}\right)^2 \frac{2^{f-2r}}{(f-2r)!} \\
&= \dim(\vec{W}_r).
\end{aligned}$$

Hence $\{\vec{a}_i\vec{w}_{i,j}\}_{1\le i,j\le N(f,r)}$, form a complete basis for $\vec{W}_r$ when $\vec{a}$ ranges through $\vec{S}_{f-2r}$.

Lemma. If $S = \{\vec{a}_i\vec{w}_{i,j} \mid 1 \le i,j \le N(f,r)\}$ is a basis for $\vec{W}_r$, then the cardinality of S is equal to the dimension of $\vec{W}_r$.

3. Multiplication of basis elements of $\vec{W}_r$

The elements $\vec{a}_i, \vec{b}_i$ and so forth correspond to signed diagrams, and therefore the products in which they occur are associative. This helps in obtaining a rule for the multiplication of basis elements.

Let $\vec{a}_i\vec{w}_{i,j}$ and $\vec{b}_k\vec{w}_{k,l}$ be in $\vec{W}_r$. Then

$$\begin{aligned}
(\vec{a}_i\vec{w}_{i,j})\left(\vec{b}_k\vec{w}_{k,l}\right) &= (\vec{a}_i\vec{w}_{i,j})\left(\vec{w}_{k,l}\vec{b}_l\right) \\
&= \vec{a}_i\left(\vec{w}_{i,j}\vec{w}_{k,l}\right)\vec{b}_l.
\end{aligned} \tag{3.1}$$

We consider now the product $\vec{w}_{i,j}\vec{w}_{k,l}$. Let $\vec{w}_{i,j}$ and $\vec{w}_{k,l}$ be elements of $\vec{W}_r$. We may assume that the product $\vec{w}_{i,j}\vec{w}_{k,l}$ is again an element $\vec{W}_r$. Then its horizontal edge structure is $\binom{i}{j}$ and which possesses the element $\vec{a} \in \vec{S}_{f-2r}$, which need not be the identity. A positive loop in $\vec{w}_{i,j}\vec{w}_{k,j}$ is replaced by n^2 and negative loop is replaced by n. We denote the power of n by $\mu(j,k)$ which occur depend only upon the indices (j,k) and we denote the element of $\vec{S}_{f-2r}$ by $\vec{a}(j,k)$. Their product $\mu(j,k)\vec{a}(j,k)$ is an element of the group algebra $k\vec{S}_{f-2r}$ of the group $\vec{S}_{f-2r}$ over k

and we denote it by $\vec{a}^*(j,k)$. We now write the product as

$$
\begin{aligned}
\vec{w}_{i,j}\vec{w}_{k,l} &= \vec{a}_i^*(j,k)\vec{w}_{i,l} \\
(\vec{a}_i\vec{w}_{i,j})(\vec{b}_k\vec{w}_{k,l}) &= \vec{a}_i(\vec{a}_i^*(j,k)\vec{w}_{i,l})\vec{b}_l \\
&= \vec{a}_i\vec{a}_i^*(j,k)\vec{b}_i\vec{w}_{i,l} \\
&= \left(\vec{a}\vec{a}_i^*(j,k)\vec{b}\right)_i \vec{w}_{i,l}.
\end{aligned}
\tag{3.2}
$$

For any $\vec{b} \in \vec{S}_{f-2r}$ and a basis element $\vec{a}_i w_{i,j}$ of $\vec{W}_r$, we define

$$\vec{b}(\vec{a}_i\vec{w}_{i,j}) = \vec{b}_i(\vec{a}_i\vec{w}_{i,j}).$$

If the definition is extended to the whole space $\vec{W}_r$ by linearity, we have a representation of $\vec{S}_{f-2r}$ on $\vec{W}_r$, i.e., for any $\vec{a}$ and $\vec{b}$ of $\vec{S}_{f-2r}$ and $\vec{w}$ of $\vec{W}_r$,$\vec{a}(\vec{b}\vec{w}) = (\vec{a}\vec{b})\vec{w}$.

