KYUNGPOOK Math. J. 47(2007), 501-517

Multiple Weakly Summing Multilinear Mappings and Polynomials

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ABSTRACT. In this paper, we introduce and study a new class containing absolutely summing multilinear mappings and polynomials, which we call *multiple weakly summing multilinear mappings* and *polynomials*. We investigate some interesting properties about multiple weakly $(p; q_1, \cdots, q_k)$-summing multilinear mappings and polynomials defined on Banach spaces: In particular, we prove a kind of Dvoretzky-Rogers' Theorem and an ideal property for multiple weakly $(p; q_1, \cdots, q_k)$-summing multilinear mappings and polynomials. We also prove that the Aron-Berner extensions of multiple weakly $(p; q_1, \cdots, q_k)$-summing multilinear mappings and polynomials are also multiple weakly $(p; q_1, \cdots, q_k)$-summing.

1. Introduction

Throughout this paper $\mathbb{K}$ denotes either the complex field $\mathbb{C}$ or the real field $\mathbb{R}$. If the field is not specified the results are valid in both cases. Let $E, E_1, \cdots, E_k$, and F be Banach spaces over the field $\mathbb{K}$. We write B_E for the unit ball of E. The dual space of E is denoted by E^*. Let $k \in \mathbb{N}$. We denote by $\mathcal{L}(E_1, \cdots, E_k : F)$ the Banach space of continuous k-linear mappings of $E_1 \times \cdots \times E_k$ endowed with the usual norm

$$\|A\| = \sup \ \{\|A(x_1, \ldots, x_k)\| \ : x_j \in B_{E_j}, j = 1, \ldots, k\}.$$

We denote $\mathcal{L}(E, \cdots, E : F)$ by $\mathcal{L}(^kE : F)$. Let A be in $\mathcal{L}(^kE : F)$. We define $\widehat{A} : E \to F$ by $\widehat{A}(x) = A(x, \cdots, x)$ for $x \in E$. A mapping $P : E \to F$ is said to be a continuous k-homogeneous polynomial if $P = \widehat{A}$ for some $A \in \mathcal{L}(^kE : F)$.

We denote by $\mathcal{P}(^kE : F)$ the Banach space of continuous k-homogeneous polynomials of E into F endowed with the polynomial norm $\|P\| = \sup_{x \in B_E} \|P(x)\|$. We denote $\mathcal{P}(^kE : \mathbb{K})$ by $\mathcal{P}(^kE)$. We refer to [7] for a general background on the theory of polynomials on an infinite dimensional Banach space.

Throughout this paper we will assume $1 \leq q_1, \cdots, q_k \leq p < \infty$. We denote the

Received April 20, 2007, and, in revised form, September 18, 2007.
2000 Mathematics Subject Classification: 46A22, 46G20, 46G25.
Key words and phrases: Multilinear mappings, polynomials.

Banach space

$$l_p^{strong}(F) := \{ \ (y_i)_{i=1}^{\infty} \ : \ (\sum_{i=1}^{\infty} \|y_i\|^p)^{1/p} < \infty \ \}$$

with the norm $\| \ (y_i)_{i=1}^{\infty} \ \|_p^{strong} := (\sum_{i=1}^{\infty} \|y_i\|^p)^{1/p}$. We denote the Banach space

$$l_p^{weak}(F) := \{ \ (y_i)_{i=1}^{\infty} \ : \ \sup_{y' \in B_{F^*}} (\sum_{i=1}^{\infty} |y'(y_i)|^p)^{1/p} < \infty \ \}$$

with the norm

$$\| \ (y_i)_{i=1}^{\infty} \ \|_p^{weak} := \sup_{y' \in B_{F^*}} (\sum_{i=1}^{\infty} |y'(y_i)|^p)^{1/p}.$$

Motivated by the growth of the theory of multilinear mappings and polynomials on Banach spaces, many authors ([2], [3], [5], [8]) began the study of the p-summing multilinear mappings and polynomials between Banach spaces. A k-linear mapping $T : E_1 \times \cdots \times E_k \to F$ is $(p; q_1, \cdots, q_k)$-summing if there exists a constant $K_1 > 0$ such that, for every choice of elements $x_i^j \in E_j$ $(j = 1, \cdots, k, 1 \le i \in \mathbb{N})$, the following relation holds:

$$\begin{aligned} (1.1) \qquad & \| \ (\ T(x_i^1, \cdots, x_i^k) \ )_{i=1}^{\infty} \ \|_p^{strong} := [\ \sum_{i=1}^{\infty} \|T(x_i^1, \cdots, x_i^k)\|^p \ \ ]^{1/p} \\ \le & \quad K_1 \cdot \| \ (x_i^1)_{i=1}^{\infty} \ \|_{q_1}^{weak} \cdots \| \ (x_i^k)_{i=1}^{\infty} \ \|_{q_k}^{weak}. \end{aligned}$$

In this case, we define the $(p; q_1, \cdots, q_k)$-summing norm of T by

$$\pi_{(p;q_1,\cdots,q_k)}(T) := \inf\{ \ K_1 > 0 : \ (1.1) \text{ holds } \}.$$

In connection with the classical definition of $(p; q_1, \cdots, q_k)$-summing, we introduce a new definition of weakly $(p; q_1, \cdots, q_k)$-summing as follows: a k-linear mapping $T : E_1 \times \cdots \times E_k \to F$ is *weakly* $(p; q_1, \cdots, q_k)$*-summing* if there exists a constant $K_2 > 0$ such that, for every choice of elements $x_i^j \in E_j$ $(j = 1, \cdots, k, 1 \le i \in \mathbb{N})$, the following relation holds:

$$\begin{aligned} (1.2) \qquad & \| \ (\ T(x_i^1, \cdots, x_i^k) \ )_{i=1}^{\infty} \ \|_p^{weak} := \sup_{y' \in B_{F^*}} [\ \sum_{i=1}^{\infty} |y'(T(x_i^1, \cdots, x_i^k))|^p \ \ ]^{1/p} \\ \le & \quad K_2 \cdot \| \ (x_i^1)_{i=1}^{\infty} \ \|_{q_1}^{weak} \cdots \| \ (x_i^k)_{i=1}^{\infty} \ \|_{q_k}^{weak}. \end{aligned}$$

In this case, we define the *weakly* $(p; q_1, \cdots, q_k)$*-summing norm* of T by

$$\pi^w_{(p;q_1,\cdots,q_k)}(T) := \inf\{ \ K_2 > 0 : \ (1.2) \text{ holds } \}.$$

We denote $\mathcal{L}^w_{(p;q_1,\cdots,q_k)}(E_1, \cdots, E_k \ : \ F)$ by the class of weakly $(p; q_1, \cdots, q_k)$-summing multilinear mappings from $E_1 \times \cdots \times E_k$ into F with the norm $\pi^w_{(p;q_1,\cdots,q_k)}$.

A weakly $(p; q, \cdots, q)$-summing mapping will be called *weakly* (p, q)*-summing* and we write $\pi^w_{(p,q)}$ for the associated norm. Moreover, a weakly (p, p)-summing mapping will be called *weakly* p*-summing* and we write π^w_p for the associated norm.

Given k and $m_1, \cdots, m_k \in \mathbb{N}$, let $(y_{i_1,\cdots,i_k})^{m_1,\cdots,m_k}_{i_1,\cdots,i_k=1}$ denote a multi-index sequence in F with the index i_j varying from 1 to m_j $(1 \le j \le k)$. Note that $\sum^{m_1,\cdots,m_k}_{i_1,\cdots,i_k=1} y_{i_1,\cdots,i_k}$ means that $\sum^{m_1}_{i_1=1} \cdots \sum^{m_k}_{i_k=1} y_{i_1,\cdots,i_k}$.

A $(p; q_1, \cdots, q_k)$-summing k-linear mapping $T : E_1 \times \cdots \times E_k \to F$ is multiple $(p; q_1, \cdots, q_k)$-summing [2] if there exists a constant $K_3 > 0$ such that, for every choice of natural numbers m_j $(1 \le j \le k)$ and for every choice of elements $x^j_{i_j} \in E_j$ $(1 \le i_j \le m_j)$, the following relation holds:

$$(1.3) \quad \| \, (\, T(x^1_{i_1}, \cdots, x^k_{i_k}) \,)^{m_1,\cdots,m_k}_{i_1,\cdots,i_k=1} \, \|^{strong}_p := [\sum_{i_1,\cdots,i_k=1}^{m_1,\cdots,m_k} \|T(x^1_{i_1}, \cdots, x^k_{i_k})\|^p \;]^{1/p}$$
$$\le \quad K_3 \cdot \| \, (x^1_i)^\infty_{i=1} \, \|^{weak}_{q_1} \cdots \| \, (x^k_i)^\infty_{i=1} \, \|^{weak}_{q_k}.$$

In this case, we define the multiple $(p; q_1, \cdots, q_k)$-summing norm of T by

$$\pi^{mul}_{(p;q_1,\cdots,q_k)}(T) = \inf\{ \, K_3 > 0 : \; (1.3) \text{ holds } \}.$$

Bombal *et al.* [2] introduce multiple p-summing multilinear mappings. Using them they proved several generalizations of Grothendieck's fundamental theorem of the metric space of tensor products.

