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One-sided Prime Ideals in Semirings

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Abstract. In this paper we define prime right ideals of semirings and prove that if every right ideal of a semiring R is prime then R is weakly regular. We also prove that if the set of right ideals of R is totally ordered then every right ideal of R is prime if and only if R is right weakly regular. Moreover in this paper we also define prime subsemimodule (generalizing the concept of prime right ideals) of an R-semimodule. We prove that if a subsemimodule K of an R-semimodule M is prime then $A_K(M)$ is also a prime ideal of R.

1. Introduction

A semiring is a set R together with two binary operations called addition "+" and multiplication "·" such that $(R,+)$ is a commutative semigroup, and $(R,\cdot)$ is a (generally) non commutative monoid with 1 as its identity element; connecting the two algebraic structures are the distributive laws: $a(b+c) = ab + ac$ and $(a+b)c = ac+bc$, for all $a, b, c \in R$. We shall assume that $(R, +, \cdot)$ has an absorbing zero 0, that is $a + 0 = 0 + a = a$ and $a \cdot 0 = 0 \cdot a = 0$ holds for all $a \in R$(cf.[8]).

A subset I of a semiring R is called a right (resp. left) ideal of R if for $a, b \in I$ and $r \in R$, $a + b \in I$ and $ar \in I$ (resp. $ra \in I$); I is a two sided ideal if it is both a right and a left ideal of R. An additively written commutative semigroup M with a neutral element θ is called a right R-semimodule written as M_R, if there is a function $f : M \times R \to M$ such that if $f(m, r)$ is denoted by mr, then the following conditions hold:

(i) $(m + m')r = mr + m'r$

(ii) $m(r + r') = mr + mr'$

(iii) $m(rr') = (mr)r'$

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(iv) $m \cdot 1 = m$

(v) $\theta r = m0 = \theta$, for all $m, m' \in M$ and $r, r' \in R$(cf.[8])

A subsemimodule N of a right R-semimodule M is a subsemigroup of $(M, +)$ such that $nr \in N$ for all $n \in N$ and $r \in R$.

2. Prime right ideals

In [10], K. Koh has defined that a right ideal I in a ring R is of prime type if $AB \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$, where A and B are the right ideal of R. In [9], F. Hansen called these ideals prime right ideals, adopting this notion, we have the following definition.

Definition 1. A right ideal P of a semiring R is called a prime right ideal if for every right ideals I, J of R, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.

The proof of the following Proposition is straight forward.

Proposition 1. *Let P be a right ideal of a semiring R. Then the following are equivalent.*

(1) *P is a prime right ideal.*

(2) *If $a, b \in R$ such that $aRb \subseteq P$ then $a \in P$ or $b \in P$.*

Proposition 2. *Any maximal right ideal of a semiring R is a prime right ideal.*

Proof. Assume that I is a maximal right ideal of a semiring R , and $aRb \subseteq I$. If $a \notin I$, then we show that $b \in I$. The maximality of I implies that right ideal generated by I and a must be the whole semiring R i.e. $R = I + aR$. Hence there exists $i \in I$ and $r_0 \in R$ such that $1 = i + ar_0$. Now $b = 1 \cdot b = (i + ar_0) \cdot b = ib + ar_0 b \in I$. Thus, I is a prime right ideal, which proves that every maximal right ideal of a semiring R is a prime right ideal. □

Proposition 3. *If I is a prime right ideal of a semiring R, then $(I : a) = \{x \in R | ax \in I\}$ is also a prime right ideal of R, for any $a \in R \backslash I$.*

Proof. As $a \cdot 0 = 0 \in I$, so $0 \in (I : a) \neq \emptyset$. Let $x, y \in (I : a)$ then $ax, ay \in I \Rightarrow ax + ay \in I \Rightarrow a(x + y) \in I \Rightarrow (x + y) \in (I : a)$. Now for any $r \in R$, and $x \in (I : a)$. We have $a(xr) = (ax)r \in I$, because $ax \in I$ and I is a right ideal. Thus $(a)(xr) \in I \Rightarrow xr \in (I : a)$, so $(I : a)$ is a right ideal of R. Let J and K be any right ideals of R such that $JK \subseteq (I : a)$ then $a(JK) \subseteq I$. As aJ and aK are right ideals of R and

$$\begin{aligned} (aJ)(aK) &= a(Ja)K \subseteq aJK \subseteq I \\ &\Rightarrow aJ \subseteq I \text{ or } aK \subseteq I \\ &\Rightarrow J \subseteq (I : a) \text{ or } K \subseteq (I : a) \end{aligned}$$

