Guaranteed Cost Control for a Class of Uncertain Delay Systems with Actuator Failures Based on Switching Method

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Abstract: This paper focuses on the problem of guaranteed cost control for a class of uncertain linear delay systems with actuator failures. When actuators suffer "serious failure" the never failed actuators can not stabilize the system, based on switching strategy of average dwell time method, under the condition that activation time ratio between the system without actuator failure and the system with actuator failures is not less than a specified constant, a sufficient condition for exponential stability and weighted guaranteed cost performance are developed in terms of linear matrix inequalities (LMIs). Finally, as an example, a river pollution control problem illustrates the effectiveness of the proposed approach.

Keywords: Actuator failures, average dwell time, guaranteed cost control, linear matrix inequalities (LMIs), switched delay system.

1. INTRODUCTION

Time delay is a common phenomenon encountered in engineering control. Also, we notice that time delay is frequently a source of instability and often deteriorates system performance. Recent years have witnessed an enormous growth of interest in stability analysis [1-5] and controller syntheses [6-8].

On the other hand, when controlling a real plant, it is always desirable to design a control system which is not only asymptotically stable but also guarantees an adequate level of performance. One way to address the robust performance problem is to consider a linear quadratic cost function. This approach is the so-called guaranteed cost control [9]. Since the work of Chang and Peng, this issue has been addressed extensively [10-13]. Owing to the growing demands of system reliability in aerospace and industrial process, the study of reliable control has recently attracted considerable attention. Therefore, the problem of reliable guaranteed cost control attracts more and more research interests in recent years [14-16]. However, these reliable control design methods are all based on a basic assumption that the never failed actuators can stabilize a given system. For the case where actuators suffer "serious failure" the never failed actuators can not stabilize the system, the existing design methods of reliable control do not work. In our recent work [17], switching technique is introduced to deal with this case and a design method is obtained by using multiple Lyapunov function method. But for the case where actuators suffer "serious failure" in delay systems, no results have been available up to now.

In this paper, we introduce switching strategy to solve the reliable guaranteed cost control problem for delay systems with "serious failed" actuators. The average dwell time method, which has been shown an effective tool in the study of stability analysis for hybrid or switched systems [18-22], is adopted to design controllers such that the closed-loop system satisfies guaranteed cost control in presence of seriously failed actuators. Finally, a numerical example is given to show the effectiveness of the proposed method. Although the idea that switching technique is introduced to deal with the problem of "serous failure" comes from [17], there are three features in this paper. First, delay effect is considered while it is neglected in [17]; second, the conditions of exponential stability and guaranteed cost control are developed while asymptotical stability and H_∞ performance are considered in [17]; the tool used in this paper is the average dwell time method while multiple Lyapunov method is adopted in [17].

In this paper, $\|x(t)\|$ denotes the usually 2-norm and $\|x_t\| = \sup_{-h\le\theta\le 0} \|x(t+\theta)\|$. "*"denotes the symmetric block in one symmetric matrix. $\lambda_{\max}(S)$ and $\lambda_{\min}(S)$ denote the maximum and minimum

Manuscript received March 23, 2006; revised November 18 2006; accepted July 26, 2007. Recommended by Editorial Board member Kim A. Stelson under the direction of Editor Jae Weon Choi. This work was supported by the NSF of China under grants 60574013.

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eigenvalues of matrix "S", respectively.

2. PROBLEM FORMULATION

We consider the following uncertain linear system

$$\dot{x}(t) = (A + \Delta A)x(t) + Ex(t-h) + Bu(t),\\ x_{t_0}(\theta) = \varphi(\theta),\ \theta \in [-h, 0], \tag{1}$$

where $x \in R^n$ is the state, $u \in R^q$ is the control input, A, B, E are constant matrices of appropriate dimensions, $\varphi(\theta)$ is a differentiable vector-valued initial function on $[-h, 0]$, $h > 0$ denotes the state delay, ΔA is a real-valued matrix representing time-varying parameter uncertainties satisfying

$$\Delta A = DF(t)N$$

for some known constant matrices D, N, $F(t)$ is an unknown matrix function satisfying

$$F^T(t)F(t) \le I.$$

Actuator failures are assumed to occur within a prescribed subset of control channels. We classify actuators of the system (1) into two groups. One is a set of actuators susceptible to failures, which is denoted by $\Omega \subseteq \{1, 2, \cdots, q\}$, these actuators may occasionally fail. The other is a set of actuators robust to failure, which is denoted by $\bar{\Omega} \subseteq \{1, 2, \cdots, q\}\ -\Omega$. Using these notations we introduce the decomposition

$$B = B_{\bar{\Omega}} + B_{\Omega}, \tag{2}$$

where $B_{\bar{\Omega}}, B_{\Omega}$ are formed from B by zeroing out columns corresponding Ω, $\bar{\Omega}$ respectively.

