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A Study of Modified H-transform and Fractional Integral Operator

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Abstract. In this paper, we establish a theorem wherein we have obtained the image of modified H-transform under the fractional integral operator involving Foxs H-function. Three corollaries of this theorem have also been derived. Further, we obtain one interesting integral by the application of the third corollary. The importance of above findings lies in the fact that our main theorem involves Fox H-function which is very general in nature. The result obtained earlier by Tariq (1998) is a special case of our main findings.

1. Introduction

The modified H-function transform was introduced by Saigo, Saxena and Ram [5] and is defined in the following manner:

$$\begin{aligned} \boldsymbol{h(s)} &= h_{P_1,Q_1}^{M_1,N_1}[F(x);\rho,s] \\ &= \int_d^\infty (sx)^{\rho-1} H_{P_1,Q_1}^{M_1,N_1}\left[(sx)^k \;\middle|\; \begin{matrix} (c_j,\gamma_j)_{1,P_1} \\ \\ (d_j,\delta_j)_{1,Q_1} \end{matrix} \right] F(x)dx \end{aligned} \tag{1.1}$$

For $k > 0$, where

$$F(x) = f(a\sqrt{x^2-d^2})U(x-d), \quad x > d > 0 \tag{1.2}$$

here $U(x-d)$ is the well-known Heaviside unit function.

Further we assume that $\boldsymbol{h(s)}$ exists and belongs to U, where U is the class of functions $f(x)$ on $R_+ = (0,\infty)$, which is infinitely differentiable with partial derivatives of any order such that

$$f(x) = \begin{bmatrix} 0 \ \ (|x|^{w_1}) & \text{as } x\to 0 \\ 0 \ \ (|x|^{-w_2}) & \text{as } x\to\infty \end{bmatrix} \tag{1.3}$$

The transform defined by (1.1) exists provided that following (sufficient) conditions are satisfied:

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(i) $|\arg s| < \frac{1}{2}\pi\Omega|k$

$$\text{where } \Omega = \sum_{j=1}^{N_1} \gamma_j - \sum_{j=N_1+1}^{P_1} \gamma_j + \sum_{j=1}^{M_1} d_j - \sum_{j=M_1+1}^{Q_1} d_j$$

(ii) $\mathrm{Re}(w_1) + 1 > 0$

(iii) $\mathrm{Re}(\rho - w_2) + k \max_{1 \le j \le N_1} \left[\mathrm{Re}\left(\frac{c_j - 1}{\gamma_j} \right) \right] < 0.$

The Fox's H-function or simply H-function was introduced by Charles Fox [1]. This function is defined and represented by means of the following Mellin-Barnes type of contour integral:

$$H_{P,Q}^{M,N}[z] = H_{P,Q}^{M,N}\left[z \,\middle|\, \begin{array}{c} (a_j, \alpha_j)_{1,P} \\ \\ (b_j, \beta_j)_{1,Q} \end{array} \right] \tag{1.4}$$

$$= \frac{1}{2\pi i} \int_L \theta(s) z^s ds \tag{1.5}$$

where $i = (-1)^{1/2}$, $z \neq 0$ and

$$\theta(s) = \frac{\prod_{j=1}^{M} \Gamma(b_j - \beta_j s) \prod_{j=1}^{N} \Gamma(1 - a_j + \alpha_j s)}{\prod_{j=M+1}^{Q} \Gamma(1 - b_j + \beta_j s) \prod_{j=N+1}^{P} \Gamma(a_j - \alpha_j s)} \tag{1.6}$$

The nature of the contour L in (1.5), the conditions of convergence of the integral (1.5), the asymptotic expansion of the H-function and some of its special cases can be referred to the work of Srivastava, Gupta and Goyal [7] and Mathai and Saxena [3].

