대한수학교육학회지 수학교육학연구 제 17 권 제 3 호
Journal of Educational Research in Mathematics
Vol. 17, No. 3, 295-308. Aug 2007

Pedagogical Content Knowledge: A Case Study of a Middle School Mathematics Teacher

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The purpose of this paper was to investigate the pedagogical content knowledge of a middle school mathematics teacher manifested in his mathematics instruction by identifying the components of the pedagogical content knowledge of the teacher. For the purpose of the study, I conducted an interpretive case study by collecting qualitative data. The results showed that the pedagogical content knowledge of the teacher was characterized by: (a) knowledge of mathematics including connection among topics and various ways of solving problems; (b) knowledge of students' understanding involving students' misconceptions, common errors, difficulties, and confusions; and (c) knowledge of pedagogy consisting of his efforts to motivate his students by providing realistic applications of mathematical topics and his use of materials.

I. Introduction

Teachers' knowledge of mathematics affects their mathematics teaching. Knowing mathematics, however, is not sufficient for teaching well. Teaching effectively is, in essence, teaching for understanding. For effective mathematics teaching, mathematics teachers should know and deeply understand mathematics, students as learners, and pedagogical strategies, as well as being capable of using that knowledge flexibly while teaching mathematics (National Council of Teachers of Mathematics [NCTM], 2000). For effective teaching, mathematics teachers should also be continuously exposed to opportunities and resources in order to enhance and refresh their knowledge (NCTM, 2000).

Various questions arise from this view: How is mathematics teachers' knowledge to be enhanced and refreshed? What knowledge makes them expert teachers of mathematics? I believe that mathematics teachers' pedagogical content knowledge is critical for them to teach mathematics well. Pedagogical content knowledge in mathematics is much more than simple knowledge of mathematics (Grossman, 1990; Kahan, Cooper, & Bethea, 2003; Marks, 1990; Tamir, 1988). However, there have been relatively few studies on the pedagogical content knowledge of mathematics teachers and how it might be characterized in their practice. One reason for so few studies is the difficulties and ambiguities that by its nature, pedagogical content knowledge

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contains. Even though the concept of pedagogical content knowledge is difficult to identify distinctly, such identification would be very useful because pedagogical content knowledge "represents a class of knowledge that is central to teachers' work and that would not typically be held by nonteaching subject matter experts or by teachers who know little of that subject" (Marks, 1990, p. 9).

II. Theoretical Framework

Teaching is a quite complex process and activity that is influenced by many kinds of teacher knowledge (Ball, 1991; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Carpenter & Franke, 1996; Even, 1993; Even & Tirosh, 1995; Fernandez, 1997; Leinhardt, 1991; Leinhardt & Greeno, 1986; Ma, 1999; Wilson, Shulman, & Richert, 1987). Although teaching is totally different from knowing, teaching any subject matter depends on knowing that subject matter. In other words, not only is teaching mathematics directly related to knowing mathematics, but also knowing and understanding mathematics is not sufficient for being able to teach mathematics. Teaching mathematics, thus, requires not only knowing mathematics but also knowing mathematics for teaching, which has been called *pedagogical content knowledge* by Shulman (1986).

Research on teacher knowledge in various knowledge domains (Cochran, DeRuiter & King, 1993; Grossman, 1990; Magnusson, Krajcik & Borko, 1999; Marks, 1990; Morine-Dershimer & Kent, 1999; Shulman, 1986, 1987, 2000; Smith & Neale, 1989; Wilson et al., 1987) recognizes the importance of pedagogical content knowledge. Shulman (1986) proposes that pedagogical content knowledge is specific to a particular subject matter and that learning to teach demands not only understanding the subject itself but also developing a large body of pedagogical content knowledge. According to Shulman, pedagogical content knowledge can be characterized as follows:

> The most useful forms of representation of [subject matter] ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others. (p. 9)

He elaborated later that pedagogical content knowledge identifies

> the distinctive bodies of knowledge for teaching. It represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to diverse interests and abilities of learners, and presented for instruction. (Shulman, 1987, p. 8)

In sum, pedagogical content knowledge is a teacher's understanding of how to help students understand mathematics (Magnusson et al., 1999; Morine-Dershimer & Kent, 1999; Wilson et al., 1987). "The idea of pedagogical content knowledge substantially improves our understanding of the knowledge required for teaching. The concept implies that not only must teachers know content deeply, know it conceptually, and know the connections among ideas, but also [they] must know the representations for and the common student difficulties with particular ideas" (Ball, Lubienski, & Mewborn, 2001, p. 449).