Let $\omega \subseteq \Omega$ correspond to a particular subset of susceptible actuators that actually experience failures. Now, introduce the decomposition similar to (2):

$$B = B_{\bar{\omega}} + B_{\omega},$$

where $B_{\bar{\omega}}, B_{\omega}$ are formed from B by zeroing out columns corresponding $\omega, \bar{\omega}$ respectively. Thus the following inequalities can be easily obtained,

$$B_{\bar{\Omega}}B_{\bar{\Omega}}^T \le B_{\bar{\omega}}B_{\bar{\omega}}^T,\ B_{\omega}B_{\omega}^T \le B_{\Omega}B_{\Omega}^T. \tag{3}$$

Remark 1: In the study of reliable control, there is a usual assumption that $(A, B_{\bar{\omega}})$ is controllable [14-16] since it is easier to design a controller in this case. Here remove this assumption to cover more general situations including both cases of $(A, B_{\bar{\omega}})$ being controllable and being uncontrollable.

Without loss of generality, we consider the case of $(A, B_{\bar{\omega}})$ being uncontrollable in this paper.

Suppose that the faulty actuators can be recovered through a time interval. Then, the state of the system is dominated by the following piecewise differential equation:

$$\dot{x} = \begin{cases} (A + \Delta A)x + Ex(t-h) + Bu \\ (A + \Delta A)x + Ex(t-h) + B_{\bar{\omega}}u. \end{cases} \tag{4}$$

We design two kinds of state feedback controllers: one is for the system without actuator failure; the other is for the system with actuator failures. Therefore, the system (4) can be rewritten into the following switched delay system

$$\dot{x}(t) = (A + \Delta A)x(t) + Ex(t-h) + B_{\sigma(t)}u_{\sigma(t)},\\ x_{t_0}(\theta) = \varphi(\theta),\ \theta \in [-h, 0], \tag{5}$$

where
$\sigma(t) : [0, +\infty) \to M = \{1, 2\}$, $B_1 = B, B_2 = B_{\bar{\omega}}$.

We design state feedback controllers for switched system (5) in the following state feedback form:

$$u_{\sigma} = K_{\sigma}x, \tag{6}$$

where K_i $(i = 1, 2)$ are controller gains.

Remark 2: For the self-repairing control systems and fault tolerant control systems, monitoring the system and detecting the instance of actuator failure (i.e., identify which discrete state i) can be realized by the adaptive detection observer or sliding mode observers, and so on (see for example, [23,24]).

Definition 1: The system (5) is said to be exponentially stable under switching signal σ if the solution $x(t)$ of the system (5) satisfies

$$\|x(t)\| \le \Gamma e^{-\gamma(t-t_0)}\|x_{t_0}\|$$

for constants $\Gamma \ge 1$ and $\gamma > 0$.

Motivated by the idea of weighted disturbance attenuation in [20], the weighted cost function associated with the system (5) is given by

$$J = \int_0^{+\infty} e^{-2\lambda t}[x(t)^T Qx(t) + u_{\sigma(t)}^T Ru_{\sigma(t)}]dt, \tag{7}$$

where λ is a positive constant, Q and R are positive definite weighted matrices.

Now, the weighted guaranteed cost control problem for the switched system (5) is stated as follows:

Definition 2: Consider the system (5). If there exist a control law u_i^* for each subsystem and a switching law $\sigma(t)$, and a positive scalar J^* such that for all admissible uncertainties, the closed-loop system is asymptotically stable and the value of the cost function (7) satisfies $J \le J^*$, then the system (5) is said to satisfy weighted guaranteed cost control, J^* is said to be a weighted guaranteed cost upper bound.

Definition 3 [18,19]: For any switching signal σ and any $t \ge \tau \ge 0$, let $N_\sigma(\tau,t)$ denote the number of discontinuities of σ on an interval (τ,t). If

$$N_\sigma(\tau,t) \le N_0 + \frac{t-\tau}{\tau_a} \tag{8}$$

holds for given $N_0 \ge 0$, $\tau_a > 0$, then the constant τ_a is called the average dwell-time and N_0 is the chatter bound. As commonly used in the literature, for convenience, we choose $N_0 = 0$ in this paper.

Let $T^+(t)$ (resp., $T^-(t)$) denote the total activation time of the system with actuator failures (resp. the system without actuator failures,) during $[t_0,t)$. For any given $\lambda \in (0,\lambda_0)$, we choose an arbitrary $\lambda^* \in (\lambda,\lambda_0)$. Motivated by the idea in [20], we propose the following switching law:

(S) Determine the switching signal $\sigma(t)$ such that the inequality

$$\frac{T^-(t)}{T^+(t)} \ge \frac{\lambda_0+\lambda^*}{\lambda_0-\lambda^*} \tag{9}$$

holds for any given initial time t_0, where λ_0 is a positive number to be chosen later.

Remark 3: The idea of switching condition (S) is in fact to constrain activation time of the system with actuator failures $T^+(t_0,t)$ relatively small compared with that of the system without actuators failure.

3. MAIN RESULTS

In this section, we first consider the non-switched delay system (1). Choose the Lyapunov functional candidate of the form

$$V(x_t) = x^T P x + \int_{t-h}^{t} e^{2\lambda_0(s-t)} x^T(s) Z x(s) ds, \tag{10}$$

where P, Z are positive definite matrices to be chosen later.