The H-function of two variables occurring in the paper was first introduced by Mittal and Gupta [4], using the following notation, which is due essentially to Srivastava and Panda [8]:

$$\begin{aligned} H[x, y] &= H\left[\begin{array}{c} x \\ y \end{array} \right] \\ &= H_{p_1,q_1:p_2,q_2;p_3,q_3}^{0,n_1:m_2,n_2;m_3,n_3}\left[\begin{array}{c} x \\ y \end{array} \,\middle|\, \begin{array}{l} (a_j;\alpha_j, A_j)_{1,p_1} : (c_j, \gamma_j)_{1,p_2}; (e_j, E_j)_{1,p_3} \\ (b_j;\beta_j, B_j)_{1,q_1} : (d_j, \delta_j)_{1,q_2}; (f_j, F_j)_{1,q_3} \end{array} \right] \\ &= -\frac{1}{4\pi^2} \int_{L_1} \int_{L_2} \theta_1(\xi, \eta)\, \theta_2(\xi)\, \theta_3(\eta)\, x^{\xi}\, y^{\xi}\, d\xi\, d\eta \end{aligned} \tag{1.7}$$

where

$$(1.8) \qquad \theta_1(\xi,\eta) = \frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j\xi+A_j\eta)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j\xi-A_j\eta)\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j\xi+B_j\eta)}$$

$$(1.9) \qquad \theta_2(\xi) = \frac{\prod_{j=1}^{n_2}\Gamma(1-c_j+\gamma_j\xi)\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j\xi)}{\prod_{j=n_2+1}^{p_2}\Gamma(c_j-\gamma_j\xi)\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\delta_j\xi)}$$

$$(1.10) \qquad \theta_3(\xi) = \frac{\prod_{j=1}^{n_3}\Gamma(1-e_j+E_j\eta)\prod_{j=1}^{m_3}\Gamma(f_j-F_j\eta)}{\prod_{j=n_3+1}^{p_3}\Gamma(e_j-E_j\eta)\prod_{j=m_3+1}^{q_3}\Gamma(1-f_j+F_j\eta)}$$

where x and y are not equal to zero and an empty product is interpreted as unity. p_i, q_i, n_i and m_j are non-negative integers such that $p_i \geq n_i \geq 0$, $q_i \geq 0$, $q_j \geq m_j \geq 0$ $(i = 1,2,3,\cdots;\ j = 2,3)$. Also, all the A's, α's, B's, β's, γ's, δ's, E's and F's are assumed to be positive quantities for standardization purposes. The definition of the H-function of two variables given by (1.7) will, however, have a meaning even if some of these quantities are zero. The details about the nature of the contours L_1, L_2, conditions of convergence of the integral given by (1.7), the special cases of this function and its other properties can be referred to in the book cited above. Throughout the paper it is assumed that this function always satisfies its appropriate conditions of convergence [7, p.252-253, Eq.(C.4C.6)].

The fractional integral operator involving Foxs H-function was defined and studied by Saxena and Kumbhat [6] in the following form:

$$(1.11) \qquad R_{x,r}^{\eta,\alpha}[f(x)] = rx^{-\eta-r\alpha-1}\int_0^x t^{\eta}(x^r-t^r)^{\alpha} H_{P,Q}^{M,N}\left[K\left(\frac{t^r}{x^r}\right)^m\left(1-\frac{t^r}{x^r}\right)^n \,\middle|\, \begin{matrix}(a_j,\alpha_j)_{1,P}\\ \\ (b_j,\beta_j)_{1,Q}\end{matrix}\right] f(t)dt$$

where M, N, P, Q are positive integers such that $1 \le M \le Q$, $0 < N \le P$

$$|\arg(K)| < \frac{1}{2}\pi\Omega' \qquad (\Omega' > 0)$$
$$\Omega' = \sum_{j=1}^{M}\beta_j - \sum_{j=M+1}^{Q}\beta_j + \sum_{j=1}^{N}\alpha_j - \sum_{j=N+1}^{P}\alpha_j$$

Here r, m, n are positive integers. The (sufficient) conditions of validity of this operator are given below:

(i) $\mathrm{Re}(\eta) + rm \min\limits_{1\le j\le M} \mathrm{Re}\left(\frac{b_j}{\beta_j}\right) + 1 > 0$

(ii) $\mathrm{Re}(\alpha) + n \min\limits_{1\le j\le M} \mathrm{Re}\left(\frac{b_j}{\beta_j}\right) + 1 > 0.$