In connection with the definition of multiple $(p; q_1, \cdots, q_k)$-summing, we introduce a new definition of multiple weakly $(p; q_1, \cdots, q_k)$-summing as follows: a weakly $(p; q_1, \cdots, q_k)$-summing k-linear mapping $T : E_1 \times \cdots \times E_k \to F$ is *multiple weakly* $(p; q_1, \cdots, q_k)$*-summing* if there exists a constant $K_4 > 0$ such that, for every choice of natural numbers m_j $(1 \le j \le k)$ and for every choice of elements $x^j_{i_j} \in E_j$ $(1 \le i_j \le m_j)$, the following relation holds:

$$(1.4) \quad \| \, (\, T(x^1_{i_1}, \cdots, x^k_{i_k}) \,)^{m_1,\cdots,m_k}_{i_1,\cdots,i_k=1} \, \|^{weak}_p := \sup_{y' \in B_{F^*}} [\sum_{i_1,\cdots,i_k=1}^{m_1,\cdots,m_k} |y'(T(x^1_{i_1}, \cdots, x^k_{i_k}))|^p \;]^{1/p}$$
$$\le \quad K_4 \cdot \| \, (x^1_i)^\infty_{i=1} \, \|^{weak}_{q_1} \cdots \| \, (x^k_i)^\infty_{i=1} \, \|^{weak}_{q_k}.$$

In this case, we define the *multiple weakly* $(p; q_1, \cdots, q_k)$*-summing norm* of T by

$$\pi^{mul-w}_{(p;q_1,\cdots,q_k)}(T) = \inf\{ \, K_4 > 0 : \; (1.4) \text{ holds } \}.$$

We denote $\mathcal{L}^{mul-w}_{(p;q_1,\cdots,q_k)}(E_1, \cdots, E_k : F)$ by the class of multiple weakly $(p; q_1, \cdots, q_k)$-summing multilinear mappings from $E_1 \times \cdots \times E_k$ into F with the norm $\pi^{mul-w}_{(p;q_1,\cdots,q_k)}$. A multiple weakly $(p; q, \cdots, q)$-summing mapping will be called *multiple weakly* (p, q)*-summing* and we write $\pi^{mul-w}_{(p,q)}$ for the associated norm. Moreover, a multiple weakly (p, p)-summing mapping will be called *multiple weakly* p*-summing* and we write π^{mul-w}_p for the associated norm.

We denote by $\mathcal{L}_{(p;q_1,\cdots,q_k)}(E_1,\cdots,E_k:F)$ and $\mathcal{L}^{mul}_{(p;q_1,\cdots,q_k)}(E_1,\cdots,E_k:F)$ the spaces of absolutely $(p;q_1,\cdots,q_k)$-summing and multiple $(p;q_1,\cdots,q_k)$-summing k-linear mappings with the norm $\pi_{(p;q_1,\cdots,q_k)}$ and $\pi^{mul}_{(p;q_1,\cdots,q_k)}$, respectively.

It is obvious that

$$\begin{aligned} & \mathcal{L}^{mul}_{(p;q_1,\cdots,q_k)}(E_1,\cdots,E_k:F) \subset \mathcal{L}^{mul-w}_{(p;q_1,\cdots,q_k)}(E_1,\cdots,E_k:F) \\ \subset\ & \mathcal{L}^{w}_{(p;q_1,\cdots,q_k)}(E_1,\cdots,E_k:F) \subset \mathcal{L}(E_1,\cdots,E_k:F), \end{aligned}$$

and that

$$\|T\| \le \pi^{w}_{(p;q_1,\cdots,q_k)}(T) \le \pi^{mul-w}_{(p;q_1,\cdots,q_k)}(T) \le \pi^{mul}_{(p;q_1,\cdots,q_k)}(T)$$

for every $T \in \mathcal{L}^{mul}_{(p;q_1,\cdots,q_k)}(E_1,\cdots,E_k:F)$.

Similar to the case of multilinear mappings, we introduce a new definition of multiple weakly $(p;q_1,\cdots,q_k)$-summing to homogeneous polynomials as follows: a k-homogeneous polynomial $P:E\to F$ is *weakly* (p,q)*-summing* if there exists a constant $C>0$ such that, for every choice of elements $x_i \in E$ $(i\in\mathbb{N})$, the following relation holds:

$$\begin{aligned} (1.5) \qquad & \| \,(P(x_i))_{i=1}^{\infty}\, \|_p^{weak} := \sup_{y'\in B_{F^*}} [\ \sum_{i=1}^{\infty} |y'(P(x_i))|^p\ ]^{1/p} \\ \le\ & C\cdot(\|\,(x_i)_{i=1}^{\infty}\,\|_q^{weak})^k. \end{aligned}$$

In this case, we define the *weakly* (p,q)*-summing norm* of P by

$$\pi^{w}_{(p,q)}(P) = \inf\{\ C>0:\ (1.5) \text{ holds }\}.$$

We denote $\mathcal{P}^{w}_{(p,q)}({}^kE:F)$ by the class of weakly (p,q)-summing k-homogeneous polynomials from E into F with the norm $\pi^{w}_{(p,q)}$. A weakly (p,p)-summing polynomial will be called *weakly* p*-summing* and we write π^{w}_{p} for the associated norm. A weakly (p,q)-summing k-homogeneous polynomial $P:E\to F$ is *multiple weakly* (p,q)*-summing* if the associated symmetric k-linear mapping $\check{P}$ is multiple weakly (p,q)-summing. We denote $\mathcal{P}^{mul-w}_{(p,q)}({}^kE:F)$ by the class of multiple weakly (p,q)-summing k-homogeneous polynomials from E into F with the norm $\pi^{w}_{(p,q)}$.

It is obvious that

$$\mathcal{P}^{mul-w}_{(p,q)}({}^kE:F) \subset \mathcal{P}^{w}_{(p,q)}({}^kE:F) \subset \mathcal{P}({}^kE:F),$$

and that

$$\|P\| \le \pi^{w}_{(p,q)}(P)$$

for every $P \in \mathcal{P}^{mul-w}_{(p,q)}({}^kE:F)$.

In section 2, we investigate some interesting properties about multiple weakly $(p;q_1,\cdots,q_k)$-summing multilinear mappings and polynomials defined on Banach spaces: In particular, we prove a kind of Dvoretzky-Rogers' Theorem and an ideal property for multiple weakly $(p;q_1,\cdots,q_k)$-summing multilinear mappings.

In section 3, we prove that the Aron-Berner extensions of multiple weakly $(p;q_1,\cdots,q_k)$-summing multilinear mappings and polynomials are also multiple weakly $(p;q_1,\cdots,q_k)$-summing.

2. Properties of multiple weakly summing multilinear mappings and polynomials

Theorem 2.1. *$P\in\mathcal{P}^w_{(p,q)}({}^kE:F)$ if and only if $\check{P}\in\mathcal{L}^w_{(p,q)}({}^kE:F)$, where $\check{P}$ is the corresponding k-linear mapping. Moreover, we have*

$$\pi^w_{(p,q)}(P)\le\pi^w_{(p,q)}(\check{P})\le\frac{(2^{(\frac{1}{p}-1)}k)^k}{k!}\pi^w_{(p,q)}(P).$$

Proof. ($\Leftarrow$): It is obvious.

($\Rightarrow$): Let $(x_i^1)_{i=1}^\infty,\cdots,(x_i^k)_{i=1}^\infty\in B_{l_q^{weak}(E)}$. By the triangle inequality,

$$\|(\ \epsilon_1x_i^1+\cdots+\epsilon_kx_i^k\ )_{i=1}^\infty\|_q^{weak}\le\|(\epsilon_1x_i^1)_{i=1}^\infty\|_q^{weak}+\cdots+\|(\epsilon_kx_i^k)_{i=1}^\infty\|_q^{weak}\le k$$

for every $\epsilon_1,\cdots,\epsilon_k=\pm1$.