(3.2) can be re written as

$$(\vec{a}\vec{w}_{i,j})(\vec{b}w_{k,l}) = \vec{a}\vec{a}^*(j,k)\vec{b}\vec{w}_{i,l}. \tag{3.3}$$

3.1. The operators $\vec{a}^*(j,k)$

For multiplication in $\vec{W}_r$ the operators $\vec{a}^*(j,k)$ have only been defined for products $\vec{w}_{i,j}\vec{w}_{k,l}$ are in $\vec{W}_r$. Define $\vec{a}^*(j,k) = 0$ for the products $\vec{w}_{i,j}\vec{w}_{k,l}$ are not in $\vec{W}_r$. In this case the product will be in $\vec{W}_s, s > r$.

Properties of $\vec{a}^*(j,k)$.

1. $\mu(j,k) = \mu(k,j)$.
2. $\vec{a}^*(k,j) = 0 \Rightarrow \vec{a}^*(j,k) = 0$.
3. $\vec{a}^*(j,j) = \mu(j,j)\varepsilon$ where ε is the identity elements of $\vec{S}_{f-2r}$.
4. $\mu(j,k) = n^s, 0 \leq s < r$ for $j \neq k$.

4. Factor algebras of $\vec{D}_f(n)$

We denote by $\vec{M}_r$, the factor algebra $\frac{\vec{V}_r}{\vec{V}_{r+1}}, 0 \leq r \leq \lfloor f/2 \rfloor$, where $\vec{V}_r$ are the ideals occuring in (3.1).

4.1. Multiplication in $\vec{M}_r$

Since $\vec{V}_r = \vec{W}_r \oplus \vec{V}_{r+1}$ by definition, a basis for $\vec{W}_r$, taken modulo $\vec{V}_{r+1}$ serve as a basis for the residue class algebra $\vec{M}_r$. We denote the residue class modulo $\vec{V}_{r+1}$

of an element $\vec{v}$ of $\vec{V}_r$ by $[\vec{v}]$. We now see the rule (4.3) may be used as a complete rule in $\vec{M}_r$: i.e., with definition given for $\vec{a}^*(j,k)$

$$[\vec{a}\vec{w}_{i,j}]\left[\vec{b}\vec{w}_{k,l}\right]=\left[\vec{a}\vec{a}^*(j,k)\vec{b}\vec{w}_{i,l}\right]. \tag{4.1}$$

We can extend the domain of the operators $\vec{a}\in\vec{S}_{f-2r}$ to the whole of $\vec{V}_r$ by defining them to operate as identity elements on $\vec{V}_{r+1}$. Then we use the usual definition of an operator on a residue class, namely $\vec{a}[\vec{w}]=[\vec{a}\vec{w}]$. Writing $[\vec{w}_{i,j}]=\vec{v}_{i,j}$, we obtain $[\vec{a}\vec{w}_{i,j}]=\vec{a}\vec{v}_{i,j}$. We now have the three rules for the use of these operators in $\vec{M}_r$:

1. $\vec{a}(\vec{b}\vec{v})=(\vec{a}\vec{b})\vec{v}$, where $\vec{a},\vec{b}\in\vec{S}_{f-2r}$ and $\vec{v}\in\vec{M}_r$; (4.2)
2. $(\vec{a}\vec{v}_{i,j})(\vec{b}\vec{v}_{k,l})=\vec{a}\vec{a}^*(j,k)\vec{b}\vec{v}_{i,l}$; (4.3)
3. $\vec{a}(\vec{v}_1\vec{v}_2)=(\vec{a}\vec{v}_1)\vec{v}_2,\vec{a}\in\vec{S}_{f-2r},\vec{v}_1,\vec{v}_2\in\vec{M}_r$. (4.4)

All these rules may be extended to elements of the group algebra. For $\vec{\alpha},\vec{\alpha}_1,\vec{\alpha}_2$, in $k\vec{S}_{f-2r}$ and $\vec{v},\vec{v}_1,\vec{v}_2$, in $\vec{M}_r$, they become

1. $\vec{\alpha}_1(\vec{\alpha}_2\vec{v})=(\vec{\alpha}_1\vec{\alpha}_2)\vec{v}$; (4.5)
2. $(\vec{\alpha}_1\vec{v}_{i,j})(\vec{\alpha}_2\vec{v}_{k,l})=\vec{\alpha}_1\vec{\alpha}^*(j,k)\vec{\alpha}_2\vec{v}_{i,l}$; (4.6)
3. $(\vec{\alpha}\vec{v}_1)\vec{v}_2=\vec{\alpha}(\vec{v}_1\vec{v}_2)$. (4.7)

4.2. The decomposition of $\vec{M}_r$

Young's theory of the decomposition of the group algebra of the symmetric group into simple ideals is well known [13]. A correspondence decomposition of $\vec{M}_r$ is obtained in this section.