Hence $(I : a)$ is a prime right ideal. □

Proposition 4. *Let I be a prime right ideal of a semiring R, then $J = \{a \in R | Ra \subseteq I\}$ is the largest two sided ideal of R contained in I.*

Proof. We start by proving $J = \{a \in R | Ra \subseteq I\}$ is a two sided ideal of R contained in I. Obviously $J \neq \emptyset$, because $0 \in J$. Next, let $a, b \in J$, then $Ra, Rb \subseteq I$. So $Ra + Rb \subseteq I \Rightarrow R(a + b) \subseteq I \Rightarrow a + b \in J$. Now let $a \in J$ and $x \in R$, then $R(ax) = (Ra)x \subseteq Ix \subseteq I \Rightarrow ax \in J$ and $R(xa) = (Rx)a \subseteq Ra \subseteq I \Rightarrow xa \in J$. So J is a two sided ideal of R. Clearly $J \subseteq I$. Let K be a two sided ideal of R such that $K \subseteq I$. Let $x \in K$, then $Rx \subseteq K \subseteq I$ (as K is a two sided ideal of R) $\Rightarrow x \in J$. Thus $K \subseteq J$. Hence $J = \{a \in R | Ra \subseteq I\}$ is the largest two sided ideal of R contained in I. □

Definition 2. A right ideal I of a semiring R is called semiprime right ideal if and only if for any right ideal H of R, $H^2 \subseteq I$ implies that $H \subseteq I$.

Obviously every prime right ideal of a semiring R is a semiprime right ideal of R.

Proposition 5. *The following conditions on a right ideal I of a semiring R are equivalent:*

(1) *I is a semiprime right ideal.*

(2) $aRa \subseteq I \Rightarrow a \in I$.

Definition 3. A right ideal I of a semiring R is called an irreducible (strongly irreducible) right ideal if $J \cap K = I (J \cap K \subseteq I)$ implies either $J = I$ or $K = I (J \subseteq I$ or $K \subseteq I)$ for every right ideal J and K of R.

Proposition 6. *Let I be a right ideal of a semiring R. If $a \notin I$, then there exist an irreducible right ideal containing I and not containing a.*

Proof. If $\{A_i : i \in \Omega\}$ is a chain of right ideals of R containing I and not containing a, then $\cup A_i$ is a right ideal of R containing I and not containing a. Therefore, by Zorn's Lemma, the set of all right ideals of R containing I and not containing a has a maximal element A. Suppose $A = B \cap C$, where B and C are both right ideals of R properly containing A. Then by the choice of A, $a \in B$ and $a \in C$. Thus $a \in B \cap C = A$, which is a contradiction. Hence A is an irreducible right ideal of the semiring R. □

Proposition 7. *Any right ideal I of a semiring R is the intersection of all the irreducible right ideals of R containing I.*

Proof. Let I be a right ideal of a semiring R and $\{A_i : i \in \Omega\}$ be the collection of irreducible right ideals of R containing I, then $I \subseteq \cap A_i$ for the reverse inclusion, let $x \notin I$, then by Proposition 6 there exists an irreducible right ideal A of R containing I but not containing x. Thus $x \notin \cap A_i$, Hence $I = \cap A_i$. □

Lemma 1. *Let R be a semiring. If I is a strongly irreducible semiprime right ideal of R, then I is a prime right ideal of R.*

Proof. Let J and K be any two right ideals of a semiring R such that $JK \subseteq I$. Then RK is a two sided ideal generated by K. Now $J \cap RK$ is a right ideal of the semiring R.

$$\begin{aligned}(J \cap RK)^2 &\subseteq J(RK) \\ &= (JR)K \\ &\subseteq JK \\ &\subseteq I\end{aligned}$$

As I is a semiprime right ideal, so $J \cap RK \subseteq I$. As I is strongly irreducible right ideal, so $J \subseteq I$ or $RK \subseteq I$. As $K \subseteq RK$, so $J \subseteq I$ or $K \subseteq I$. Hence I is a prime right ideal. □

Proposition 8. *Intersection of prime right ideals of a semiring R is a semiprime right ideal.*

3. Fully prime right semirings

A semiring R is called right weakly regular if for each $x \in R$, $x \in (xR)^2$ (cf. [2]). The following theorem is from [2].