Claim: $\|(\ \check{P}(x_i^1,\cdots,x_i^k)\ )_{i=1}^\infty\|_q^{weak}\le\dfrac{(2^{(\frac{1}{p}-1)}k)^k}{k!}\pi^w_{(p,q)}(P)$

Indeed, we have

$$\begin{aligned}
&\|(\ \check{P}(x_i^1,\cdots,x_i^k)\ )_{i=1}^\infty\|_q^{weak}\\
=\ &\sup_{y'\in B_{F^*}}[\ \sum_{i=1}^\infty|\ y'(\check{P}(x_i^1,\cdots,x_i^k))\ |^p\ \ ]^{1/p}\\
=\ &\frac{1}{2^kk!}\sup_{y'\in B_{F^*}}[\ \sum_{i=1}^\infty|\sum_{\epsilon_1,\cdots,\epsilon_k=\pm1}\epsilon_1\cdots\epsilon_k\cdot y'(P(\epsilon_1x_i^1+\cdots+\epsilon_kx_i^k))\ |^p\ \ ]^{1/p}\\
&\text{(by the Polarization Formula)}\\
\le\ &\frac{1}{2^kk!}\sup_{y'\in B_{F^*}}[\ \sum_{i=1}^\infty(\sum_{\epsilon_1,\cdots,\epsilon_k=\pm1}|\ y'(P(\epsilon_1x_i^1+\cdots+\epsilon_kx_i^k))\ |^p)\ \ ]^{1/p}\\
=\ &\frac{1}{2^kk!}\cdot[\ \sum_{\epsilon_1,\cdots,\epsilon_k=\pm1}\sup_{y'\in B_{F^*}}(\ \sum_{i=1}^\infty|\ y'(P(\epsilon_1x_i^1+\cdots+\epsilon_kx_i^k))\ |\ )^p\ \ ]^{1/p}\\
\le\ &\frac{1}{2^kk!}\cdot\pi^w_{(p,q)}(P)\cdot[\ \sum_{\epsilon_1,\cdots,\epsilon_k=\pm1}(\ \ \|(\ \epsilon_1x_i^1+\cdots+\epsilon_kx_i^k\ )_{i=1}^\infty\ \|_q^w\ \ )^{kp}\ \ ]^{1/p}\\
\le\ &\frac{1}{2^kk!}\cdot\pi^w_{(p,q)}(P)\cdot[\ \sum_{\epsilon_1,\cdots,\epsilon_k=\pm1}k^{kp}\ \ ]^{1/p}\\
=\ &\frac{(2^{(\frac{1}{p}-1)}k)^k}{k!}\pi^w_{(p,q)}(P).\ \ \square
\end{aligned}$$

Note that $2^{(\frac{1}{p}-1)k} \cdot \frac{k^k}{k!} \leq \frac{k^k}{k!}$. By a similar proof of Theorem 2.1, we see that $P \in \mathcal{P}_{(p,q)}(^k E : F)$ if and only if $\check{P} \in \mathcal{L}_{(p,q)}(^k E : F)$. Moreover, we have

$$\pi_{(p,q)}(P) \leq \pi_{(p,q)}(\check{P}) \leq \frac{(2^{(\frac{1}{p}-1)}k)^k}{k!} \pi_{(p,q)}(P).$$

Examples. (a) Let $\frac{1}{q} - \frac{1}{p} < 1/2$. We show that $\mathcal{L}_{(p,q)}(E : E) \neq \mathcal{L}^w_{(p,q)}(E : E)$ for every infinite dimensional Banach space E and $1 \leq p < \infty$.

Claim: $id_E \in \mathcal{L}^w_{(p,q)}(E : E) \backslash \mathcal{L}_{(p,q)}(E : E)$.

Indeed, obviously $id_E \in \mathcal{L}^w_{(p,q)}(E : E)$. If $id_E \in \mathcal{L}_{(p,q)}(E : E)$, then $id_E^2 = id_E$ would be compact, which is a contradiction.

(b) Let $k/(k-1) < q \leq p < \infty$.

We show that $\mathcal{L}^{mul-w}_{(p,q)}(^k l_{q/(q-1)}) \neq \mathcal{L}(^k l_{q/(q-1)})$ $(k \geq 2)$

Indeed, we define a continuous k-linear form $A : l^k_{q/(q-1)} \to \mathbb{K}$ by

$$A(\ (x_i^{(1)})_{i=1}^{\infty}, \cdots, (x_i^{(k)})_{i=1}^{\infty}\ ) = \sum_{i=1}^{\infty} x_i^{(1)} \cdots x_i^{(k)}.$$

Claim: A is not multiple weakly (p,q)-summing.

Let $m_1, \cdots, m_k \in \mathbb{N}, e_1, \cdots, e_n, \cdots$ be the canonical unit vectors in $l_{q/(q-1)}$. Let $m \in \mathbb{N}, e_1, \cdots, e_m$ be the first m canonical unit vectors in $l_{q/(q-1)}$. Then

$$\|(e_i)_{i=1}^{\infty}\|_q^{weak} = \sup_{x' \in B_{l^*_{q/(q-1)}} = B_{l_q}} \sum_{i=1}^{\infty} |x'(e_i)|^q = 1.$$

But we have

$$\|(\ A(e_{i_1}, \cdots, e_{i_k})\ )_{i_1, \cdots, i_k=1}^{m_1, \cdots, m_k}\|_p^{weak} = \min\{m_1, \cdots, m_k\} \to \infty,$$

as $m_1, \cdots, m_k \to \infty$.

By definition, $\hat{A} \in \mathcal{P}(^k l_{q/(q-1)}) \backslash \mathcal{P}^{mul-w}_{(p,q)}(^k l_{q/(q-1)})$ $(k \geq 2)$.

Lemma 2.2. *Let $1 \leq t < s < \infty$. Then*

$$\sup_{y' \in B_{l_s}} \sum_{i=1}^{m} |y'(e_i)|^t \geq m^{\frac{s-t}{s}}.$$

Thus

$$\sup_{\sum_{n=1}^{\infty} |x_n|^s = 1} \sum_{n=1}^{\infty} |x_n|^t = \infty.$$

Proof. Let $m \in \mathbb{N}, e_1, \cdots, e_m$ be the first m canonical unit vectors in l_s. Let

$y' = \left(\frac{1}{m}\right)^{\frac{1}{s}}(e_1 + \cdots + e_m)$. Then $\|y'\|_s = 1$ and $\sum_{i=1}^m |y'(e_i)|^t = m^{\frac{s-t}{s}}$. It completes the proof. □

Remarks. Let $1 < q \le p < \infty$, $\max\left\{1, \dfrac{q}{k(q-1)}\right\} < r < p/(p-1)$.

Claim: $\mathcal{L}^{mul-w}_{(p,q)}({}^k l_{q/(q-1)} : l_r) \neq \mathcal{L}({}^k l_{q/(q-1)} : l_r)$ $(k \ge 2)$

Indeed, we define a continuous k-linear mapping $A : l^k_{q/(q-1)} \to l_r$ by

$$A(\ (x_i^{(1)})_{i=1}^\infty, \cdots, (x_i^{(k)})_{i=1}^\infty\ ) = (x_i^{(1)} \cdots x_i^{(k)})_{i=1}^\infty.$$

We claim: A is not multiple weakly (p,q)-summing.

Let $m_1, \cdots, m_k \in \mathbb{N}$, $e_1, \cdots, e_n, \cdots$ be the canonical unit vectors in $l_{q/(q-1)}$. We have

$$\begin{aligned}
&\|(\ A(e_{i_1}, \cdots, e_{i_k})\ )_{i_1,\cdots,i_k=1}^{m_1,\cdots,m_k}\|_p^{weak} \\
= &\sup_{y' \in B_{l_r^*} = B_{l_{r/(r-1)}}} \sum_{i_1,\cdots,i_k=1}^{m_1,\cdots,m_k} |\ y'(A(e_{i_1}, \cdots, e_{i_k}))\ |^p \\
= &\sup_{y' \in B_{l_{r/(r-1)}}} \sum_{i=1}^{\min\{m_1,\cdots,m_k\}} |y'(e_i)|^p \to \infty \text{ (by Lemma 2.2)},
\end{aligned}$$

as $m_1, \cdots, m_k \to \infty$, while $\|(e_i)_{i=1}^\infty\|_q^{weak} = 1, \cdots, \|(e_i)_{i=1}^\infty\|_q^{weak} = 1$.

By definition, $\hat{A} \in \mathcal{P}({}^k l_{q/(q-1)} : l_r) \backslash \mathcal{P}^{mul-w}_{(p,q)}({}^k l_{q/(q-1)} : l_r)$ $(k \ge 2)$.