Denote by $\vec{I}^\lambda$, the simple ideal of the group algebra $k\vec{S}_f$ of the symmetric group $\vec{S}_f$ over the field k of characteristic 0. There is a 1-1 correspondence between the partition λ of $2f+1$ and the simple ideals $\vec{I}^\lambda$ of $k\vec{S}_f$. $k\vec{S}_f$ is a direct sum of these simple ideals :

$$k\vec{S}_f=\bigoplus_{\lambda}\vec{I}^\lambda,$$

where $\lambda\in B^1_{2f+1}=\{\lambda\vdash 2f+1$ whose $2-$ core is $[1]\}$. The simple ideal $\vec{I}^\lambda$ is a total matrix algebra over k. Let $\left\{E^\lambda_{\alpha,\beta}\right\}_{1\le\alpha,\beta\le f_\lambda}$ forms a basis for $\vec{I}^\lambda,\lambda\in B^1_{2f+1}$. For each partitions λ of $2f+1$, the whole set $\left\{E^\lambda_{\alpha,\beta}\right\}_{1\le\alpha,\beta\le f_\lambda}$ forms a basis for the group algebra $k\vec{S}_f$. The basis elements have the multiplication properties

$$\begin{aligned}E^\lambda_{\alpha,\beta}E^\mu_{\gamma,\delta} &= 0 \text{ for all } \alpha,\beta,\gamma,\delta, \text{ if } \lambda\neq\mu,\lambda,\mu\in B^1_{2f+1}.\\ E^\lambda_{\alpha,\beta}E^\lambda_{\gamma,\delta} &= \delta_{\beta,\gamma}E^\lambda_{\alpha,\delta} \text{ where } \delta_{\beta,\gamma} \text{ is the Kronecker delta.}\end{aligned}$$

Each element $\vec{\xi} \in k\vec{S}_f$ can be written in the form

$$\vec{\xi} = \sum_{\lambda} \sum_{\alpha,\beta} \xi^{\lambda}_{\alpha,\beta} E^{\lambda}_{\alpha,\beta},$$

where $\lambda \in B^1_{2f+1} = \{\lambda \vdash 2f+1$ whose $2-$ core is $[1]\}$,$1 \leq \alpha, \beta \leq f_\lambda$ and $\xi^{\lambda}_{\alpha,\beta} \in k$. It follows that

$$E^{\lambda}_{\alpha,\beta} \vec{\xi} E^{\lambda}_{\gamma,\delta} = \xi^{\lambda}_{\beta,\gamma} E^{\lambda}_{\alpha,\delta}. \tag{4.8}$$

$\xi^{\lambda}_{\alpha,\beta}$ is called (α, β) coefficient of $\vec{\xi}$ in $\vec{I}^{\lambda}, \lambda \in B^1_{2f+1}$.

We now replace f by $f - 2r$. Denote by $\vec{M}^{\lambda}_r$, a linear subset of $\vec{M}_r$ spanned by $\{E^{\lambda}_{\alpha,\beta} \vec{v}_{i,j}\}_{1 \leq \alpha,\beta \leq f_\lambda}$, $1 \leq i, j \leq N(f, r)$ and $\lambda \in B^1_{2(f-2r)+1} = \{\lambda \vdash 2(f-2r)+1$ whose 2 core is [1] such that $0 \leq r \leq \lfloor f/2 \rfloor\}$.

We have the following :

Theorem 4.3.