2. Main theorem

If

$$\boldsymbol{h(s)} = \int_d^\infty (sx)^{\rho-1} H_{P_1,Q_1}^{M_1,N_1}\left[(sx)^k \,\middle|\, \begin{matrix} (c_j,\gamma_j)_{1,P_1} \\ (d_j,\delta_j)_{1,Q_1} \end{matrix} \right] F(x)\,dx \tag{2.1}$$

and

$$R_{x,r}^{\eta,\alpha}[f(x)] = rx^{-\eta-r\alpha-1}\int_0^x t^\eta (x^r - t^r)^\alpha H_{P,Q}^{M,N} \left[K\left(\frac{t^r}{x^r}\right)^m \left(1-\frac{t^r}{x^r}\right)^n \,\middle|\, \begin{matrix} (a_j,\alpha_j)_{1,P} \\ (b_j,\beta_j)_{1,Q} \end{matrix} \right] f(t)dt \tag{2.2}$$

then

$$R_{s,r}^{\eta,\alpha}[\boldsymbol{h(s)}] \tag{2.3}$$
$$= \int_0^\infty (sx)^{\rho-1} H_{1,1:P+1,Q;P_1,Q_1}^{0,1:M,N+1;M_1,N_1}\left[\begin{matrix} K \\ (sx)^k \end{matrix} \,\middle|\, \begin{matrix} \left(1-\frac{\eta}{r}-\frac{\rho}{r}; m, \frac{k}{r}\right) \\ \left(-\alpha-\frac{\eta}{r}-\frac{\rho}{r}; m+n, \frac{k}{r}\right) \end{matrix} \right.$$
$$\left. \begin{matrix} (-\alpha,n)(a_j,\alpha_j)_{1,P} & ;(c_j,\gamma_j)_{1,P_1} \\ (b_j,\beta_j)_{1,Q} & ;(d_j,\delta_j)_{1,Q_1} \end{matrix} \right] F(x)dx$$

where $F(x) = f(a\sqrt{x^2-d^2}U(x-d)$, $x > d > 0$ *as defined in* (1.2) *provided that*

(i) $\Omega > 0,\ |\arg s| < \frac{1}{2}\pi\Omega/k,$

(ii) $\Omega' > 0,\ |\arg K| < \frac{1}{2}\Omega'\pi,$

(iii) r, m *and* n *are positive integers,*

(iv) $\mathrm{Re}(w_1)+1>0,$ *and* $\mathrm{Re}(\rho-w_2)+k\max_{1\le j\le N_1}\mathrm{Re}\left(\frac{c_j-1}{\gamma_j}\right)<0$

(v) $\mathrm{Re}(\alpha)+n\min_{1\le j\le M}\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)+1>0$ *and* $\mathrm{Re}(\eta)+rm\min_{1\le j\le M}\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)+1>0$

(vi) $\mathrm{Re}\left(\frac{\eta+\rho}{r}\right)+m\min_{1\le j\le M}\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)+\frac{k}{r}\min_{1\le j\le M_1}\mathrm{Re}\left(\frac{d_j}{\delta_j}\right)>0.$

Proof. In order to prove the main theorem, substituting the value of $\boldsymbol{h(s)}$ from (2.1) in the left hand side of (2.2), we find that

$$(2.4)\quad R_{s,r}^{\eta,\alpha}[\boldsymbol{h(s)}]$$

$$= rs^{-\eta-r\alpha-1}\int_{t=0}^{s} t^{\eta}(s^r-t^r)^{\alpha}H_{P,Q}^{M,N}\left[K\left(\frac{t^r}{s^r}\right)^m\left(1-\frac{t^r}{s^r}\right)^n \left| \begin{matrix}(a_j,\alpha_j)_{1,P}\\ \\ (b_j,\beta_j)_{1,Q}\end{matrix}\right.\right]$$

$$\times\left\{\int_{x=d}^{\infty}(tx)^{\rho-1}H_{P_1,Q_1}^{M_1,N_1}\left[(tx)^k\left|\begin{matrix}(c_j,\gamma_j)_{1,P_1}\\ \\ (d_j,\delta_j)_{1,Q_1}\end{matrix}\right.\right]F(x)\,dx\right\}dt$$

Now interchanging the orders of x and t-integrals which is permissible under given conditions, we get

$$(2.5)\quad R_{s,r}^{\eta,\alpha}[\boldsymbol{h(s)}] = rs^{-\eta-r\alpha-1}\int_{x=d}^{\infty}x^{\rho-1}F(x)\left\{\int_{t=0}^{s}t^{\eta+\rho-1}(s^r-t^r)^{\alpha}\right.$$

$$\times H_{P,Q}^{M,N}\left[K\left(\frac{t^r}{s^r}\right)^m\left(1-\frac{t^r}{s^r}\right)^n\left|\begin{matrix}(a_j,\alpha_j)_{1,P}\\ \\ (b_j,\beta_j)_{1,Q}\end{matrix}\right.\right]$$

$$\left.H_{P_1,Q_1}^{M_1,N_1}\left[(tx)^k\left|\begin{matrix}(c_j,\gamma_j)_{1,P_1}\\ \\ (d_j,\delta_j)_{1,Q_1}\end{matrix}\right.\right]dt\right\}dx$$