Lemma 2.3. *Let $x_1^* \in E_1^*, \cdots, x_k^* \in E_k^*$ and $u \in \mathcal{L}^w_{(p,q_{k+1})}(E_{k+1} : F)$. Then the k-linear mapping $T : E_1 \times \cdots \times E_k \to F : T(x_1, \cdots, x_{k+1}) = x_1^*(x_1) \cdots x_k^*(x_k) u(x_{k+1})$ is multiple weakly $(p; q_1, \cdots, q_{k+1})$-summing and*

$$\pi^{w-mul}_{(p;q_1,\cdots,q_{k+1})}(T) \le \pi^w_{(p,q_{k+1})}(u) \cdot \|x_1^*\| \cdots \|x_k^*\|.$$

Proof. Let $x_i^j \in E_j$ $(i \in \mathbb{N}, j = 1, \cdots, k+1)$ be such that

$$\|(x_i^1)_{i=1}^\infty\|_{q_1}^{weak} \le 1, \cdots, \|(x_i^{k+1})_{i=1}^\infty\|_{q_{k+1}}^{weak} \le 1.$$

Then $\|x_i^j\| \le 1$ for every $(i \in \mathbb{N}, j = 1, \cdots, k+1)$. We have, for $m_j \in \mathbb{N}$ $(j =$

$1, \cdots, k+1)$,

$$
\begin{aligned}
& \| \ (\ T(x^1_{i_1}, \cdots, x^{k+1}_{i_{k+1}}) \ )^{m_1, \cdots, m_{k+1}}_{i_1, \cdots, i_{k+1}=1} \ \|^{weak}_p \\
= & \sup_{y' \in B_{F^*}} [\sum_{i_1, \cdots, i_{k+1}=1}^{m_1, \cdots, m_{k+1}} |y'(T(x^1_{i_1}, \cdots, x^{k+1}_{i_{k+1}}))|^p \ \ ]^{1/p} \\
= & \sup_{y' \in B_{F^*}} [\sum_{i_1, \cdots, i_{k+1}=1}^{m_1, \cdots, m_{k+1}} |x^*_1(x^1_{i_1})|^p \cdots |x^*_k(x^k_{i_k})|^p \cdot |y'(u(x^{k+1}_{i_{k+1}}))|^p \ \ ]^{1/p} \\
\leq & [\sum_{i_1, \cdots, i_k=1}^{m_1, \cdots, m_k} |x^*_1(x^1_{i_1})|^p \cdots |x^*_k(x^k_{i_k})|^p \ \ ]^{1/p} \sup_{y' \in B_{F^*}} [\sum_{i=1}^{\infty} |y'(u(x^{k+1}_i))|^p \ \ ]^{1/p} \\
\leq & \|(\ u(x^{k+1}_i) \ )^{\infty}_{i=1}\|^{weak}_{(p,q_{k+1})} \cdot \|x^*_k\| \cdot \\
\cdot & [\sum_{i_1, \cdots, i_{k-1}=1}^{m_1, \cdots, m_{k-1}} |x^*_1(x^1_{i_1})|^p \cdots |x^*_{k-1}(x^{k-1}_{i_{k-1}})|^p \cdot [\ \sum_{i=1}^{\infty} | \frac{x^*_k}{\|x^*_k\|}(x^k_i)|^p \ \ ] \ \ ]^{1/p} \\
\leq & \pi^w_{(p,q_{k+1})}(u) \cdot \|x^*_k\| \cdot \|(x^k_i)^{\infty}_{i=1}\|^{weak}_{(p,q_k)} \cdot [\sum_{i_1, \cdots, i_{k-1}=1}^{m_1, \cdots, m_{k-1}} |x^*_1(x^1_{i_1})|^p \cdots |x^*_{k-1}(x^{k-1}_{i_{k-1}})|^p \ \ ]^{1/p} \\
\leq & \pi^w_{(p,q_{k+1})}(u) \cdot \|x^*_k\| \cdot [\sum_{i_1, \cdots, i_{k-1}=1}^{m_1, \cdots, m_{k-1}} |x^*_1(x^1_{i_1})|^p \cdots |x^*_{k-1}(x^{k-1}_{i_{k-1}})|^p \ \ ]^{1/p} \\
& \cdots \\
\leq & \pi^w_{(p,q_{k+1})}(u) \cdot \|x^*_k\| \cdots \|x^*_1\| < \infty. \qquad \square
\end{aligned}
$$

By a similar proof of Lemma 2.3, we see that: *let* $x^*_1 \in E^*_1, \cdots, x^*_k \in E^*_k$ *and* $u \in \mathcal{L}_{(p,q_{k+1})}(E_{k+1} : F)$. *Then the* k*-linear mapping* $T : E_1 \times \cdots \times E_k \to F : T(x_1, \cdots, x_{k+1}) = x^*_1(x_1) \cdots x^*_k(x_k) u(x_{k+1})$ *is multiple* $(p; q_1, \cdots, q_{k+1})$*-summing and*

$$\pi^{mul}_{(p;q_1, \cdots, q_{k+1})}(T) \leq \pi_{(p,q_{k+1})}(u) \cdot \|x^*_1\| \cdots \|x^*_k\|.$$

Recall that the weak Dvoretzky-Rogers Theorem ([6], Theorem 2.18) shows that for every infinite dimensional Banach space F, there exists $(y_i)^{\infty}_{i=1} \in l^{weak}_p(F) \backslash l^{strong}_p(F)$. We prove a kind of Dvoretzky-Rogers' Theorem for multiple weakly $(p; q_1, \cdots, q_k)$-summing multilinear mappings and polynomials.

Theorem 2.4. (a) *If* $E_1, \cdots, E_k$ *are finite dimensional Banach spaces and* F *is a Banach space and* $k \in \mathbb{N}$, *then* $\mathcal{L}^{mul}_{(p;q_1, \cdots, q_k)}(E_1, \cdots, E_k : F) = \mathcal{L}^{mul-w}_{(p;q_1, \cdots, q_k)}(E_1, \cdots, E_k : F) = \mathcal{L}^{w}_{(p;q_1, \cdots, q_k)}(E_1, \cdots, E_k : F) = \mathcal{L}(E_1, \cdots, E_k : F)$. *The polynomial version is also valid.*
(b) *If* $E_1, \cdots, E_k, F$ *are infinite dimensional Banach spaces and* $k \geq 2$, *then* $\mathcal{L}^{mul-w}_{(p;q_1, \cdots, q_k)}(E_1, \cdots, E_k : F) \neq \mathcal{L}_{(p;q_1, \cdots, q_k)}(E_1, \cdots, E_k : F)$. *The polynomial version is also valid.*

Proof. (a): Suppose that $dim(E_i) = n_i$ for $i = 1, \cdots, k$. Let $\{e_1^i, \cdots, e_{n_i}^i\}$ and $\{(e_1^i)^*, \cdots, (e_{n_i}^i)^*\}$ be basis for E_i and E_i^* for $i = 1, \cdots, k$, respectively. Let $T \in \mathcal{L}(E_1, \cdots, E_k : F)$. Then for $x_1 \in E_1, \cdots, x_k \in E_k$, we have

$$\begin{aligned} T(x_1, \cdots, x_k) &= T\Big(\sum_{i_1=1}^{n_1} (e_{i_1}^1)^*(x_1) e_{i_1}^1, \cdots, \sum_{i_k=1}^{n_k} (e_{i_k}^k)^*(x_k) e_{i_k}^k \Big) \\ &= \sum_{i_1,\cdots,i_k=1}^{n_1,\cdots,n_k} (e_{i_1}^1)^*(x_1) \cdots (e_{i_k}^k)^*(x_k)\ T(e_{i_1}^1, \cdots, e_{i_k}^k). \end{aligned}$$

Thus T has finite rank.

Claim: T is multiple $(p; q_1, \cdots, q_k)$-summing.

For simplicity in the notation we assume that

$$T(x_1, \cdots, x_k) = x_1^*(x_1) \cdots x_k^*(x_k)\ y_0 \ (\text{for } x_1 \in E_1, \cdots, x_k \in E_k)$$

for some $x_i^* \in E_i^*$ and some $y_0 \in F$. Let $m_j \in \mathbb{N}, x_i^j \in E_j$ $(i \in \mathbb{N}, j = 1, \cdots, k)$, such that $\|(x_i^1)_{i=1}^\infty\|_{q_1}^{weak} \le 1, \cdots, \|(x_i^k)_{i=1}^\infty\|_{q_k}^{weak} \le 1$. Then $\|x_i^j\| \le 1$ for every $(i \in \mathbb{N}, j = 1, \cdots, k)$. We have