(i) *$\vec{M}^{\lambda}_r$ is a two sided ideal of $\vec{M}_r$;*

(ii) *$\vec{M}_r$ is a direct sum of these ideals*

$$\vec{M}_r = \bigoplus_{\lambda} \vec{M}^{\lambda}_r,$$

where $\lambda \in B^1_{2(f-2r)+1} = \{\lambda \vdash 2(f-2r)+1$ *whose* 2*-core is* [1] *such that* $0 \leq r \leq \lfloor f/2 \rfloor\}$.

Proof. Let $\vec{a}, \vec{b} \in \vec{S}_{f-2r}$. From (4.3) we have that $(\vec{a}\vec{v}_{i,j}) \left(\vec{b}\vec{v}_{k,l}\right) = \vec{a}\vec{a}^*(j,k)\vec{b}\vec{v}_{i,l}$, where $\vec{v}_{i,j} = [\vec{w}_{i,j}]$ and $\vec{a}^\star(j,k) \in k\vec{S}_{f-2r}$. This rule may be extended linearly to elements of the group algebra $k\vec{S}_{f-2r}$. For $\vec{\alpha}_1, \vec{\alpha}_2$ in $k\vec{S}_{f-2r}$ we have that

$$(\vec{\alpha}_1 \vec{v}_{i,j}) (\vec{\alpha}_2 \vec{v}_{k,l}) = \vec{\alpha}_1 \vec{\alpha}^*(j,k) \vec{\alpha}_2 \vec{v}_{i,l}, \tag{4.9}$$

where $\vec{a}^*(j,k) \in k\vec{S}_{f-2r}$. First we shall prove that $\vec{M}^{\lambda}_r, \lambda \in B^1_{2(f-2r)+1} = \{\lambda \vdash 2(f-2r)+1$ whose 2 core is [1] such that $0 \leq r \leq \lfloor f/2 \rfloor\}$ is a two sided ideal of $\vec{M}_r$. Let $\vec{\alpha}_1 \vec{v}_{i,j} \in \vec{M}^{\lambda}_r$. Then $\vec{\alpha}_1 \vec{v}_{i,j}$ can be written in the form

$$\vec{\alpha}_1 \vec{v}_{i,j} = \sum_{\alpha,\beta} \xi^{\lambda}_{\alpha,\beta} E^{\lambda}_{\alpha,\beta} \vec{v}_{i,j},$$

where $\lambda \in B^1_{2(f-2r)+1}, 1 \leq \alpha, \beta \leq f_\lambda$ and $\xi^{\lambda}_{\alpha,\beta} \in k$. This implies that $\vec{\alpha}_1 = \sum_{\alpha,\beta} \xi^{\lambda}_{\alpha,\beta} E^{\lambda}_{\alpha,\beta} \in \vec{M}^{\lambda}_r$ and $E^{\lambda}_{\alpha,\beta} \in \vec{I}^{\lambda}, \lambda \in B^1_{2(f-2r)+1}$. If follows that $\vec{\alpha}_1 \in \vec{I}^{\lambda}$. Let $\vec{\alpha}_2 \vec{v}_{k,l} \in \vec{M}_r$ Therefore the expression on the right side of (5.9), $\vec{\alpha}_1 \alpha^*(j,k) \vec{\alpha}_2 \vec{v}_{i,l} \in \vec{M}^{\lambda}_r$, where $\vec{\alpha}^*(j,k) \in k\vec{S}_{f-2r}$. Hence $\vec{M}^{\lambda}_r$ is a right ideal of $\vec{M}_r$.

If $\vec{\alpha}_2\vec{v}_{k,l} \in \vec{M}_r^\lambda$ and $\vec{\alpha}_1\vec{v}_{i,j} \in \vec{M}_r$ then $\alpha_2 \in \vec{I}^\lambda, \lambda \in B^1_{2(f-2r)+1}$. Therefore $(\vec{\alpha}_1\vec{v}_{i,j})(\vec{\alpha}_2\vec{v}_{k,l}) = \vec{\alpha}_1\vec{\alpha}^*(j,k)\vec{\alpha}_2\vec{v}_{i,l} \in \vec{M}_r^\lambda$. Thus $\vec{M}_r^\lambda$ is a left ideal of $\vec{M}_r$ and hence $\vec{M}_r^\lambda$ is a two sided ideal of $\vec{M}_r$.