By extending Shulman's characterization of pedagogical content knowledge through an investigation of teachers of English and by emphasizing the importance of pedagogical content knowledge (Calderhead, 1996), as not being a subset of content knowledge and pedagogy, Grossman (1990) identified pedagogical content knowledge as including: conceptions of purposes for teaching subject matter; knowledge of students' understanding, including common misconceptions and difficulties; knowledge of instructional strategies, and curricular knowledge. She further contends that knowledge of subject matter "includes beliefs and knowledge about the purposes for teaching a subject" (p. 8) since the "conceptions are reflected in teachers' goals" (p. 8) for teaching the subject. More to the point, Magnusson, Krajcik, and Borko (1999) adopted Grossman's work with modifications and an additional component for the case of science teaching: orientations to teaching a subject, knowledge of curricula, knowledge of students' understanding, knowledge of instructional strategies, and knowledge of assessment.

Shulman uses pedagogical content knowledge as a subset of what teachers need to know about content and about pedagogy (Lampert, 1991) and as static knowledge (Cochran et al., 1993), but Cochran et al. propose to rename pedagogical content knowledge as *pedagogical content knowing* to recognize its dynamic nature. They define pedagogical content knowing as "a teacher's integrated understanding of four components of pedagogy, subject matter content, student characteristics, and the environmental context of learning"(p.266).

Gess-Newsome (1999) explains that pedagogical content knowledge is "a transformation of at least two constituent knowledge domains: general pedagogical knowledge and subject matter knowledge" (p. 5). In the view, pedagogical content knowledge as the only form of knowledge that impacts teaching is the transformation of subject matter, pedagogical, and contextual knowledge into a unique form, which is called as the *transformative model.* According to Gess-Newsome, "the transformative model implies that these initial knowledge bases are inextricably combined into a new form of knowledge, pedagogical content knowledge, in which the parent domain may be discovered only through complicated analysis. Pedagogical content knowledge, therefore, should be "well structured and easily accessible" (p. 13) in order that teachers can help and foster students' understanding concepts and ideas in classroom instruction.

Marks (1990) suggests that pedagogical content knowledge is the offspring of general pedagogical knowledge and subject matter knowledge. He argues, however, that pedagogical content knowledge is not a separate category of teacher knowledge; instead, it is inextricable from content knowledge. By investigating 8 fifth-grade teachers' teaching of equivalence of fractions, he presents a structure of pedagogical content knowledge: (a) subject matter for instructional purposes, (b) students' understanding of the subject matter, (c) media for instruction in the subject matter (texts and materials), and (d) instructional processes for the subject matter. In particular, he proposes that the subject matter knowledge consist of "purposes of math instruction, justifications for learning a given topic, important ideas to teach in a given topic, prerequisite knowledge for a given topic, and typical 'school math' problems" (p. 5).

From the review of the literature, I conceptualize pedagogical content knowledge as knowledge for teaching that is a transformation of knowledge of subject matter, knowledge of students' understanding, and knowledge of pedagogy. This conceptualization. As it is not knowledge for just knowing, pedagogical content knowledge should be not only knowing but also being able to use that knowing in the real contexts of teaching practice (Ball, 1999, 2000; Ball & Bass, 2000; Ball et al., 2001).

I conduct an interpretive case study to examine a middle school mathematics teacher's teaching practice to examine the pedagogical content knowledge of mathematics teachers in terms of how it is manifested in their classroom instruction and ultimately to identify its components. This study investigates the research question: What are the components of the pedagogical content knowledge that is manifested in instruction of a middle school mathematics teacher?

III. Methods

Mr. Cochran (pseudonym) participated in this study. He has taught mathematics in the middle grades—eighth grade primarily—in a southeastern city in the US, for 13 years. He was among about 20 teachers whom I contacted by e-mail and the only one to respond to it.

I collected data for the study from interviews, lesson observations, and classroom artifacts. I observed Mr. Cochran's classroom teaching for 8 consecutive class periods. The purpose of the lesson observations was to examine the pedagogical content knowledge he manifested while teaching. During the classroom observations, I took field notes that described almost everything that happened in the classes. The notes included his introduction to a topic, his representations on the board or overhead, his questions for students, students' responses and questions, verbal and written explanations, and so on. The field notes were later typed. When possible, classroom observations were audio taped.

I interviewed Mr. Cochran twice with semi-structured questions. The interviews were conducted after I observed 4 classes. The interviews consisted of two parts: (1) questions designed to discover his background, conceptions about learning and teaching mathematics, and beliefs about mathematics; and (2) questions designed to elicit his goals for teaching certain topics, his views of common students' conceptions or misconceptions about the topics, and his reflections on the lessons that I observed. The interviews were audio taped and later transcribed for analysis.