$$\begin{aligned} & \| \ (\ T(x_{i_1}^1, \cdots, x_{i_k}^k) \ )_{i_1,\cdots,i_k=1}^{m_1,\cdots,m_k} \ \|_p^{strong} \\ = \ & \Big[\sum_{i_1,\cdots,i_k=1}^{m_1,\cdots,m_k} \|T(x_{i_1}^1, \cdots, x_{i_k}^k)\|^p \ \Big]^{1/p} \\ = \ & \Big[\sum_{i_1,\cdots,i_k=1}^{m_1,\cdots,m_k} |x_1^*(x_{i_1}^1)|^p \cdots |x_k^*(x_{i_k}^k)|^p \cdot \|y_0\|^p \ \Big]^{1/p} \\ \le \ & \|y_0\| \cdot \|x_k^*\| \cdot \Big[\sum_{i_1,\cdots,i_{k-1}=1}^{m_1,\cdots,m_{k-1}} |x_1^*(x_{i_1}^1)|^p \cdots |x_{k-1}^*(x_{i_{k-1}}^{k-1})|^p \cdot \Big[\sum_{i=1}^{\infty} \Big|\frac{x_k^*}{\|x_k^*\|}(x_i^k)\Big|^p \ \Big] \ \Big]^{1/p} \\ \le \ & \|y_0\| \cdot \|x_k^*\| \cdot \|(x_i^k)_{i=1}^\infty\|_{(p,q_k)}^{weak} \cdot \Big[\sum_{i_1,\cdots,i_{k-1}=1}^{m_1,\cdots,m_{k-1}} |x_1^*(x_{i_1}^1)|^p \cdots |x_{k-1}^*(x_{i_{k-1}}^{k-1})|^p \ \Big]^{1/p} \\ \le \ & \|y_0\| \cdot \|x_k^*\| \cdot \Big[\sum_{i_1,\cdots,i_{k-1}=1}^{m_1,\cdots,m_{k-1}} |x_1^*(x_{i_1}^1)|^p \cdots |x_{k-1}^*(x_{i_{k-1}}^{k-1})|^p \ \Big]^{1/p} \\ \cdots & \\ \le \ & \|y_0\| \cdot \|x_k^*\| \cdots \|x_1^*\| < \infty. \end{aligned}$$

Thus $\mathcal{L}^w(p; q_1, \cdots, q_k)(E_1, \cdots, E_k : F) = \mathcal{L}(E_1, \cdots, E_k : F)$. Since

$$\begin{aligned} \mathcal{L}^{mul}(p; q_1, \cdots, q_k)(E_1, \cdots, E_k : F) &\subset \mathcal{L}^{mul-w}_{(p;q_1,\cdots,q_k)}(E_1, \cdots, E_k : F) \\ &\subset \mathcal{L}^{w}_{(p;q_1,\cdots,q_k)}(E_1, \cdots, E_k : F) \\ &\subset \mathcal{L}(E_1, \cdots, E_k : F). \end{aligned}$$

(b): Let $x^* \in E_k^*, y_0 \in F$. We define a bounded linear operator $u : E_k \to F$ by $u(x) = x^*(x)y_0$ for $x \in E_k$.

Claim: $u \in \mathcal{L}^{mul-w}_{(p,q_k)}(E_k : F)$

Indeed, for $\|(x_i)_{i=1}^{\infty}\|_p^{weak} \le 1$, we have

$$\begin{aligned} & \| \ (u(x_i))_{i=1}^{\infty} \ \|_p^{weak} \\ = & \sup_{y' \in B_{F^*}} [\ \sum_{i=1}^{\infty} |y'(u(x_i))|^p \ ]^{1/p} = \sup_{y' \in B_{F^*}} |y'(y_0)| \cdot [\ \sum_{i=1}^{\infty} |x^*(x_i)|^p \ ]^{1/p} \\ \le & \ \|y_0\| \cdot \|(x_i)_{i=1}^{\infty}\|_p^{weak} \le \|y_0\| < \infty. \end{aligned}$$

Fix $x_1^* \in E_1^*, \cdots, x_{k-1}^* \in E_{k-1}^*$. We define a continuous k-linear mapping $T : E_1 \times \cdots \times E_k \to F$ by

$$T(x_1, \cdots, x_k) = x_1^*(x_1) \cdots x_{k-1}^*(x_{k-1}) \cdot u(x_k) \ \ (x_i \in E_i, i = 1, \cdots, k).$$

By a result of Matos ([8], Theorem 5), $T \notin \mathcal{L}_{(p;q_1,\cdots,q_k)}(E_1, \cdots, E_k : F)$. By Lemma 2.3, $T \in \mathcal{L}^{w-mul}_{(p;q_1,\cdots,q_k)}(E_1, \cdots, E_k : F)$. □

Proposition 2.5. *Consider a k-linear mapping $T : E_1 \times \cdots \times E_k \to F$. The following are equivalent:*

- (a) $T \in \mathcal{L}^{mul-w}_{(p;q_1,\cdots,q_k)}(E_1, \cdots, E_k : F)$;
- (b) *for every choice of elements* $x_i^j \in l_{q_j}^{weak}(E_j)$ $(i \in \mathbb{N}, 1 \le j \le k)$, *we have* $(\ T(x_{i_1}^1, \cdots, x_{i_k}^k) \ )_{i_1,\cdots,i_k=1}^{\infty} \in l_p^{weak}(\mathbb{N}^k; F)$.

In that case, the induced multilinear mapping

$$\tilde{T} : l_{q_1}^{weak}(E_1) \times \cdots \times l_{q_k}^{weak}(E_k) \to l_p^{weak}(\mathbb{N}^k; F)$$

given by $\tilde{T}(\ (x_{i_1}^1)_{i_1=1}^{\infty}, \cdots, (x_{i_k}^1)_{i_k=1}^{\infty} \ ) = (\ T(x_{i_1}^1, \cdots, x_{i_k}^k) \ )_{i_1,\cdots,i_k=1}^{\infty}$ *is continuous and satisfies* $\|\tilde{T}\| = \pi^{mul-w}_{(p;q_1,\cdots,q_k)}(T)$.

Proof. It follows from the definition. □

Proposition 2.6. *Consider a multilinear mapping $T : E_1 \times \cdots \times E_k \to F$. The following are equivalent :*

- (a) $T \in \mathcal{L}^{mul-w}_{(p;q_1,\cdots,q_k)}(E_1, \cdots, E_k : F)$*;*
- (b) *there exists a constant $K > 0$ such that for every choice of natural numbers m_j $(2 \le j \le k)$ and for every choice of elements $x_{i_j}^j \in l_{q_j}^{weak}(E_j)$ $(2 \le j \le k, 1 \le i_j \le m_j)$, with $\|(x_{i_j}^j)_{i_j=1}^{m_j}\|_{q_j}^{weak} \le 1$ we have that the associated linear operator $S : E_1 \to l_p^{weak}(\{1, \cdots, m_2\} \times \cdots \times \{1, \cdots, m_k\}; F)$ given by*

$$S(x) = (\ T(x, x_{i_2}^2, \cdots, x_{i_k}^k) \ )_{i_2,\cdots,i_k=1}^{m_2,\cdots,m_k}$$

is (p, q_1)-summing and satisfies

$$(*) \quad \pi^w_{(p,q_1)}(S) \le K.$$

In that case, $\pi^w_{(p;q_1,\cdots,q_k)}(T) = \inf\{\ K > 0 :\ (*)$ *holds* $\}$.

Proof. It follows from the definition. □

Theorem 2.7. *(a)* $\mathcal{L}^{mul-w}_{(p;q_1,\cdots,q_k)}(E_1,\cdots,E_k : F)$ $[\mathcal{L}^{w}_{(p;q_1,\cdots,q_k)}(E_1,\cdots,E_k : F)$ *respectively* $]$ *is a Banach space under the norm* $\pi^{w}_{(p;q_1,\cdots,q_k)}$ $[\pi^{mul-w}_{(p;q_1,\cdots,q_k)}$ *respectively* $]$.
(b) $\mathcal{P}^w_{(p,q)}({}^kE : F)$ *is a Banach space under the norm* $\pi^w_{(p,q)}$.

Proof. (a): Let $(T_n)_{n=1}^\infty$ be a Cauchy sequence in $\mathcal{L}^{mul-w}_{(p;q_1,\cdots,q_k)}(E_1,\cdots,E_k : F)$. Since $\lim_{n,m\to\infty} \|T_n - T_m\| \le \lim_{n,m\to\infty} \pi^{mul-w}_{(p;q_1,\cdots,q_k)}(T_n - T_m) = 0$, $(T_n)_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{L}(E_1,\cdots,E_k : F)$. Since $\mathcal{L}(E_1,\cdots,E_k : F)$ is complete, there exists some $T \in \mathcal{L}(E_1,\cdots,E_k : F)$ such that $\|T_n - T\| \to 0$.

Claim: $T \in \mathcal{L}^{mul-w}_{(p;q_1,\cdots,q_k)}(E_1,\cdots,E_k : F)$.