Since the elements $E^\lambda_{\alpha,\beta}\vec{v}_{i,j}, 1 \le \alpha, \beta \le f_\lambda, 1 \le i,j \le N(f,r)$ spans the two sided ideal $\vec{M}_r^\lambda, \lambda \in B^1_{2(f-2r)+1}$, the set of elements $E^\lambda_{\alpha,\beta}\vec{v}_{i,j}, 1 \le i,j \le N(f,r)$ forms a basis for $\vec{M}_r$, where $\lambda \in B^1_{2(f-2r)+1} = \{\lambda \vdash 2(f-2r)+1$ whose 2 core is $[1]$ such that $0 \le r \le \lfloor f/2 \rfloor\}$. Hence $\vec{M}_r$ has the decomposition

$$\vec{M}_r = \bigoplus_{\lambda \in B^1_{2(f-2r)+1}} \vec{M}_r^\lambda$$

. $\square$

Notation 4.4. We now take a particular ideal $\vec{M}_r^\lambda, \lambda \in B^1_{2(f-2r)+1}$, we drop the index λ, write $E^\lambda_{\alpha,\beta} = E_{\alpha,\beta}$, and so forth. We denote the (β,γ) coefficient of $\vec{a}^*(j,k)$ in $\vec{I}^\lambda$ by $a^*_{j_\beta,k_\gamma}$. Let $E_{\alpha,\beta}\vec{v}_{i,j} = T_{i_\alpha,j_\beta}$. These elements form a basis for $\vec{M}_r^\lambda$. With this new notation, the multiplication rule

$$(E_{\alpha,\beta}\vec{v}_{i,j})\,(E_{\gamma,\delta}\vec{v}_{k,l}) = E_{\alpha,\beta}\vec{a}^*(j,k)E_{\gamma,\delta}\vec{v}_{i,l} \tag{4.10}$$

becomes

$$\begin{aligned} T_{i_\alpha,j_\beta}T_{k_\gamma,l_\delta} &= E_{\alpha,\beta}\vec{a}^*(j,k)E_{\gamma,\beta}E_{\alpha,\delta}\vec{v}_{i,l} \\ &= a^*_{j_\beta,k_\gamma}E_{\alpha,\delta}\vec{v}_{i,l}\ ,\ \text{where } E_{\alpha,\beta}\vec{a}^*(j,k)E_{\gamma,\beta} = a^*_{j_\beta,k_\gamma} \\ &= a^*_{j_\beta,k_\gamma}T_{i_\alpha,l_\delta}. \end{aligned}$$

Remark 4.5. $a^*_{j_\beta,k_\gamma} = \mu(j,k)a_{j_\beta,k_\gamma}$, where $a_{j_\beta,k_\gamma} = E_{\alpha,\beta}\vec{a}(j,k)E_{\gamma,\beta}$.

$$\begin{aligned} \text{Indead,}\quad a^*_{j_\beta,k_\gamma} &= E_{\alpha,\beta}\vec{a}^*(j,k)E_{\gamma,\beta} \\ \Rightarrow a^*_{j_\beta,k_\gamma} &= E_{\alpha,\beta}\mu(j,k)\vec{a}(j,k)E_{\gamma,\beta} \text{ because } \vec{a}^*(j,k) = \mu(j,k)\vec{a}_{j,k} \\ &= \mu(j,k)E_{\alpha,\beta}\vec{a}(j,k)E_{\gamma,\beta} \\ &= \mu(j,k)a_{j_\beta,k_\gamma},\ \text{where } a_{j_\beta,k_\gamma} = E_{\alpha,\beta}\vec{a}(j,k)E_{\gamma,\beta}. \end{aligned}$$

Theorem 4.6. *For a given integer f, the algebra $\vec{M}_r$ is semisimple for all sufficiently large n.*