Mr. Cochran taught three courses to eighth graders: Algebra, Pre-algebra, and General Mathematics. I observed 6 Algebra and 2 Pre-algebra class periods on consecutive days. During my observations of the Algebra class, he covered a chapter called Radicals, Functions &

Coordinate Geometry. In the Pre-algebra class, he covered Theoretical probability of a chapter called Probability. I obtained copies of the chapters of the textbooks and practice workbooks for Algebra and Pre-algebra. I also got copies of the materials Mr. Cochran designed and used such as review sheets, a quiz from the previous years for a review or practice, and a quiz conducted during my observations.

The data were analyzed using the constant comparative method. The process of the data analysis began with finding keywords from the interview transcripts and categorizing the keywords into a theme emerging from the interviews. Then I classified every problem, comment, question, representation, or symbol into categories derived from the theoretical framework, which included knowledge of subject matter, knowledge of students' understanding, and knowledge of pedagogy. Also, the field notes were analyzed according to the framework. The data were organized with regard to such characterizations. Each category consisted of subcategories that emerged from the data. I used the participant's terms in the data. After I had analyzed those data, I constantly compared the interview data with the field notes, the textbook, and other materials.

IV. Results

1. Knowledge of Mathematics

This study attempts to identify the pedagogical content knowledge of a middle school mathematics teacher as manifested in his mathematics instruction. As stated above, Grossman (1990) and Marks' (1990) suggest that subject matter knowledge include purposes for teaching the subject. In particular, Marks found in his study that knowledge of mathematics include subcomponents such as purposes of math instruction, justifications for learning a given topic, and important ideas to teach in a given topic. Such finding was used for the analyses of the teacher's pedagogical content knowledge manifested in his teaching.

During my observations, Mr. Cochran was teaching radicals and radical equations in the Algebra class and probability in the Pre-Algebra class. He said that the topics of radicals and radical equations are pieces of the puzzle in Algebra I and that the purpose of his teaching those topics was to expose students to abstract thoughts or concepts of mathematics. He held the view that the students in Algebra I must understand the concepts of what squaring a number is, what square root means, how to take a square root, what the square of a number is, and how to get rid of a square root sign. One of the goals of his instruction on radicals was to "bridge the gap" between taking the square root of any positive integer and applying that process to taking the square root of a squared number. Second, he wanted the students to realize that the radical sign was going to disappear when squaring a number that had a radical sign and that they would be able to solve the resulting equation without the radical sign.

His illustration that solving equations with radicals would mean that the students could apply the concepts of radicals to equations with radicals reveals his concerns for the students' deeper understanding of those concepts (Hiebert & Carpenter, 1992; NCTM, 2000). The students should be able to apply the knowledge of how to take the square root not only to a number but also to variables and terms in equations. In other words, the students must be able to transfer the concept of radicals to radical equations. In addition, he said that there were a variety of ways of solving radical equations; there is always one more tool or method to solve the equation. Mr. Cochran's knowledge of mathematics enabled him to explore a problem in several different ways (Sowder & Schappelle, 1995).

Mr. Cochran also declared that concepts of probability are fairly practical, and that they help people see different options and possibilities, which supports the assertion of *Principles and Standards*(NCTM, 2000) that "teachers should give middle-grades students numerous opportunities to engage in probabilistic thinking about simple situations from which students can develop notions of chance"(p.253). To do this, Mr. Cochran engaged his students in an activity of rolling dice and flipping coins that allowed them to recognize how to get a probability. Learning probability can take learners through various processes; for example, they sometimes have to make diagrams or chart data. In Mr. Cochran's explanation, a basic understanding of probability enabled him and his students to think through decisions and apply their knowledge to realistic situations.

Importantly, Mr. Cochran regarded himself as still developing his knowledge about mathematics:

> As I have taught more, as I've read more, I think I've also been challenged by students more, and I just think I increased my knowledge through courses that I've taken. This is my, I think, thirteenth year of teaching, and [I] certainly am just challenged and grow in my math abilities each and every year. I think staff development [activities] is another way that I can become more confident in my knowledge of mathematics.

On the other hand, it was interesting to note that Mr. Cochran also saw himself as having "limited knowledge" or "more of a textbook knowledge" of the concept of radicals; he described that knowledge as follows:

> I guess I would explain that to some degree like a history buff who all of his life has loved history and researched history, gone places, been a world traveler, and he teaches history; he's got a great knowledge to tap into, a great depth [of] knowledge. Another teacher who may be asked to teach one year, who maybe teaches science, for example, who [is] asked to teach history. Their knowledge is probably limited with just what they have been exposed to. They don't have the background knowledge or the depth of knowledge. And I think that I feel that way with some of my content. Radicals would probably be that way. I know what I have read from a book, and I know that I've never been taught as a student myself. But probably having a great depth of knowledge of how radicals can be used in research or the scientific world or in different areas of development [is needed]. I just feel that I would be confident in saying that I'm limited in that.