By Proposition 2.5, it suffices to show that $\tilde{T} \in \mathcal{L}(l^{weak}_{q_1}(E_1),\cdots,l^{weak}_{q_k}(E_k) : l^{weak}_p(\mathbb{N}^k;F)\ )$. It is obvious that $(\tilde{T_n})_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{L}(l^{weak}_{q_1}(E_1),\cdots,l^{weak}_{q_k}(E_k) : l^{weak}_p(\mathbb{N}^k;F)\ )$. Since $l^{weak}_p(\mathbb{N}^k;F)$ is complete, $\mathcal{L}(l^{weak}_{q_1}(E_1),\cdots,l^{weak}_{q_k}(E_k) : l^{weak}_p(\mathbb{N}^k;F)\ )$ is complete. Thus there exists some $\tilde{S} \in \mathcal{L}(l^{weak}_{q_1}(E_1),\cdots,l^{weak}_{q_k}(E_k) : l^{weak}_p(\mathbb{N}^k;F)\ )$ such that $\tilde{T}_n \to \tilde{S}$. It is obvious that $\tilde{T} = \tilde{S}$. By Proposition 2.5,

$$\lim_{n\to\infty} \pi^{mul-w}_{(p;q_1,\cdots,q_k)}(T_n - T) = \lim_{n\to\infty} \|\widetilde{T_n - T}\| = \lim_{n\to\infty} \|\tilde{T}_n - \tilde{T}\| = 0.$$

Similarly, we see that $\mathcal{L}^{w}_{(p;q_1,\cdots,q_k)}(E_1,\cdots,E_k : F)$ is a Banach space under the norm $\pi^w_{(p;q_1,\cdots,q_k)}$.
(b): By (a), it follows. □

Proposition 2.8. *If $v \in \mathcal{L}(F : G)$ and $T \in \mathcal{L}^{mul-w}_{(p;q_1,\cdots,q_k)}(E_1,\cdots,E_k : F)$, then $v{\circ}T \in \mathcal{L}^{mul-w}_{(p;q_1,\cdots,q_k)}(E_1,\cdots,E_k : G)$ and $\pi^{mul-w}_{(p;q_1,\cdots,q_k)}(v{\circ}T) \le \pi^{mul-w}_{(p;q_1,\cdots,q_k)}(T)\ \cdot \|v\|$.*

Proof. Let $(x^j_i)_{i=1}^\infty \in B_{l^{weak}_{q_j}(E_j)}$ for $j = 1,\cdots,k$.

Then we have, for $m_1,\cdots,m_k \in \mathbb{N}$,

$$\begin{aligned} &\|\ (\ v{\circ}T(x^1_{i_1},\cdots,x^k_{i_k})\ )^{m_1,\cdots,m_k}_{i_1,\cdots,i_k=1}\ \|^{weak}_p \\ =\ &\sup_{z' \in B_{G^*}} [\ \sum_{i_1,\cdots,i_k=1}^{m_1,\cdots,m_k} |\ z'(v{\circ}T(x^1_{i_1},\cdots,x^k_{i_k}))\ |^p\ \ ]^{1/p} \end{aligned}$$

$$= \sup_{z' \in B_{G^*}} [\sum_{i_1,\cdots,i_k=1}^{m_1,\cdots,m_k} | z' \circ \frac{v}{\|v\|}(T(x^1_{i_1},\cdots,x^k_{i_k})) |^p]^{1/p} \cdot \|v\|$$
$$= \sup_{y' \in B_{F^*}} [\sum_{i_1,\cdots,i_k=1}^{m_1,\cdots,m_k} | y'(T(x^1_{i_1},\cdots,x^k_{i_k})) |^p]^{1/p} \cdot \|v\|$$
$$\leq \pi^{mul-w}_{(p;q_1,\cdots,q_k)}(T) \cdot \|v\|.$$

□

Proposition 2.9. *If $v \in \mathcal{L}(F : G)$ and $P \in \mathcal{P}^{mul-w}_{(p,q)}({}^kE : F)$, then $v \circ P \in \mathcal{P}^{mul-w}_{(p,q)}({}^kE : G)$ and $\pi^{mul-w}_{(p,q)}(v \circ P) \leq \|v\| \cdot \pi^{mul-w}_{(p,q)}(P)$.*

Proof. It is similar as the proof of Proposition 2.8. □

We have a composition theorem, which shows the good behavior of this class in relation with the p-summing linear operators.

Theorem 2.10. *Let $u_j \in \mathcal{L}_q(E_j : Y_j)$ $(j = 1,\cdots,k)$, $T \in \mathcal{L}^{mul-w}_p(Y_1,\cdots,Y_k : F)$ and $1 \leq r < \infty$ with $1/r = 1/p+1/q$. Then $S = T(u_1,\cdots,u_k)$ is weak-type multiple r-summing and $\pi^{mul-w}_r(S) \leq \pi^{mul-w}_p(T) \cdot \pi_q(u_1)\cdots\pi_q(u_k)$.*

Proof. We follows along the lines of ([6], Composition Theorem 2.22). Let $(x^j_i)^\infty_{i=1} \in B_{l^{weak}_{q_j}(E_j)}$ for $j = 1,\cdots,k$. By ([6], Lemma 2.2), there exist $(\lambda^j_i)^\infty_{i=1} \in B_{l_q}$ and $(y^j_i)^\infty_{i=1} \in B_{l^{weak}_p(y_j)}$ such that $u_j(x^j_i) = \lambda^j_i y^j_i$ and $\|(y^j_i)^\infty_{i=1}\|^{weak}_p \leq \pi_q(u_j)$.

Then we have, for $m_1,\cdots,m_k \in \mathbb{N}$,

$$\| (S(x^1_{i_1},\cdots,x^k_{i_k}))^{m_1,\cdots,m_k}_{i_1,\cdots,i_k=1} \|^{weak}_r$$
$$= \sup_{z' \in B_{F^*}} [\sum_{i_1,\cdots,i_k=1}^{m_1,\cdots,m_k} | z'(S(x^1_{i_1},\cdots,x^k_{i_k})) |^r]^{1/r}$$
$$= \sup_{z' \in B_{F^*}} [\sum_{i_1,\cdots,i_k=1}^{m_1,\cdots,m_k} | z'(T(\lambda^1_{i_1} y^1_{i_1},\cdots,\lambda^k_{i_k} y^k_{i_k})) |^r]^{1/r}$$
$$= \sup_{z' \in B_{F^*}} [\sum_{i_1,\cdots,i_k=1}^{m_1,\cdots,m_k} | \lambda^1_{i_1}\cdots\lambda^k_{i_k} |^r \cdot | z'(T(y^1_{i_1},\cdots,y^k_{i_k})) |^r]^{1/r}$$
$$\leq [\sum_{i_1,\cdots,i_k=1}^{m_1,\cdots,m_k} | \lambda^1_{i_1}\cdots\lambda^k_{i_k} |^{r\cdot\frac{q}{r}}]^{\frac{r}{q}\cdot\frac{1}{r}} \sup_{z' \in B_{F^*}} [\sum_{i_1,\cdots,i_k=1}^{m_1,\cdots,m_k} | z'(T(y^1_{i_1},\cdots,y^k_{i_k})) |^{r\cdot\frac{p}{r}}]^{\frac{r}{p}\cdot\frac{1}{r}}$$

(by Hölder's inequality because $1 = \frac{1}{q/r} + \frac{1}{p/r}$)

$$\leq \| (\lambda^1_i)^\infty_{i=1} \|_q \sup_{z' \in B_{F^*}} [\sum_{i_1,\cdots,i_k=1}^{m_1,\cdots,m_k} | z'(T(y^1_{i_1},\cdots,y^k_{i_k})) |^p]^{1/p}$$

$$\leq \pi_p^{mul-w}(T) \cdot \|(y_i^1)_{i=1}^\infty\|_p^{weak} \cdots \|(y_i^k)_{i=1}^\infty\|_p^{weak}$$
$$\leq \pi_p^{mul-w}(T) \cdot \pi_q(u_1) \cdots \pi_q(u_k),$$

showing $\pi_r^{mul-w}(S) \leq \pi_p^{mul-w}(T) \cdot \pi_q(u_1) \cdots \pi_q(u_k)$. □

By Proposition 2.8 and Theorem 2.10, we have an ideal property for multiple weakly summing multilinear mappings.