Proof. As in Theorem 4.3, we have $\vec{M}_r$ is the direct sum of ideals

$$\vec{M}_r = \bigoplus_\lambda \vec{M}_r^\lambda,$$

where $\lambda \in B^1_{2(f-2r)+1} = \{\lambda \vdash 2(f-2)+1 \text{ core is } [1] \text{ such that } 0 \le r \le \lfloor f/2 \rfloor\}$. So we have to prove that $\vec{M}^\lambda_r$ is simple. Also the group algebra $k\vec{S}_{f-2r}$ has the decomposition

$$k\vec{S}_{f-2r} = \bigoplus_\lambda \vec{I}^\lambda,$$

where $\lambda \in B^1_{2(f-2r)+1}$. Let $\{E_{\alpha,\beta}\}_{1\le\alpha,\beta\le f_\lambda}$, be a basis for the group algebra $k\vec{S}_{f-2r}$. Each element $\vec{\eta} \in k\vec{S}_{f-2r}$ can be written in the form

$$\vec{\eta} = \sum_\lambda \sum_{\alpha,\beta} \eta_{\alpha,\beta} E_{\alpha,\beta},$$

where $\eta_{\alpha,\beta} \in k, 1 \le \alpha, \beta \le f_\lambda$ and $\lambda \in B^1_{2(f-2r)+1}$. It follows that $E_{\alpha,\beta}\vec{\eta}E_{\gamma,\delta} = \eta_{\beta,\gamma}E_{\alpha,\delta}\eta_{\alpha,\beta}$ is called the (α,β) coefficient of $\vec{\eta}$ in $\vec{I}^\lambda, \lambda \in B^1_{2(f-2r)+1}$. We denote (α,β) coefficient of $\vec{a}^*(i,j)$ in $\vec{I}^\lambda$ by $a^*_{i_\alpha,j_\beta}$. The algebra $\vec{M}^\lambda_r$ are generalized matrix algebras. The multiplication matrix of $\vec{M}^\lambda_r, \lambda \in B^1_{2(f-2r)+1}$ is denoted by $\phi = \left(a^*_{j_\alpha,j_\beta}\right)$ where $a^*_{j_\alpha,j_\beta}$ is the (α,β) coefficient of $\vec{a}^*(i,j)$ in the ideal $\vec{I}^\lambda, \lambda \in B^1_{2(f-2r)+1}$, of $k\vec{S}_{f-2r}$. To prove $\vec{M}^\lambda_r$ is simple, it is surfices to prove that the matrix $\phi = \left(a^*_{j_\alpha,j_\beta}\right)$ is non singular, i.e., to prove that the determinant $\left|\mu(i,j)a_{i_\alpha,j_\beta}\right| \neq 0$. The diagonal submatrices of ϕ are obtained by fixing $i=j$ and letting α,β range through $1 \le \alpha,\beta \le f_\lambda$. Then $\vec{a}^*(i,i) = \mu(i,i)\varepsilon$ where ε denote the identity of $\vec{S}_{f-2r}$. If follows that $\vec{a}^*(i,i) = n^r\varepsilon$. Therefore

$$\begin{aligned} a^*_{i_\alpha,j_\beta} &= \mu(i,i)a_{i_\alpha,j_\beta} \\ &= \mu(i,i)\delta_{\alpha,\beta} \text{ (the Kronecker delta)} \\ &= n^r, \end{aligned}$$

so that each diagonal submatrix of ϕ is $n^r \cdot I$, where I is the unit matrix of order $f_\lambda \times f_\lambda$. The off -diagonal submatrices with $i \neq j$, have $a^*_{i_\alpha,j_\beta} = \vec{a}^*_{i_\alpha,j_\beta}\mu(i,j)a_{i_\alpha,j_\beta}$, where $\mu(i,j) = 0$ or $n^s, 0 \le s < r$. Hence the main diagonal term provides us with a power of n in the determinant is strictly higher than the power of any other term of the expansion. Hence $det(\phi)$ is a non zero polynomial in n with coefficients in k. The integral roots of the polynomial are the values of n for which $\vec{M}^\lambda_r$ is non-semisimple. For all other integers $\vec{M}^\lambda_r$ is simple. Since $det(\phi)$ has only a finite number of roots, it follows that $\vec{M}^\lambda_r$ is simple for sufficiently large n. □