Lemma 1: Given constant delay h and positive constants λ_0, ε, if there exist positive definite matrices P and Z such that the following matrix inequality

$$\begin{pmatrix} \Theta & PE \\ E^T P & -e^{-2\lambda_0 h} Z \end{pmatrix} < 0 \tag{11}$$

hold, where

$$\Theta = Q + Z + PA + A^T P + \varepsilon PDD^T P + \varepsilon^{-1} N^T N + 2\lambda_0 P - PBR^{-1}B^T P,$$

then, under the state feedback control $u = Kx$ with $K = -R^{-1}B^T P$, we have

$$V(x_t) \le e^{-2\lambda_0(t-t_0)} V(x_{t_0}) - \int_{t_0}^{t} e^{-2\lambda_0(t-s)} [x^T(s) Q x(s) + u^T(s) R u(s)] ds.$$

Proof: See the Appendix.

When actuators failure occurs, the system (1) becomes the form

$$\begin{aligned} \dot{x}(t) &= (A+\Delta A)x(t) + Ex(t-h) + B_{\bar{\omega}} u, \\ x_{t_0}(\theta) &= \varphi(\theta), \quad \theta \in [-h,0]. \end{aligned} \tag{12}$$

By designing state feedback control $u = Kx$ with $K = -R^{-1}B_{\bar{\omega}}^T P$, we have the following result.

Lemma 2: Given constant delay h and positive constant λ_0, ε, assume that there exist positive definite matrices P and Z such that the following matrix inequality

$$\begin{pmatrix} \Pi & PE \\ E^T P & -e^{-2\lambda_0 h} Z \end{pmatrix} < 0 \tag{13}$$

hold, where

$$\Pi = Q + Z + PA + A^T P + \varepsilon PDD^T P + \varepsilon^{-1} N^T N - 2\lambda_0 P - PB_{\bar{\Omega}} R^{-1} B_{\bar{\Omega}}^T P,$$

then, we have

$$V(x_t) \le e^{2\lambda_0(t-t_0)} V(x_{t_0}) - \int_{t_0}^{t} e^{2\lambda_0(t-s)} [x^T(s) Q x(s) + u^T(s) R u(s)] ds.$$

Proof: See the Appendix.

Remark 4: Lemma 1 gives some decay estimate for Lyapunov functional candidate $V(x_t)$ in (10), while Lemma 2 gives some estimate of exponential growth for $V(x_t)$. These estimates will be used to develop the main result.

Theorem 1: If there exist a set of positive scalars ε, λ_0 and positive definite matrices P_i, Z_i $(i = 1,2)$ such that the matrix inequalities

$$\begin{pmatrix} \Xi_1 & P_1E \\ E^TP_1 & -e^{-2\lambda_0 h}Z_1 \end{pmatrix} < 0, \tag{14}$$

$$\begin{pmatrix} \Xi_2 & P_2E \\ E^TP_2 & -e^{-2\lambda_0 h}Z_2 \end{pmatrix} < 0, \tag{15}$$

hold, where

$$\begin{aligned}\Xi_1 &= Q + P_1A + A^TP_1 + \varepsilon P_1DD^TP_1 + \varepsilon^{-1}N^TN + Z_1 \\ &\quad + 2\lambda_0P_1 - P_1BR^{-1}B^TP_1, \\ \Xi_2 &= Q + P_2A + A^TP_2 + \varepsilon P_2DD^TP_2 + \varepsilon^{-1}N^TN + Z_2 \\ &\quad - 2\lambda_0P_2 - P_2B_{\bar{\Omega}}R^{-1}B_{\bar{\Omega}}^TP_2.\end{aligned}$$

Then, the system (5) under the state feedback controllers (6) with $K_i = -R^{-1}B_i^TP_i$ is exponentially stable for any switching signal σ satisfying the condition (S) and the average dwell time

$$\tau_a \ge \tau_a^* = \frac{\ln\mu}{2\lambda}, \tag{16}$$

where $\mu \ge 1$ satisfies

$$P_i \le \mu P_j, \quad Z_i \le \mu Z_j, \quad \forall i,j \in M. \tag{17}$$

Moreover, a weighted guaranteed cost upper bound is given by

$$J^* = \frac{\lambda_0}{\lambda^*}[x_0^TP_{\sigma(t_0)}x_0 + \int_{-h}^{0} e^{2\lambda_0 s}x^T(s)Z_{\sigma(t_0)}x(s)ds]. \tag{18}$$

Proof: Define a piecewise Lyaponov functional candidate for system (5) as follows

$$\begin{aligned}V(x_t) &= V_{\sigma(t)}(x_t) \\ &= x^TP_{\sigma(t)}x + \int_{t-h}^{t} e^{2\lambda_0(s-t)}x^T(s)Z_{\sigma(t)}x(s)ds,\end{aligned} \tag{19}$$

where P_i, Z_i $(i=1,2)$ satisfying (14) and (15).
According to (17) and the definition of $V_{\sigma(t)}(x_t)$ in (19), we can easily obtain

$$V_i \le \mu V_j, \quad \forall i,j \in M. \tag{20}$$

For any given $t > 0$, we let

$$0 = t_0 < t_1 < \cdots < t_k = t_{N_{\sigma(t_0,t)}}$$

denote the switching time instants of σ over the interval (t_0,t). Using (14), (15), Lemma 1 and Lemma 2, we have