To evaluate the t-integral, we express both the H-functions in terms of Mellin-Barnes contour integrals with the help of (1.5) and change the order of contour integrations and t-integral. After evaluating the t-integral and reinterpreting the result thus obtained in terms of H-function of two variables, we easily arrive at the right hand side of (2.3) after a little simplification. $\square$

3. Special Cases

(I) If we reduce the Fox's H-function involved in (2.2) of our main theorem to ${}_2F_1$ with the help of a known result [7, p.19; Eq.(2.6.8)], we arrive at the following corollary after little simplification

Corollary 1. *If*

$$\boldsymbol{h(s)} = \int_d^\infty (sx)^{\rho-1} H_{P_1,Q_1}^{M_1,N_1}\left[\ (sx)^k \ \middle|\begin{matrix}(c_j,\gamma_j)_{1,P_1}\\ \\ (d_j,\delta_j)_{1,Q_1}\end{matrix}\right] F(x)\,dx \tag{3.1}$$

and

$$\begin{aligned} &R_{x,r}^{\eta;\alpha}[f(x)] \\ &= rx^{-\eta-r\alpha-1}\int_0^x t^\eta (x^r-t^r)^\alpha \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)}\,{}_2F_1\left(a,b;\,c;\,1-\frac{t^r}{s^r}\right) f(t)dt \end{aligned} \tag{3.2}$$

then

$$\begin{aligned} &R_{s,r}^{\eta,\alpha}[\boldsymbol{h(s)}] \\ &= \int_d^\infty (sx)^{\rho-1} H_{0,1:3,2;P_1+1,Q_1}^{0,0:1,3;M_1,N_1+1}\left[\begin{matrix}-1\\ (sx)^k\end{matrix}\ \middle|\begin{matrix}\overline{\qquad} & :\\ \left(-\alpha-\frac{\eta}{r}-\frac{\rho}{r};1;\frac{k}{r}\right) & :\end{matrix}\right. \\ &\qquad \left.\begin{matrix}(-\alpha,1)(1-a,1),(1-b,1) & ;\left(1-\frac{\eta}{r}-\frac{\rho}{r},-\frac{k}{r}\right),(c_j,\gamma_j)_{1,P_1}\\ (0,1),(1-c,1) & ;(d_j,\delta_j)_{1,Q_1}\end{matrix}\right] F(x)dx. \end{aligned} \tag{3.3}$$

The conditions of validity of the aforementioned corollary can be easily derived from our main theorem.

(II) Again reducing Fox's H-function involved in (2.1) to Whittaker function by using a known result [3, p.155], a little simplification will give the following

Corollary 2. *If*

$$\boldsymbol{h(s)} = \int_d^\infty (sx)^{\rho-1} e^{-\frac{1}{2}sx}\, W_{a,b}(sx)\, F(x)dx$$

and

$$\begin{aligned} R_{x,r}^{\eta;\alpha}[f(x)] &= rx^{-\eta-r\alpha-1}\int_0^x t^\eta (x^r-t^r)^\alpha H_{P,Q}^{M,N} \\ &\left[\ K\left(\frac{t^r}{s^r}\right)^m\left(1-\frac{t^r}{s^r}\right)^n\ \middle|\begin{matrix}(a_j,\alpha_j)_{1,P}\\ \\ (b_j,\beta_j)_{1,Q}\end{matrix}\right] f(t)dt \end{aligned}$$

then

$$
\begin{aligned}
& R_{s,r}^{\eta,\alpha}[\boldsymbol{h(s)}] \\
= & \int_d^\infty (sx)^{\rho-1} H_{1,1:P+1,Q;1,2}^{0,1:M,N+1;2,0}\left[\begin{array}{c|c} K & \left(1-\frac{\eta}{r}-\frac{\rho}{r};m,\frac{1}{r}\right) : \\ (sx) & \left(-\alpha-\frac{\eta}{r}-\frac{\rho}{r};m+n,\frac{1}{r}\right) : \end{array}\right. \\
& \left.\begin{array}{ll} (-\alpha,n),(a_j,\alpha_j)_{1,P} & ;(1-a,1) \\ (b_j,\beta_j)_{1,Q} & ;\left(\frac{1}{2}+b,1\right),\left(\frac{1}{2}-b,1\right) \end{array}\right] F(x)dx
\end{aligned}
$$