Theorem 4.7. *Let $\vec{M}_r$ be semisimple. Then there is an embedding of $k\vec{S}_{f-2r}$ in the factor algebra $\vec{M}_r$.*

Proof. Let $\vec{a} \in \vec{S}_{f-2r}$. Then the set $\{\frac{1}{n^r}\vec{a}\vec{w}_{ij}\}$ of elements of $\vec{M}_r$ forms a subgroup of $\vec{M}_r$. We denote such elements by $\hat{a}$. We define $\hat{a} = \vec{a}\varepsilon_r$, where ε_r is the identity element in $\vec{M}_r$. The mapping $\vec{a} \xrightarrow{\Psi} \hat{a}$ is a homomorphism of $\vec{S}_{f-2r}$ onto a subgroup of $\vec{M}_r$. Indeed,

$$\begin{aligned}\Psi(\vec{a}\vec{b}) &= \hat{a}\hat{b} = (\vec{a}\vec{b})\varepsilon_r, \text{ for any } \vec{a}, \vec{b} \in \vec{S}_{f-2r}\\ &= \vec{a}(\vec{b}\varepsilon_r) \text{ by } (4.5)\\ &= \vec{a}(\varepsilon_r(\vec{b}\varepsilon_r))\\ &= \vec{a}(\varepsilon_r)\vec{b}(\varepsilon_r) \text{ by } (4.7)\\ &= \hat{a}\hat{b}\\ &= \Psi(\vec{a})\Psi(\vec{b}).\end{aligned}$$

The mapping Ψ is an isomorphism, since the further mapping $\hat{a} \longrightarrow \frac{1}{n^r}\vec{a}\vec{w}_{i,j}$ is onto a group isomorphic to $\vec{S}_{f-2r}$. The embedding is now extended to elements $\vec{\alpha}$ of the group algebra $k\vec{S}_{f-2r}$, by defining $\hat{\alpha} = \vec{\alpha}\varepsilon_r$, where ε_r is the identity of $\vec{M}_r$ and $\hat{\alpha} = \{\frac{1}{n^r}\vec{\alpha}\vec{w}_{i,j}\}$, a subalgebra of $\vec{M}_r$. Hence the elements $\{\frac{1}{n^r}\vec{\alpha}\vec{w}_{i,j}\}$ of $\vec{M}_r$ isomorphic to $k\vec{S}_{f-2r}$, which proves the theorem. $\square$

5. Structure of $\vec{M}_r$ in the semisimple case

Let $\vec{M}_r$ be semisimple. By Theorem 4.6, we have $\vec{M}_r$ has the decomposition

$$\vec{M}_r = \bigoplus \vec{M}_r^{\lambda}, \tag{5.1}$$

where $\lambda \in B^1_{2(f-2r)+1} = \{\lambda \vdash 2(f-2r)+1\}$ whose 2 core is [1] such that $0 \leq r \leq \lfloor f/2 \rfloor$.

The ideals $\vec{M}_r^{\lambda}$ are total matrix algebras over k. Since $\{E_{\alpha,\beta}\vec{w}_{i,j}\}_{1\leq\alpha,\beta\leq f_\lambda}, 1 \leq i,j \leq N(f,r)$, forms a basis $\vec{M}_r^{\lambda}, \vec{M}_r^{\lambda}$ has degree $f_\lambda N(f,r)$ over k. In the semisimple case $\hat{a}\vec{w}_{i,j} = \vec{a}w_{i,j}$. Therefore the products $\hat{a}\vec{w}_{i,j}$ as a basis for $\vec{M}_r$. Clearly any basis $\{\vec{\alpha}\}$ of $k\vec{S}_{f-2r}$ provides a basis $\{\hat{\alpha}\vec{w}_{i,j}\}$ for $\vec{M}_r$, where $1 \leq i,j \leq N(f,r)$. The ideal $\vec{M}_r^{\lambda}, \lambda \in B^1_{2(f-2r)+1}$, is spanned by the products $\hat{\alpha}\vec{w}_{i,j}$ where $\vec{\alpha} \in \vec{I}^{\lambda}, 1 \leq i,j \leq N(f,r)$. The set of elements $\hat{\alpha}$ with $\vec{\alpha} \in \vec{I}^{\lambda}, \lambda \in B^1_{2(f-2r)+1}$ form a subalgebra $\hat{I}^{\lambda}$ of $\vec{M}_r$ which is isomorphic to $\vec{I}^{\lambda}$. Hence $\vec{I}^{\lambda}, \lambda \in B^1_{2(f-2r)+1}$ is a total matrix algebra of degree f_λ that is a subalgebra of the total matrix algebra $\vec{M}_r^{\lambda}$ contains a total matrix subalgebra $\vec{N}_r^{\lambda}$ of degree $N(f,r)$ over k such that $\vec{M}_r^{\lambda}$ may be written as a product :