$$V(x_t) = V_{\sigma(t)}(x_t) = V_i(x_t)$$

$$\le \begin{cases} e^{-2\lambda_0(t-t_k)}V_i(x_{t_k}) - \int_{t_k}^{t} e^{-2\lambda_0(t-s)}\Psi_i(s)ds, & i=1, \\ e^{2\lambda_0(t-t_k)}V_i(x_{t_k}) - \int_{t_k}^{t} e^{2\lambda_0(t-s)}\Psi_i(s)ds, & i=2, \end{cases} \tag{21}$$

where $\Psi_i(s) = x^T(s)Qx(s) + u_i^T(s)Ru_i(s)$.
Combining (20) and (21) leads to

$$\begin{aligned}V(x_t) &\le e^{2\lambda_0T^+(t_k,t)-2\lambda_0T^-(t_k,t)}V_{\sigma(t_k)}(x_{t_k}) \\ &\quad - \int_{t_k}^{t} e^{2\lambda_0T^+(s,t)-2\lambda_0T^-(s,t)}\Psi_{\sigma(t_k)}(s)ds \\ &\le e^{2\lambda_0T^+(t_k,t)-2\lambda_0T^-(t_k,t)}\mu V_{\sigma(t_k^-)}(x_{t_k^-}) \\ &\quad - \int_{t_k}^{t} e^{2\lambda_0T^+(s,t)-2\lambda_0T^-(s,t)}\Psi_{\sigma(t_k)}(s)ds \\ &\le e^{2\lambda_0T^+(t_{k-1},t)-2\lambda_0T^-(t_{k-1},t)}\mu V_{\sigma(t_{k-1})}(x_{t_{k-1}}) \\ &\quad - e^{2\lambda_0T^+(t_k,t)-2\lambda_0T^-(t_k,t)}\mu \cdot \\ &\quad \int_{t_{k-1}}^{t_k} e^{2\lambda_0T^+(s,t)-2\lambda_0T^-(s,t)}\Psi_{\sigma(t_{k-1})}(s)ds \\ &\quad - \int_{t_k}^{t} e^{2\lambda_0T^+(s,t)-2\lambda_0T^-(s,t)}\Psi_{\sigma(t_k)}(s)ds \\ &\le \cdots \\ &\le \mu^{N_\sigma(0,t)}e^{2\lambda_0T^+(0,t)-2\lambda_0T^-(0,t)}V_{\sigma(t_0)}(x_0) \\ &\quad - \int_0^t \mu^{N_\sigma(s,t)}e^{2\lambda_0T^+(s,t)-2\lambda_0T^-(s,t)}\Psi_{\sigma(s)}(s)ds \\ &= e^{2\lambda_0T^+(0,t)-2\lambda_0T^-(0,t)+N_\sigma(0,t)\ln\mu}V_{\sigma(t_0)}(x_0) \\ &\quad - \int_0^t e^{2\lambda_0T^+(s,t)-2\lambda_0T^-(s,t)+N_\sigma(s,t)\ln\mu}\Psi_{\sigma(s)}(s)ds.\end{aligned} \tag{22}$$

First, we give the proof of the exponential stability for switched delay system (5).
According to (8) and (16), we have

$$N_\sigma(0,t)\ln\mu \le 2\lambda t, \quad \forall t > 0. \tag{23}$$

Therefore, it follows from (22) and (23) that

$$\begin{aligned}V(x_t) &\le e^{2\lambda_0T^+(0,t)-2\lambda_0T^-(0,t)+N_\sigma(0,t)\ln\mu}V_{\sigma(t_0)}(x_0) \\ &\le e^{2\lambda_0T^+(0,t)-2\lambda_0T^-(0,t)+2(\lambda^*-\lambda)t}V_{\sigma(t_0)}(x_0) \\ &\le e^{-2\lambda^*t+2(\lambda^*-\lambda)t}V_{\sigma(t_0)}(x_0) \\ &= e^{-2\lambda t}V_{\sigma(t_0)}(x_0).\end{aligned} \tag{24}$$

From the Lyapunov functional in (19), we have

$$a\|x(t)\|^2 \le V(x_t) \le b\|x_t\|^2, \tag{25}$$

where

$$a = \min_{i\in M} \lambda_{\min}(P_i),$$
$$b = \max_{i\in M} \lambda_{\max}(P_i) + h \max_{i\in M} \lambda_{\max}(Z_i).$$

Using (24) and (25), we get

$$\|x(t)\|^2 \le \frac{1}{a} V(x_t) \le \frac{b}{a} e^{-2\lambda t} \|x_0\|^2.$$

Therefore,

$$\|x(t)\| \le \sqrt{\frac{b}{a}} e^{-\lambda t} \|x_0\|, \tag{26}$$

which implies the system (5) is exponentially stable.