Corollary 2.11. *Let $v \in \mathcal{L}_q(F : G), u_j \in \mathcal{L}_q(E_j : Y_j)$ $(j = 1, \cdots, k)$, $T \in \mathcal{L}_p^{mul-w}(Y_1, \cdots, Y_k : F)$ and $1 \leq r < \infty$ with $1/r = 1/p + 1/q$. Then $v \circ T(u_1, \cdots, u_k)$ is multiple weakly r-summing and*

$$\pi_r^{mul-w}(v \circ T) \leq \pi_p^{mul-w}(T) \cdot \pi_q(u_1) \cdots \pi_q(u_k) \ \cdot \|v\|.$$

We have also an ideal property for multiple weakly summing homogeneous polynomials:

Theorem 2.12. *Let $v \in \mathcal{L}(F : G), u \in \mathcal{L}_q(E : Y)$, $P \in \mathcal{P}_p^{mul-w}$ $({}^kY : F)$ and $1 \leq r < \infty$ with $1/r = 1/p + 1/q$. Then $S = v \circ P \circ u$ is multiple weakly r-summing and $\pi_r^{mul-w}(S) \leq \|v\| \cdot \pi_p^{mul-w}(P) \cdot (\pi_q(u))^k$.*

Proof. It follows from Proposition 2.9 and a similar proof of Theorem 2.10. □

3. Aron-Berner extensions of multiple weakly summing mappings

A bounded k-homogeneous polynomial P has an extension $\overline{P} \in \mathcal{P}({}^kE^{**} : F^{**})$ to the bidual E^{**} of E, which is called the Aron-Berner extension of P (see [1]). In fact, $\overline{P}$ is defined in the following way: We first start with the complex-valued bounded k-homogeneous polynomial $P \in \mathcal{P}({}^kE)$. Let A be the bounded symmetric k-linear form on E corresponding to P. We can extend A to an k-linear form $\overline{A}$ on the bidual E^{**} in such a way that for each fixed j, $1 \leq j \leq k$ and for each fixed $x_1, \ldots, x_{j-1} \in E$ and $z_{j+1}, \ldots, z_k \in E^{**}$, the linear form

$$z \rightarrow \overline{A}(x_1, \ldots, x_{j-1}, z, z_{j+1}, \ldots, z_k), \quad z \in E^{**},$$

is weak-star continuous. By this weak-star continuity A can be extended to an k-linear form $\overline{A}$ on E^{**}, beginning with the last variable and working backwards to the first. Then the restriction

$$\overline{P}(z) = \overline{A}(z, \ldots, z)$$

is called the Aron-Berner extension of P. In particular, Davie and Gamelin [4] proved that $\|P\| = \|\overline{P}\|$. Next, for a vector-valued k-homogeneous polynomial $P \in \mathcal{P}({}^kE : F)$, the Aron-Berner extension $\overline{P} \in \mathcal{P}({}^kE^{**} : F^{**})$ is defined as follows: Given $z \in E^{**}$ and $w \in F^*$,

$$\overline{P}(z)(w) = \overline{w \circ P}(z).$$

For $x \in E$, we define $\delta_x : E^* \to \mathbb{C}$ by $\delta_x(x^*) = x^*(x)$ for each $x^* \in E^*$. Let $< x_\alpha >$ is a net in E and $x_0^{**} \in E^{**}$. We say that $< x_\alpha >$ converges polynomial-star to x_0^{**} if for every $P \in \mathcal{P}(^kE)(k \in \mathbb{N})$, we have $P(x_\alpha)$ converges to $\overline{P}(x_0^{**})$, where $\overline{P}$ is the Aron-Berner extension of P. Recall that Davie and Gamelin [4] proved that B_E is polynomial-star dense in $B_{E^{**}}$.

The following shows that the Aron-Berner extensions of multiple weakly $(p; q_1, \cdots, q_k)$-summing multilinear mappings and polynomials are also multiple weakly $(p; q_1, \cdots, q_k)$-summing.

Theorem 3.1. *We have:*

(*a*) $T \in \mathcal{L}^{mul-w}_{(p;q_1,\cdots,q_k)}(E_1, \cdots, E_k : F)$ *if and only if* $\overline{T} \in \mathcal{L}^{mul-w}_{(p;q_1,\cdots,q_k)}(E_1^{**}, \cdots, E_k^{**} : F^{**})$, *where* $\overline{T}$ *is any extension of* T *to* $E_1^{**} \times \cdots \times E_k^{**}$ *by the Aron-Berner Extension Method. In this case,* $\pi^{mul-w}_{(p;q_1,\cdots,q_k)}(T) = \pi^{mul-w}_{(p;q_1,\cdots,q_k)}(\overline{T})$.

(*b*) $P \in \mathcal{P}^{mul-w}_{(p,q)}(^kE : F)$ *if and only if* $\overline{P} \in \mathcal{P}^{mul-w}_{(p,q)}(^kE^{**} : F^{**})$, *where* $\overline{P}$ *is the Aron-Berner extension of* P *to* E^{**}. *In this case,* $\pi^{mul-w}_{(p,q)}(P) = \pi^{mul-w}_{(p,q)}(\overline{P})$.

Proof. For simplicity in the notation we write the proof for $k = 2$.
(a): $(\Leftarrow)$: Since $\overline{T}$ is an extension of T, $\pi^{mul-w}_{(p;q_1,\cdots,q_k)}(T) \le \pi^{mul-w}_{(p;q_1,\cdots,q_k)}(\overline{T}) < \infty$. Thus $T \in \mathcal{L}^{mul-w}_{(p;q_1,\cdots,q_k)}(E_1, \cdots, E_k : F)$.
$(\Rightarrow)$: Let $m_1, m_2 \in \mathbb{N}$. Let $z^{'''} \in B_{F^{***}}, (x_1^{''}, \cdots, x_{m_1}^{''}) \in B_{l^{weak}_{q_1}(E_1^{**})}$ and $(y_1^{''}, \cdots, y_{m_2}^{''}) \in B_{l^{weak}_{q_2}(E_2^{**})}$. By Goldstine's Theorem there exist nets $(z_l^{'})_\Gamma$ in B_{F^*}, nets $(x^{i_1}_{\alpha_{i_1}})_{\Omega_{i_1}} (1 \le i_1 \le m_1)$ in B_{E_1}, and nets $(y^{i_2}_{\beta i_2})_{\Lambda_{i_2}}$ $(1 \le i_2 \le m_2)$ in B_{E_2} such that $(z_l^{'})_\Gamma$ converges weak-star to $z^{'''}$, and that $(x^{i_1}_{\alpha_{i_1}})_{\Omega_{i_1}}$ $(1 \le i_1 \le m_1)$ converges weak-star to $x^{''}_{i_1}$, and that $(y^{i_2}_{\beta i_2})_{\Lambda_{i_2}}$ $(1 \le i_2 \le m_2)$ converges weak-star to $y^{''}_{i_2}$. Hence

$$\lim_{\alpha_{i_1} \in \Omega_{i_1} \to \infty} x^{'}(x^{i_1}_{\alpha_{i_1}}) = x^{''}_{i_1}(x^{'}) \ (x^{'} \in E_1^*, 1 \le i_1 \le m_1)$$

and

$$\lim_{\beta_{i_2} \in \Lambda_{i_2} \to \infty} y^{'}(y^{i_2}_{\beta_{i_2}}) = y^{''}_{i_2}(y^{'}) \ (y^{'} \in E_2^*, 1 \le i_2 \le m_2) \quad (A)$$

Claim : $\limsup_{\alpha_1 \in \Omega_1 \to \infty} \cdots \limsup_{\alpha_{m_1} \in \Omega_{m_1} \to \infty} \|(x^1_{\alpha_{i_1}}, \cdots, x^{m_1}_{\alpha_{m_1}})\|^{weak}_{q_1} \le 1$,
$\limsup_{\beta_1 \in \Lambda_1 \to \infty} \cdots \limsup_{\beta_{m_2} \in \Lambda_{m_2} \to \infty} \|(y^1_{\beta_1} \cdots, y^{m_2}_{\beta_{m_2}})\|^{weak}_{q_2} \le 1$.

Indeed, we have

$$\begin{aligned} & \limsup_{\alpha_1 \in \Omega_1 \to \infty} \cdots \limsup_{\alpha_{m_1} \in \Omega_{m_1} \to \infty} \|(x^1_{\alpha_{i_1}}, \cdots, x^{m_1}_{\alpha_{m_1}})\|^{weak}_{q_1} \\ = & \sup_{x^{'} \in B_{E_1^*}} \Big(\lim_{\alpha_1 \in \Omega_1 \to \infty} |x^{'}(x^1_{\alpha_1})|^{q_1} + \cdots + \lim_{\alpha_{m_1} \in \Omega_{m_1} \to \infty} |x^{'}(x^{m_1}_{\alpha_{m_1}})|^{q_1} \Big)^{1/q_1} \end{aligned}$$

$$= \sup_{x' \in B_{E_1^*}} (|x_1''(x')|^{q_1} + \cdots + |x_{m_1}''(x')|^{q_1})^{1/q_1} \quad \text{(by (A))}$$
$$= \sup_{\delta_{x'} \in B_{E_1^{***}}} (|\delta_{x'}(x_1'')|^{q_1} + \cdots + |\delta_{x'}(x_{m_1}'')|^{q_1})^{1/q_1}$$
$$\leq \|(x_1'', \cdots, x_{m_1}'')\|_{q_1}^{weak} \leq 1.$$

Similarly, we can show that

$$\limsup_{\beta_1 \in \Lambda_1 \to \infty} \cdots \limsup_{\beta_{m_2} \in \Lambda_{m_2} \to \infty} \|(y_{\beta_1}^1 \cdots, y_{\beta_{m_2}}^{m_2})\|_{q_2}^{weak} \leq 1.$$