Theorem 1. *The following assertions for a semiring R are equivalent:*

(1) *R is right weakly regular;*

(2) *$J^2 = J$ for each right ideal J of R;*

(3) *For each ideal I of R; $J \cap I = JI$, for any right ideal J of R.*

Definition 4. A semiring R is said to be a fully prime (semiprime) right semiring if all its right ideals are prime (semiprime) right ideals.

Theorem 2. *For a semiring R the following are equivalent:*

(1) *R is right weakly regular;*

(2) *Every right ideal of R is semiprime.*

Proof. (1) ⇒ (2) : Let I be a right ideal of a semiring R and $J^2 \subseteq I$, where J is a right ideal of R. By above Theorem , $J^2 = J$, so $J \subseteq I$. Thus I is a semiprime right ideal of R.

(2) ⇒ (1) : Let I be a right ideal of R, then I^2 is also a right ideal of R. Also $I^2 \subseteq I^2$. By (2) $I \subseteq I^2$. Hence $I = I^2$. □

Proposition 9. *Let R be a semiring. If R is fully prime right semiring then R is right weakly regular and the set of ideals of R is totally ordered.*

Proof. Let R be fully prime right semiring and I be any right ideal of R then $I^2 \subseteq I^2 \Rightarrow I \subseteq I^2$. Thus $I = I^2$. Hence by Theorem 1, R is right weakly regular. Let A, B be ideals of R then $AB \subseteq A \cap B \Rightarrow A \subseteq A \cap B$ or $B \subseteq A \cap B$ that is, either $A \subseteq B$ or $B \subseteq A$. □

Proposition 10. *If R is a right weakly regular semiring such that the set of right ideals of R is totally ordered then every right ideal of R is prime.*

Proof. Let I, J, K be three right ideals of the semiring R, such that $IJ \subseteq K$. As the set of right ideals of R is totally ordered, so without loss of generality, we assume that $I \subseteq J$. Now $IJ \subseteq K \Rightarrow I = I^2 = I \cdot I \subseteq I \cdot J \subseteq K$. So $I \subseteq K$. Hence K is a prime right ideal. □

Theorem 3. *Let R be a semiring such that the set of right ideals of R is totally ordered, then R is fully prime right semiring if and only if R is right weakly regular.*

Proof. The proof of the theorem follows as a direct consequence of Proposition 10 and Proposition 11. □

4. Prime subsemimodules

In this section we extend the notions of prime and semiprime right ideals of a semiring R to arbitrary R-semimodules and develope some of their basic properties.

Proposition 11. *Let R be a semiring. If K is a subsemimodule of a right R-semimodule M, the set $A_K(M) = \{a \in R : Ma \subseteq K\}$ is a two-sided ideal of R.*

Proof. As $0 \in R$ and $M0 = \theta \in K$. So $0 \in A_K(M)$ and $A_K(M) \neq \emptyset$. Let $a, b \in A_K(M)$, then $Ma \subseteq K$ and $Mb \subseteq K \Rightarrow M(a+b) \subseteq Ma + Mb \subseteq K$. So $a + b \in A_K(M)$. Let $a \in A_K(M)$ then $Ma \subseteq K$. Now $M(ar) = (Ma)r \subseteq Kr \subseteq K$, for all $r \in R$. Thus $ar \in A_K(M)$. Again $M(ra) = (Mr)a \subseteq Ma \subseteq K$. Thus $ra \in A_K(M)$, so $A_K(M)$ is a two sided ideal of R. □

Definition 5. Let R be a semiring. If K is a subsemimodule of a right R-semimodule M, then the ideal, $A_K(M) = \{a \in R : Ma \subseteq K\}$ is called the associated ideal of K. If $K = (\theta); A_{(\theta)}(M)$ is called annihilator of M in R; M is called faithful if $A_{(\theta)(M)} = (0)$.

Definition 6. An R-subsemimodule K of a right R-semimodule M is a prime R-subsemimodule of M if for any $v \in M$ and $a \in R$, $vRa \subseteq K \Rightarrow v \in K$ or $a \in A_K(M)$, K is semiprime R-subsemimodule of M, if for any $v \in M$ and $a \in R$, $vaRa \subseteq K \Rightarrow va \in K$. The right R-semimodule M itself is called prime (resp. semiprime) if the zero subsemimodule (θ) of M is prime (resp. semiprime). Moreover, the semiring R is prime (resp. semiprime) if the zero ideal (0) of R is prime (resp. semiprime).