In the following, we show that the closed-loop system satisfies the performance upper bound. Multiplying both sides of (22) by $e^{-N_\sigma(0,t)\ln\mu}$ results in

$$\begin{aligned} &e^{-N_\sigma(0,t)\ln\mu} V(x_t) \\ &\le e^{2\lambda_0 T^+(0,t) - 2\lambda_0 T^-(0,t)} V_{\sigma(t_0)}(x_0) \\ &\quad - \int_0^t e^{2\lambda_0 T^+(s,t) - 2\lambda_0 T^-(s,t) - N_\sigma(0,s)\ln\mu} \Psi_{\sigma(s)} ds \qquad (27) \\ &\le e^{-2\lambda^* t} V_{\sigma(t_0)}(x_0) \\ &\quad - \int_0^t e^{2\lambda_0 T^+(s,t) - 2\lambda_0 T^-(s,t) - N_\sigma(0,s)\ln\mu} \Psi_{\sigma(s)}(s) ds. \end{aligned}$$

From (23), (27) can be written as

$$e^{-2\lambda t} V(x_t) \le e^{-2\lambda^* t} V_{\sigma(t_0)}(x_0) - \int_0^t e^{2\lambda_0 T^+(s,t) - 2\lambda_0 T^-(s,t) - 2\lambda s} \Psi_{\sigma(s)}(s) ds.$$

Note that $e^{-2\lambda t} V(x_t) \ge 0$, we know

$$\int_0^t e^{2\lambda_0 T^+(s,t) - 2\lambda_0 T^-(s,t) - 2\lambda s} \Psi_{\sigma(s)}(s) ds \le e^{-2\lambda^* t} V_{\sigma(t_0)}(x_0).$$

Therefore,

$$\int_0^t e^{-2\lambda_0 (t-s) - 2\lambda s} \Psi_{\sigma(s)}(s) ds \le e^{-2\lambda^* t} V_{\sigma(t_0)}(x_0). \tag{28}$$

Now, integrating (28) from $t = 0$ to ∞ yields

$$\begin{aligned} &\int_0^\infty \left(\int_0^t e^{-2\lambda_0 (t-s) - 2\lambda s} \Psi_{\sigma(s)}(s) ds \right) dt \\ &= \int_0^\infty e^{-2\lambda s} \Psi_{\sigma(s)}(s) \left(\int_s^\infty e^{-2\lambda_0 (t-s)} dt \right) ds \\ &= \frac{1}{2\lambda_0} \int_0^\infty e^{-2\lambda s} \Psi_{\sigma(s)}(s) ds. \end{aligned}$$

It obviously holds that

$$\begin{aligned} &\frac{1}{2\lambda_0} \int_0^\infty e^{-2\lambda s} \Psi_{\sigma(s)}(s) ds \\ &\le \int_0^\infty e^{-2\lambda^* t} V_{\sigma(t_0)}(x_0) dt = \frac{1}{2\lambda^*} V_{\sigma(t_0)}(x_0). \end{aligned}$$

Thus, we have

$$\begin{aligned} &\int_0^\infty e^{-2\lambda s} \Psi_{\sigma(s)}(s) ds \\ &\le \frac{\lambda_0}{\lambda^*} V_{\sigma(t_0)}(x_0) \\ &= \frac{\lambda_0}{\lambda^*} [x_0^T P_{\sigma(t_0)} x_0 + \int_{-h}^0 e^{2\lambda_0 s} x^T(s) Z_{\sigma(t_0)} x(s) ds]. \end{aligned}$$

This is the end of proof. □

Remark 5: The controller designed is switching controller, in which switching law must satisfy two conditions. One is the condition (S), which constrains the "serious failure" time not too large; the other condition is about average dwell time, which is a constraint of "serious failure" frequency on actuators. Therefore, Theorem 1 indicates that the system (1) can satisfy guaranteed cost control on condition that "serious failure" time is correspondingly shorter and "serious failure" frequency is also correspondingly lower.

Remark 6: When $\mu = 1$, namely, $\tau_a^* = 0$, which implies that switching signals can be arbitrary ones and a common Lypunov functional is formed. In this case, the switched system (1) satisfies guaranteed cost control under arbitrary switching. Moreover, setting $\lambda = 0$, which means no switching between subsystems, the weighted guaranteed cost control degenerates into a regular guaranteed cost control problem without weighting $e^{-\lambda\tau}$ for a single subsystem.

Theorem 1 provides a sufficient condition for the solution to the weighted guaranteed cost control problem. However, inequalities (14) and (15) are not easy to solve since they are not LMIs. The following remark shows how to turn (14) and (15) into LMIs, which can be easily solved by Matlab.