$$\vec{M}_r^{\lambda} = \vec{I}^{\lambda} \times \vec{N}_r^{\lambda}. \tag{5.2}$$

Then (5.1) becomes

$$\vec{M}_r = \bigoplus_{\lambda} \left(\vec{I}^{\lambda} \times \vec{N}_r^{\lambda}\right),$$

where $\lambda \in B^1_{2(f-2r)+1} = \{\lambda \vdash 2(f-2r)+1$ whose 2 core is [1] such that $0 \le r \le \lfloor f/2 \rfloor$.

While the $\vec{N}^\lambda_r$ are not unique, we may select one in each $\vec{M}^\lambda_r$. So that they are all isomorphic since each is a total matrix algebra of degree $N(f,r)$ over k. Let $\vec{N}_r$ be an algebra which is isomorphic to them. Since $k\vec{S}_{f-2r}$ has the decomposition

$$k\vec{S}_{f-2r} = \bigoplus_{\lambda} \vec{I}^\lambda,$$

where $\lambda \in B^1_{2(f-2r)+1} = \{\lambda \vdash 2(f-2r)+1$ whose 2 core is [1] such that $0 \le r \le \lfloor f/2 \rfloor\}$, a corresponding decomposition can be given for the direct product of $k\vec{S}_{f-2r}$ and $\vec{N}_r$. So we have that

$$\begin{aligned} k\vec{S}_{f-2r} \times \vec{N}_r &= \left(\bigoplus_{\lambda \in B^1_{2(f-2r)+1}} \vec{I}^\lambda \right) \times \vec{N}_r \\ &\cong \bigoplus_{\lambda \in B^1_{2(f-2r)+1}} \left(\vec{I}^\lambda \times \vec{N}_r \right). \end{aligned}$$

Let $\vec{\alpha} \in k\vec{S}_{f-2r}$ and $\vec{\alpha} = \sum \vec{\alpha}^\lambda$, where $\vec{\alpha}^\lambda \in \vec{I}^\lambda$, for each $\lambda, \lambda \in B^1_{2(f-2r)+1}$. Let $\vec{s}, \vec{s}^\lambda$ be the elements of $\vec{N}_r$ and $\vec{N}^\lambda_r$. Then the mapping given by

$$\vec{\alpha}\vec{s} = \left(\sum \vec{\alpha}^\lambda \right) \vec{s} \to \sum \vec{\alpha}^\lambda \vec{s} \to \sum \vec{\alpha}^\lambda \vec{s}^\lambda$$

is an isomorphism between $k\vec{S}_{f-2r} \times \vec{N}_r$ and $\vec{M}_r$. Under this mapping the set $k\vec{S}_{f-2r} \times I_r$, where I_r is the identity element of $\vec{N}_r$, corresponds to the subalgebra $k\vec{H}$ of $\vec{M}_r$. Let μ_r be the identity element of $k\vec{S}_{f-2r}$. We denote the set corresponding to $\mu_r \times \vec{N}_r$ by $\hat{N}_r$ and we have

$$\vec{M}_r \cong k\vec{H} \times \hat{N}_r.$$

We have the main theorem :

Theorem 5.1. *If $\vec{M}_r$ is semi simple then $\vec{M}_r$ contains an isomorphic copy of the group algebra $k\vec{H}$ and a total matrix algebra $\hat{N}_r$ of degree $N(f,r)$ over k and $\vec{M}_r \cong k\vec{H} \times \hat{N}_r$.*

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