Proposition 12. *A right ideal I of a semiring R is prime if and only if I is prime as an R-subsemimodule of R_R.*

Proof. Let I be a prime right ideal of R. Let $a, b \in R$ such that $aRb \subseteq I$, then

$aRb \subseteq aRbR = (aR)(RbR) \subseteq IR \subseteq I$. Since I is a prime right ideal, so either $aR \subseteq I$ or $RbR \subseteq I$. As $Rb \subseteq RbR$, so either $aR \subseteq I$ or $Rb \subseteq I$. Thus either $a \in I$ or $b \in A_I(R)$. Hence I is prime R-subsemimodule of R_R. Conversely, suppose that I is a prime R-subsemimodule of R_R and $a, b \in R$ such that $aRb \subseteq I$ this implies $a \in I$ or $b \in A_I(R)$, but $A_I(R) \subseteq I$ which implies that $a \in I$ or $b \in I$. Hence I is a prime right ideal. □

Remark 1. If we replace the notion of prime with semiprime in the above Proposition, the proof follows analogously.

Proposition 13. *Every non-zero R-subsemimodule N of a prime R-semimodule M_R is a prime R-semimodule.*

Proof. Suppose M_R is a prime R-semimodule, and N a non-zero subsemimodule of M_R.We show that N is a prime R-semimodule. Let $v \in N$ and $a \in R$ such that $vRa = (\theta)$. If $v \neq \theta$, then since M is a prime R-semimodule, we have (θ) to be a prime subsemimodule of M. So $a \in A_{(\theta)}(M) = \{a \in R : Ma = (\theta)\} \subseteq \{a \in R : Na = (\theta)\} = A_{(\theta)}(N)$. The above set inclusion exist, because of the fact $N \subseteq M$. Thus (θ) as a subsemimodule of N is also prime. Hence N is prime. □

Proposition 14. *Let R be a semiring, M be a right R-semimodule and K be a proper subsemimodule of M. If K is a prime subsemimodule of M then $A_K(M)$ is a prime ideal of R.*

Proof. Let for $a, b \in R$, $aRb \subseteq A_K(M)$. Assume that $a \notin A_K(M)$, then $Ma \nsubseteq K$, so there exists $v \in M$ such that $va \notin K$. Since $aRb \subseteq A_K(M)$, $M(aRb) \subseteq K$, therefore $v(aRb) \subseteq K$, for all $v \in M \Rightarrow (va)Rb \subseteq K$. Since K is a prime subsemimodule, and $va \notin K$, therefore $b \in A_K(M)$. Hence $A_K(M)$ is a prime ideal of R. □

Remark 2. Above result holds, even if, we replace the notion of prime with semiprime.

Proposition 15. *Let K be a subsemimodule of an R-semimodule M, then for $m \in M$, the set $A_K(m) = \{a \in R : ma \in K\}$ is a right ideal of R.*

Proof. Since $0 \in R$, and $m \cdot 0 = \theta \in K$, so $0 \in A_K(m)$ and so $A_K(m) \neq \emptyset$. Let $a, b \in A_K(m)$ then $ma, mb \in K \Rightarrow ma + mb \in K \Rightarrow m(a + b) \in K$, which implies that $a + b \in A_K(m)$. Now, for $a \in AK(m), ma \in K$, then $(ma)R \subseteq KR$ or $m(aR) \subseteq K$. So, $aR \subseteq AK(m)$, for all $a \in A_K(m) \Rightarrow A_K(m)R \subseteq A_K(m)$. Hence $A_K(m)$ is a right ideal of R. □

Remark 3. Unlike $A_K(M)$, $A_K(m)$ is one sided ideal. Moreover $A_K(M) \subseteq A_K(m)$, because $\{a \in R : Ma \subseteq K\} \subseteq \{a \in R : ma \in K\}$.