Remark 7: Pre- and post-multiplying both sides of inequalities (14) and (15) by $diag\{X_1, X_1\}$ and $diag\{X_2, X_2\}$, respectively, where $X_i = P_i^{-1}$ ($i = 1,2$), we obtain

$$\begin{pmatrix} X_1 Q X_1 + A X_1 + X_1 A^T \\ + \varepsilon D D^T + \varepsilon^{-1} X_1 N^T N X_1 & E X_1 \\ + X_1 Z_1 X_1 + 2\lambda_0 X_1 - B R^{-1} B^T & \\ * & -e^{-2\lambda_0 h} X_1 Z_1 X_1 \end{pmatrix} < 0, \tag{29}$$

$$\begin{pmatrix} X_2QX_2 + AX_2 + X_2A^T + \\ \varepsilon DD^T + \varepsilon^{-1}X_2N^TNX_2 + & EX_2 \\ X_2Z_2X_2 - 2\lambda_0 X_2 - B_{\bar{\Omega}}R^{-1}B_{\bar{\Omega}}^T & \\ * & -e^{-2\lambda_0 h}X_2Z_2X_2 \end{pmatrix} < 0. \tag{30}$$

We define $X_1Z_1X_1 = Y_1, X_2Z_2X_2 = Y_2$. According to Schur complement Lemma, matrix inequalities (29) and (30) are equivalent to the following LMIs

$$\begin{pmatrix} AX_1 + X_1A^T + \varepsilon DD^T \\ +Y_1 + 2\lambda_0 X_1 - BR^{-1}B^T & EX_1 & X_1 & X_1N^T \\ * & -e^{-2\lambda_0 h}Y_1 & 0 & 0 \\ * & * & -Q^{-1} & 0 \\ * & * & * & -\varepsilon I \end{pmatrix} < 0, \tag{31}$$

$$\begin{pmatrix} AX_2 + X_2A^T + Y_2 \\ +\varepsilon DD^T - 2\lambda_0 X_2 & EX_2 & X_2 & X_2N^T \\ -B_{\bar{\Omega}}R^{-1}B_{\bar{\Omega}}^T \\ * & -e^{-2\lambda_0 h}Y_2 & 0 & 0 \\ * & * & -Q^{-1} & 0 \\ * & * & * & -\varepsilon I \end{pmatrix} < 0. \tag{32}$$

Remark 8: Note that in LMIs (31) and (32) ε can be regarded as a variable. In addition, in order to get a lower guaranteed bound, we need a larger μ and a smaller λ_0, which can be realized by parameter iterative method. But this results in larger average dwell time in (16), which is of course undesirable. Thus, we need to select the parameters μ and λ_0 according to practical requirement.

4. EXAMPLE

In this section, we apply the proposed design method to illustrate the river pollution control problem.

Let $z(t)$ and $q(t)$ denote the concentrations per unit volume of biochemical oxygen demand (BOD) and dissolved oxygen (DO), respectively, at time t, in a reach of a polluted river. Let z^* and q^*, corresponding to some measure of water quality standards, denote the desired steady values of z and q, respectively. Define

$$x_1(t) = z(t) - z^*, x_2(t) = q(t) - q^*,$$
$$x(t) = \begin{pmatrix} x_1(t) & x_2(t) \end{pmatrix}^T.$$

Then the dynamic equation for x is described by [25,26].

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A)x(t) + (E_1 + \Delta E_1)x(t - h_1) \\ &\quad + (E_2 + \Delta E_2)x(t - h_2) + Bu + w, \end{aligned} \tag{33}$$

where

$$A = \begin{pmatrix} -k_{10} - \eta_1 - \eta_2 & 0 \\ -k_{30} & -k_{20} - \eta_1 - \eta_2 \end{pmatrix},$$

$$\Delta A(t) = \begin{pmatrix} -\Delta k_1(t) & 0 \\ -\Delta k_3(t) & -\Delta k_2(t) \end{pmatrix}, \quad B = \begin{pmatrix} \eta_1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} \beta_0 \eta_2 & 0 \\ 0 & \beta_0 \eta_2 \end{pmatrix}, \quad \Delta E_1 = \begin{pmatrix} \eta_2 \Delta\beta & 0 \\ 0 & \eta_2 \Delta\beta \end{pmatrix},$$

$$E_2 = \begin{pmatrix} (1-\beta_0)\eta_2 & 0 \\ 0 & (1-\beta_0)\eta_2 \end{pmatrix},$$

$$\Delta E_2 = \begin{pmatrix} -\eta_2 \Delta\beta & 0 \\ 0 & -\eta_2 \Delta\beta \end{pmatrix},$$

$$w = \begin{pmatrix} v_1(t) - \Delta k_1(t) z^* \\ v_2(t) - \Delta k_3(t) z^* + \Delta k_2(t)(q^s - q^*) \end{pmatrix},$$

$u(t) = \begin{pmatrix} u_1(t) & u_2(t) \end{pmatrix}^T$ is the control variables of river pollution. The physical meaning of these parameters can be found in [25].

When $w = 0, h_1 = h_2$, (33) can be rewritten as the following uncertain linear delay system

$$\dot{x}(t) = (A + \Delta A)x(t) + Ex(t - h) + Bu, \tag{34}$$

where $E = \begin{pmatrix} \eta_2 & 0 \\ 0 & \eta_2 \end{pmatrix}$.

In this simulation, we choose $k_{10} = 0.8$, $k_{20} = 1$, $k_{30} = 1.5$, $\eta_1 = -0.4$, $\eta_2 = -0.6$, $\Delta k_3(t) = -0.1\sin t$, $\Delta k_1(t) = \Delta k_2(t) = -0.04\sin t$, $h = 0.2$. Thus

$$A = \begin{pmatrix} 0.2 & 0 \\ -1.6 & -0.5 \end{pmatrix}, \quad B = \begin{pmatrix} -0.4 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Omega = \{1\},$$

$$B_{\bar{\Omega}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} -0.6 & 0 \\ 0 & -0.6 \end{pmatrix},$$

the parameter uncertainties $\Delta A(t) = DF(t)N$, where $F(t) = \sin t$, $D = \begin{pmatrix} 0.2 & 0 \\ 0.5 & 0.1 \end{pmatrix}$, $N = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.4 \end{pmatrix}$.