provided that

(*i*) $\mathrm{Re}(s)>0,\ \mathrm{Re}(w_1)+1>0,\ \Omega'>0,\ |\arg K|<\frac{1}{2}\Omega'\pi,$

(*ii*) r, m *and* n *are positive integers,* $\Omega'>0,\ |\arg K|<\frac{1}{2}\Omega'\pi,$

(*iii*) $\mathrm{Re}(\alpha)+n\min\limits_{1\le j\le M}\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)+1>0$ *and* $\mathrm{Re}\left(\frac{\eta+\rho}{r}\right)+m\min\limits_{1\le j\le M}\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)$ $+\frac{1}{r}\min\mathrm{Re}\left(\frac{1}{2}+b,\frac{1}{2}-b\right)>0.$

(III) Now if we reduce Fox's H-function involved in (2.1) to Exponential function and Fox's H-function involved in (2.2) to ${}_1F_0$ by using known results [7, p.18, (2.6.2) and (2.6.4)] then we get the following

Corollary 3. *If*

$$\boldsymbol{h(s)}=\int_d^\infty (sx)^{\rho-1}e^{-sx}F(x)dx$$

and

$$R_{x,1}^{\eta,\alpha}[f(x)]=x^{-\eta-r\alpha-1}\int_0^x t^\eta(x-t)^\alpha\left[1+\frac{Kt}{x}\right]^{-c}f(t)dt$$

then

$$
\begin{aligned}
(3.4)\qquad & R_{s,1}^{\eta,\alpha}[\boldsymbol{h(s)}] \\
= & B(1+\alpha,\eta+\rho) \\
& \int_d^\infty (sx)^{\rho-1}\phi_1(\eta+\rho,c;1+\alpha+\eta+\rho;-K-sx)F(x)dx
\end{aligned}
$$

where ϕ_1 *is known as Confluent form of Appell's function* $[7,p.89;(6.4.8)]$ *where* $|K|<1,\ \mathrm{Re}(s)>0,\ \mathrm{Re}(w_1)+1>0,\ \mathrm{Re}(\eta+\rho)>0$ *and* $\mathrm{Re}(\alpha+1)>0.$

4. Application

If we take $F(x)$ to be unity in the corollary 3, then with the help of [2, p.317, (3); p.286, (3)] and [7, p.244, (A.26)], we get the following interesting integral

$$
\begin{aligned}
&\int_d^{\infty} \phi_1(\eta+\rho, c; 1+\alpha+\eta+\rho; -K-sx)dx \\
= &\ \frac{d^{\frac{\rho-1}{2}}\, s^{-\left(\frac{\rho+1}{2}\right)}\Gamma(1+\alpha+\eta+\rho)}{\Gamma(c)\,\Gamma(\eta+\rho)} \\
= &\ H^{0,1:2,0;1,1}_{1,1:1,2;1,1}\left[\begin{array}{c|lll} sd & \left(\frac{3}{2}-\eta-\frac{\rho}{2};1,1\right) & : \left(\frac{3}{2}-\frac{\rho}{2},1\right) & ;(1-c,1) \\ K & \left(\frac{1}{2}-\eta-\alpha-\frac{\rho}{2};1,1\right) & : \left(\frac{\rho}{2}+\frac{1}{2},1\right),\left(-\frac{\rho}{2}+\frac{1}{2},1\right) & ;(0,1) \end{array}\right]
\end{aligned}
$$

provided

(i) $\mathrm{Re}\left(\eta+\frac{\rho}{2}\right)+\min\left(\frac{1}{2}\pm\frac{\rho}{2}\right)>0,$

(ii) $\mathrm{Re}(\alpha+1)>0$ and $|K|<1$, $d>0$, $\mathrm{Re}(s)>0$.

In the last, if we reduce Fox's H-function to ${}_2F_1$ in (2.2), then we get a known result given by Tariq [9] after a little simplification.

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