Proposition 16. *Let M be an R-semimodule and K be a subsemimodule of M, then $A_K(M) = \cap_{m \in M} A_K(m)$.*

Proof. Let K be a subsemimodule of a right R-semimodule M, then we have to show that $A_K(M) = \cap\{A_K(m) : m \in M\}$. Let $a \in A_K(M)$ which implies that $Ma \subseteq K \Rightarrow ma \in K$, for all $m \in M$, therefore $a \in A_K(m)$, for all $m \in M$. Thus $a \in \cap\{A_K(m) : m \in M\}$. Hence $A_K(M) = \cap\{A_K(m) : m \in M\}$. □

Theorem 4. *Let K be a subsemimodule of an R-semimodule M. If K is prime the $A_K(m)$ for every $m \in M$, is prime right ideal.*

Proof. $A_K(m) = \{a \in R : ma \in K\}$ is a right ideal of R. Now we prove that $A_K(m)$ is prime. Let $a, b \in R$ such that $aRb \subseteq A_K(m)$, with $a \notin A_K(m)$. Therefore $m(aRb) \subseteq K$, so $(ma) \in Rb \subseteq K$. As K is a prime subsemimodule, so $ma \in K$ or $b \in A_K(M) \subseteq A_K(m)$. But $ma \notin K$, as $a \notin A_K(m)$, therefore $b \in A_K(m)$. If $b \notin A_K(m)$, then $m(aRb) \subseteq K \Longrightarrow (ma)Rb \subseteq K$. As K is prime subsemimodule, so $ma \in K$ or $b \in A_K(M) \subseteq A_K(m)$, but $b \notin A_K(m)$, so $ma \in K$, thus $a \in A_K(m)$. Hence $A_K(m)$ is a prime right ideal of R. □

Proposition 17. *Let R be a semiring and M be a right R-semimodule then K is a prime right subsemimodule of M if and only if for all right ideals A of R and for all subsemimodules N of M, $NA \subseteq K$ implies $N \subseteq K$ or $A \subseteq A_K(M)$.*

Proof. Suppose K is a prime right subsemimodule of M_R. If N is a right R-subsemimodule of M and A is a right ideal of R with $NA \subseteq K$. On contrary suppose that $N \subsetneq K$ and $A \subsetneq A_K(M)$. Let $v \in N \backslash K$ and $a \in A \backslash A_K(M)$. Now $NA \subseteq K \Rightarrow (NR)A \subseteq K$, as N is right R-subsemimodule. Therefore $vRa \subseteq K$. But neither $v \in K$ nor $a \in A_K(M) \Rightarrow K$ is not prime, a contradiction. Conversely, suppose that for all right subsemimodules N of M and right ideals A of R, $NA \subseteq K$ implies $N \subseteq K$ or $A \subseteq A_K(M)$. Let $v \in M$, $a \in R$ such that $vRa \subseteq K$. Now $vRa \subseteq (vR)(aR) \subseteq KR \subseteq K$ so $vR \subseteq K$ or $aR \subseteq A_K(M)$. (By hypothesis). So $v \in K$ or $a \in AK(M)$. Hence K is a prime subsemimodule. □

Corollary 1. *For every prime subsemimodule K of R-semimodule M_R, if a subsemimodule I of M_R properly contains K and a right ideal B of R properly contains $A_K(M)$, then $IB \subsetneq K$.*

Proposition 18. *A semiring R is prime if and only if there exists a faithful prime right (left) semimodule M_R.*

Proof. Suppose R is prime, then by definition, the zero ideal (0) of R is prime as an R-subsemimodule of R_R. Thus $A_{(\theta)}(R) = \{a \in R : Ra = (\theta)\} = (0) \Rightarrow R_R$ is faithful. Conversely, suppose that M_R is a faithful prime right semimodule. We have to show that R is a prime semiring, that is (0) is a prime ideal of R. Suppose that $aRb = (0)$, for $a, b \in R$. If $a \neq 0$ then $MaR \neq (\theta)$. For if $MaR = (\theta)$, then $aR \subseteq \{x \in R : Mx = (\theta)\} = (0)$. Thus $a = 0$, which is a contradiction to the assumption. Hence there exits $v \in M$ such that $vaR \neq (\theta)$. But $aRb = (\theta)$. Hence $vaRb = (\theta)$ is a proper R-subsemimodule of M. As M is a prime right R-semimodule and $vaRb = (\theta)$ with $va \neq \theta$, so $b \in A_{(\theta)}(R) = \{x \in R : Mx = (\theta)\} = (0)$. Hence (0) is a prime ideal of R, showing that R is a prime semiring. □

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