Suppose that the faulty actuators can be recovered through a time interval, the guaranteed cost control problem of system (34) can be solved by using Theorem 1 Choosing $\lambda_0 = 2$, $\varepsilon = 1$, we get positive definition matrices P_i, Z_i $(i = 1,2)$ by solving LMIs (31), (32)

$$P_1=\begin{pmatrix} 1.9001 & -0.0416 \\ -0.0416 & 0.4128 \end{pmatrix},$$
$$P_2=\begin{pmatrix} 1.3576 & -0.0265 \\ -0.0265 & 0.2541 \end{pmatrix},$$
$$Z_1=\begin{pmatrix} 10.8364 & -0.2885 \\ -0.2885 & 0.5164 \end{pmatrix},$$
$$Z_2=\begin{pmatrix} 7.3753 & -0.1710 \\ -0.1710 & 0.2611 \end{pmatrix}.$$

Choosing $\mu = 2.4$, $\lambda = 0.5$, $\lambda^* = 2$, from (16), we get $\tau_a^* = \frac{\ln\mu}{2\lambda} = 0.8755$. According to Theorem 1, if the activation time ratio between the system without actuator failure and the system with actuator failures is not less than

$$\frac{T^-(t)}{T^+(t)} \geq \frac{\lambda_0+\lambda^*}{\lambda_0-\lambda^*} = 9,$$

exponential stability is achieved.
Moreover, from (26), we have

$$\|x(t)\| \leq 4.0078e^{-0.5(t-t_0)} \| x_{t_0} \|.$$

Let the initial state of system (34) be $x(t)=(1\ -0.5)^T$ for $-0.2 \leq t \leq 0$. From (18), the weighted guaranteed cost upper bound is

$$J^* = \frac{\lambda_0}{\lambda^*}[x_0^T P_1 x_0 + \int_{-h}^{0} e^{2\lambda_0 s} x^T(s) Z_1 x(s) ds] = 4.3346.$$

5. CONCLUSIONS

In this paper, we have investigated the problem of guaranteed cost control for a class of linear delay systems for the case where actuators suffer failures. We focused on the case that the never failed actuators are inadequate to stabilize the systems by a single state feedback. Suppose that the faulty actuators can be self-repaired through a time interval, the entire system can be regarded as a switched system Based on average dwell time scheme, we have designed the switching state feedback controllers in terms of LMIs such that the considered delay systems is exponentially stable and a weighted guaranteed cost upper is derived.

APPENDIX

Proof of Lemma 1: The derivative of $V(x_t)$ along the trajectory of the delay system (1) is given by

$$\begin{aligned}\dot V(x) &= 2x^T(t)P\dot x(t) + x^T(t)Zx(t) - e^{-2\lambda_0 h}x^T(t-h) \\ &\quad \times Zx(t-h) - 2\lambda_0 \int_{t-h}^{t} e^{2\lambda_0(s-t)} x^T(s)Zx(s)ds \\ &= 2x^T(t)P[(A+\Delta A)x(t) + Ex(t-h) + Bu] \\ &\quad + x^T(t)Zx(t) - e^{-2\lambda_0 h}x^T(t-h)Zx(t-h) \\ &\quad - 2\lambda_0 \int_{t-h}^{t} e^{2\lambda_0(s-t)} x^T(s)Zx(s)ds \\ &\leq x^T(PA + A^TP + \varepsilon PDD^TP + \varepsilon^{-1}N^TN + Z \\ &\quad - 2PBR^{-1}B^TP)x + x^T(t)PEx(t-h) \\ &\quad + x^T(t-h)E^TPx(t) - e^{-2\lambda_0 h}x^T(t-h)Zx(t-h) \\ &\quad - 2\lambda_0 \int_{t-h}^{t} e^{2\lambda_0(s-t)} x^T(s)Zx(s)ds.\end{aligned}$$

Therefore,

$$\begin{aligned}&\dot V + 2\lambda_0 V \\ &\leq x^T(PA + A^TP + \varepsilon PDD^TP + \varepsilon^{-1}N^TN + Z \\ &\quad + 2\lambda_0 P - 2PBR^{-1}B^TP)x + x^T(t)PEx(t-h) \\ &\quad + x^T(t-h)E^TPx(t) - e^{-2\lambda_0 h}x^T(t-h)Zx(t-h) \\ &= \begin{pmatrix} x(t) \\ x(t-h)\end{pmatrix}^T \begin{pmatrix} \Theta - Q - PBR^{-1}B^TP & PE \\ E^TP & -e^{-2\lambda_0 h}Z \end{pmatrix} \begin{pmatrix} x(t) \\ x(t-h)\end{pmatrix}.\end{aligned}$$