Since $\overline{T}(x_{i_1}'', y_{i_2}'') \in B_{F^{**}}$, by the weak-star convergence of the net $(z_l')_\Gamma$, we have

$$\lim_{l \in \Gamma \to \infty} \overline{T}(x_{i_1}'', y_{i_2}'')(z_l') = z'''(\overline{T}(x_{i_1}'', y_{i_2}'')) \ (1 \leq i_1 \leq m_1, 1 \leq i_2 \leq m_2) \quad (B)$$

It follows that

$$[\sum_{i_1, i_2 = 1}^{m_1, m_2} |\ z'''(\overline{T}(x_{i_1}'', y_{i_2}''))\ |^p\ ]^{1/p}$$
$$= [\sum_{i_1, i_2 = 1}^{m_1, m_2} \lim_{l \in \Gamma \to \infty} |\ \overline{T}(x_{i_1}'', y_{i_2}'')(z_l')\ |^p\ ]^{1/p}$$
$$= \lim_{l \in \Gamma \to \infty} [\sum_{i_1, i_2 = 1}^{m_1, m_2} |\ \overline{T}(x_{i_1}'', y_{i_2}'')(z_l')\ |^p\ ]^{1/p} \quad \text{(by } (B))$$
$$= \lim_{l \in \Gamma \to \infty} [\sum_{i_1, i_2 = 1}^{m_1, m_2} \lim_{\alpha_{i_1} \in \Omega_{i_1} \to \infty} \lim_{\beta_{i_2} \in \Lambda_{i_2} \to \infty} |\ z_l'(T(x_{\alpha_{i_1}}^{i_1}, y_{\beta_{i_2}}^{i_2}))|^p\ ]^{1/p}$$

(by the definition of $\overline{T}$ and (A))

$$= \lim_{l \in \Gamma \to \infty} \lim_{\alpha_1 \in \Omega_1 \to \infty} \cdots \lim_{\alpha_{m_1} \in \Omega_{m_1} \to \infty} \lim_{\beta_1 \in \Lambda_1 \to \infty} \cdots \lim_{\beta_{m_2} \in \Lambda_{m_2} \to \infty} [\sum_{i_1, i_2 = 1}^{m_1, m_2} |\ z_l'(T(x_{\alpha_{i_1}}^{i_1}, y_{\beta_{i_2}}^{i_2}))|^p\ ]^{1/p}$$
$$\leq \pi_{(p;q_1,\cdots,q_k)}^{mul-w}(T)(\limsup_{\alpha_1 \in \Omega_1 \to \infty} \cdots \limsup_{\alpha_{m_1} \in \Omega_{m_1} \to \infty} \|(x_{\alpha_{i_1}}^1, \cdots, x_{\alpha_{m_1}}^{m_1})\|_{q_1}^{weak})$$
$$\cdot \ (\limsup_{\beta_1 \in \Lambda_1 \to \infty} \cdots \limsup_{\beta_{m_2} \in \Lambda_{m_2} \to \infty} \|(y_{\beta_1}^1 \cdots, y_{\beta_{m_2}}^{m_2})\|_{q_2}^{weak})$$
$$\leq \pi_{(p;q_1,\cdots,q_k)}^{mul-w}(T) < \infty \quad \text{(by } (**)),$$

which shows $\overline{T} \in \mathcal{L}_{(p;q_1,\cdots,q_k)}^{mul-w}(E_1^{**}, \cdots, E_k^{**} : F^{**})$ and $\pi_{(p;q_1,\cdots,q_k)}^{mul-w}(T) = \pi_{(p;q_1,\cdots,q_k)}^{mul-w}(\overline{T})$.

(b): ($\Leftarrow$): Since $\overline{P}$ is an extension of P, $\pi_{(p,q)}^{mul-w}(P) \leq \pi_{(p,q)}^{mul-w}(\overline{P}) < \infty$. Thus $P \in \mathcal{P}_{(p,q)}^{mul-w}({}^kE : F)$.

($\Rightarrow$): Let $m \in \mathbb{N}$. Let $z^{'''} \in B_{F^{***}}, (x_1^{''}, \cdots, x_m^{''}) \in B_{l_q^{weak}(E^{**})}$. By the Davie-Gamelin Theorem [B_{F^*} (B_E, resp.) is polynomial-star dense in $B_{F^{***}}$ ($B_{E^{**}}$, resp.)] there exist nets $(z_l^{'})_\Gamma$ in B_{F^*}, nets $(x_{\alpha_i})_{\Omega_i}$ $(1 \le i \le m)$ in B_E such that $(z_l^{'})_\Gamma$ converges polynomial-star to $z^{'''}$, and that $(x_{\alpha_i})_{\Omega_i}$ $(1 \le i \le m)$ converges polynomial-star to $x_i^{''}$. Hence

$$\lim_{\alpha_i \in \Omega_i \to \infty} Q(x_{\alpha_i}) = \overline{Q}(x_i^{''}) \ (Q \in \mathcal{P}(^n E), n \in \mathbb{N}, 1 \le i \le m) \quad (C)$$

Claim: $\limsup_{\alpha_1 \in \Omega_1 \to \infty} \cdots \limsup_{\alpha_m \in \Omega_m \to \infty} \|(x_{\alpha_1}, \cdots, x_{\alpha_m})\|_q^{weak} \le 1$
Indeed, we have

$$\begin{aligned}
& \limsup_{\alpha_1 \in \Omega_1 \to \infty} \cdots \limsup_{\alpha_m \in \Omega_m \to \infty} \|(x_{\alpha_1}, \cdots, x_{\alpha_m})\|_q^{weak} \\
= & \sup_{x^{'} \in B_{E^*}} \Big(\lim_{\alpha_1 \in \Omega_1 \to \infty} |x^{'}(x_{\alpha_1})|^q + \cdots + \lim_{\alpha_m \in \Omega_m \to \infty} |x^{'}(x_{\alpha_m})|^q \Big)^{1/q} \\
= & \sup_{x^{'} \in B_{E^*}} (|x_1^{''}(x^{'})|^q + \cdots + |x_{m_1}^{''}(x^{'})|^q)^{1/q} \quad \text{(by (C))} \\
= & \sup_{\delta_{x^{'}} \in B_{E^{***}}} (|\delta_{x'}(x_1^{''})|^q + \cdots + |\delta_{x'}(x_m^{''})|^q)^{1/q} \\
\le & \|(x_1^{''}, \cdots, x_m^{''})\|_q^{weak} \le 1.
\end{aligned}$$

Since $\overline{P}(x_i^{''}) \in B_{F^{**}}$, by the weak-star convergence of the net $(z_l^{'})_\Gamma$, we have

$$\lim_{l \in \Gamma \to \infty} \overline{P}(x_i^{''})(z_l^{'}) = z^{'''}(\overline{P}(x_i^{''})) \ (1 \le i \le m) \quad (D)$$

It follows that

$$\begin{aligned}
& \Big[\sum_{i=1}^m | \ z^{'''}(\overline{P}(x_i^{''})) \ |^p \ \Big]^{1/p} \\
= & \Big[\sum_{i=1}^m \lim_{l \in \Gamma \to \infty} | \ \overline{P}(x_i^{''})(z_l^{'}) \ |^p \ \Big]^{1/p} = \lim_{l \in \Gamma \to \infty} \Big[\sum_{i=1}^m | \ \overline{P}(x_i^{''})(z_l^{'}) \ |^p \ \Big]^{1/p} \quad (\text{by } (D)) \\
= & \lim_{l \in \Gamma \to \infty} \Big[\sum_{i=1}^m \lim_{\alpha_i \in \Omega \to \infty} | \ z_l^{'}(P(x_{\alpha_i})|^p \ \Big]^{1/p} \quad \text{(by the definition of } \overline{P} \text{ and (C))} \\
= & \lim_{l \in \Gamma \to \infty} \lim_{\alpha_i \in \Omega \to \infty} \Big[\sum_{i=1}^m | \ z_l^{'}(P(x_{\alpha_i})|^p \ \Big]^{1/p} \\
\le & \pi_{(p,q)}^{mul-w}(P) \ \Big(\limsup_{\alpha_1 \in \Omega_1 \to \infty} \cdots \limsup_{\alpha_m \in \Omega_m \to \infty} \|(x_{\alpha_1}, \cdots, x_{\alpha_m})\|_q^{weak} \Big)^k \\
\le & \pi_{(p,q)}^{mul-w}(P) < \infty \quad (\text{by } (**)),
\end{aligned}$$

which shows $\overline{P} \in \mathcal{P}_{(p,q)}^{mul-w}(^k E^{**} : F^{**})$ and $\pi_{(p,q)}^{mul-w}(P) = \pi_{(p,q)}^{mul-w}(\overline{P})$. □

Acknowledgment. The author would like to thank the referee for helpful comments.

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