From (11), we have

$$\begin{aligned}\dot V &\leq -2\lambda_0 V - x^T(Q + PBR^{-1}B^TP)x \\ &\leq -2\lambda_0 V - (x^TQx + u^TRu).\end{aligned} \tag{35}$$

By using the differential theory and (35) for (10), we have

$$V(x_t) \leq e^{-2\lambda_0(t-t_0)}V(x_{t_0}) - \int_{t_0}^{t} e^{-2\lambda_0(t-s)}[x^T(s)Qx(s) + u^T(s)Ru(s)]ds.$$

Proof of Lemma 2: Similarly to the proof of Lemma 1, differentiating $V(x_t)$ along the trajectory of system (12) results in

$$\begin{aligned}\dot V(x_t) &= 2x^TP[(A+\Delta A)x + Ex(t-h) + BKx] \\ &\quad + x^T(t)Zx(t) - e^{-2\lambda_0 h}x^T(t-h)Zx(t-h) \\ &\quad - 2\lambda_0 \int_{t-h}^{t} e^{2\lambda_0(s-t)} x^T(s)Zx(s)ds.\end{aligned}$$

Note that the outputs of faulty actuators corresponding to any $\omega \subseteq \Omega$ are assumed to be zero, i.e., the control input $u_i(x)$ may be applied to the plant only through normal actuators, we have

$$BK = -B_{\bar{\varpi}}R^{-1}B_{\bar{\varpi}}^{T}P.$$

From (3), we have

$$B_{\bar{\Omega}}R^{-1}B_{\bar{\Omega}}^{T} \le B_{\bar{\varpi}}R^{-1}B_{\bar{\varpi}}^{T}. \tag{36}$$

Therefore,

$$\begin{aligned}\dot{V}(x_t) &= 2x^T P\big[(A+\Delta A)x + Ex(t-h) + B_{\bar{\varpi}}u\big] \\ &\quad + x^T(t)Zx(t) - e^{-2\lambda_0 h}x^T(t-h)Zx(t-h) \\ &\quad - 2\lambda_0\int_{t-h}^{t} e^{2\lambda_0(s-t)}x^T(s)Zx(s)ds \\ &\le x^T(PA + A^T P + \varepsilon PDD^T P + \varepsilon^{-1}N^T N \\ &\quad + Z - 2PB_{\bar{\varpi}}R^{-1}B_{\bar{\varpi}}^{T}P)x + x^T(t)PEx(t-h) \\ &\quad + x^T(t-h)E^T Px(t) - e^{-2\lambda_0 h}x^T(t-h)Z \\ &\quad \times x(t-h) - 2\lambda_0\int_{t-h}^{t} e^{2\lambda_0(s-t)}x^T(s)Zx(s)ds.\end{aligned}$$

According to (36) and (13), we have

$$\begin{aligned}&\dot{V} - 2\lambda_0 V \\ &\le x^T(PA + A^T P + \varepsilon PDD^T P + \varepsilon^{-1}N^T N + Z \\ &\quad - 2\lambda_0 P - 2PB_{\bar{\Omega}}R^{-1}B_{\bar{\Omega}}^{T}P)x + x^T(t)PEx(t-h) \\ &\quad + x^T(t-h)E^T Px(t) - e^{-2\lambda_0 h}x^T(t-h)Zx(t-h) \\ &\quad - 4\lambda_0\int_{t-h}^{t} e^{2\lambda_0(s-t)}x^T(s)Zx(s)ds \\ &\le x^T(PA + A^T P + \varepsilon PDD^T P + \varepsilon^{-1}N^T N + Z \\ &\quad - 2\lambda_0 P - 2PB_{\bar{\Omega}}R^{-1}B_{\bar{\Omega}}^{T}P)x + x^T(t)PEx(t-h) \\ &\quad + x^T(t-h)E^T Px(t) - e^{-2\lambda_0 h}x^T(t-h)Zx(t-h) \\ &= \begin{pmatrix} x(t) \\ x(t-h)\end{pmatrix}^T \begin{pmatrix} \Pi - Q - PB_{\bar{\Omega}}R^{-1}B_{\bar{\Omega}}^{T}P & PE \\ E^T P & -e^{-2\lambda_0 h}Z\end{pmatrix}\begin{pmatrix} x(t) \\ x(t-h)\end{pmatrix} \\ &\le 0.\end{aligned}$$

Therefore,

$$\dot{V} \le 2\lambda_0 V - (x^T Qx + u^T Ru),$$

which in turn gives

$$\begin{aligned}V(x_t) &\le e^{2\lambda_0(t-t_0)}V(x_{t_0}) \\ &\quad - \int_{t_0}^{t} e^{2\lambda_0(t-s)}[x^T(s)Qx(s) + u^T(s)Ru(s)]ds.\end{aligned}$$

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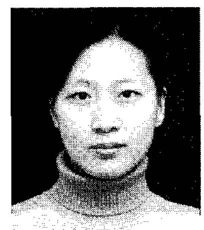

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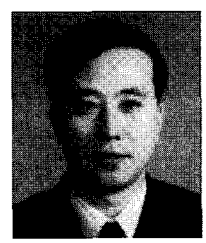

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