2. Knowledge of Students' Understanding

In general, Mr. Cochran appeared to emphasize to his students that there are a variety of ways to get an answer, which comes from his beliefs about mathematics. He repeatedly told the students that they needed to learn different approaches to solve a problem and to understand those approaches. Although he did not ask them to try a different way for each problem, he seemed to try to draw various ways of solution from the students. He would get a student to show his or her way. Sometimes students would volunteer to talk or present their approach on the board if they had a different one from the teachers' presentation or other peers' methods. He very often told the students that they did not always have to work in the same way. He urged that they look for different methods and be able to find a lot of ways to arrive at an answer.

In many occasions, Mr. Cochran appeared to apply his understanding of what the students would feel confused about and what was difficult about the topic. When explaining a topic, he always mentioned, "One student may do it in this way," "Another student might do it using cross cancel later," or "One student may confuse it with cross cancel later." He usually asked the students whether they had different approaches. If anyone did, he encouraged the student to present on the board and explain what he or she had done, which revealed his effort to understand a particular student's understanding. Moreover, he told the students that they needed to learn different approaches and needed to find the easiest way to get the answer to a problem.

Mr. Cochran, however, did not make a connection between his claims that there were lots of approaches to an answer and the practice of checking or verifying answers. Although he stressed that students should make sure that the answers were correct, he did not go further than that. It would have been effective if he had guided them to use a different way from one that they already tried to verify answers to a problem. Then the students could have tried and experienced how different approaches would work as compared to the first approach.

When explaining how to simplify radicals, he connected the concept of fractions and used various examples, some of which were rule based. For instance, he asked the students to think about what they did when learning about simplifying fractions and what the rules were in doing so. He reminded and showed the "rules" for fractions, using examples of $\frac{2}{4}=\frac{1}{2}$, $\frac{4}{2}=2$, and $\frac{7}{3}=2\frac{1}{3}$. Further extensions of the multiplication of fractions for radicals were given with examples like $\frac{2}{4}\times\frac{1}{2}$ and $\frac{16}{30}\times\frac{10}{12}$; doing cross cancel first and multiplying later— $\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$, $\frac{4}{3}\times\frac{1}{3}=\frac{4}{9}$ —or multiplying fractions first and doing cross cancel later— $\frac{2}{8}=\frac{1}{4}$. He explained the possible processes that students might perform according to how comfortable they were with each.

Mr. Cochran often asked questions to elicit the students' thoughts, misconceptions, or errors as examples. After he had given a test on radicals, he went through common errors that the students made on the test. He picked out examples from the test and the students asked questions. For instance, he wrote $\frac{4+\sqrt{20}}{2}=\frac{4}{2}+\frac{\sqrt{20}}{2}$ on the board, explaining that simplifying and factoring skills were required. The students were confused when simplifying radicals in this form. When a student asked if her way would be fine, he examined it without implying whether her solution was right or wrong: $\frac{4+\sqrt{20}}{2}=2+\sqrt{20}=2+2\sqrt{5}=4+\sqrt{5}$. After going over this, he asked, "Is that, $2+\sqrt{20}$, possible?" to draw out the students' misconceptions and errors. He presented an example of a fraction in such a form: $\frac{2+6}{2}$. He demonstrated in two ways: first, 6 from canceling out both 2 in numerator and denominator; and $\frac{2}{2}+\frac{6}{2}=1+3=4$ by asking what $2+6$ is. Then he showed the problem in the way

$$\begin{aligned}&\frac{4+\sqrt{20}}{2}\\ &=\frac{4+2\sqrt{5}}{2}\\ &=\frac{4}{2}+\frac{2\sqrt{5}}{2}\\ &=2+\sqrt{5}\end{aligned}$$

It was apparent that he was using his knowledge about what the students would feel confused about and what was difficult about the topic. Extending the example above, he demonstrated another problem $\frac{2\sqrt{3}+2\sqrt{7}}{2}=\frac{2(\sqrt{3}+\sqrt{7})}{2}=\sqrt{3}+\sqrt{7}$ (he then crossed out the two 2 in both numerator and denominator). One of the students said, "It is confusing." Mr. Cochran acknowledged that it was confusing. Then he presented several more examples that might be confusing to the students. He revisited fractions and showed a wrong solution. Also, when a student asked if an answer to a radical equation problem $\sqrt{12-x}=x$ could be -4 as well(the students already knew 3 was an answer), he posed the question "Can you have negative value of square root?" showing $\sqrt{12-(-4)}=4\neq-4$. This provided the student with an opportunity to see and reflect upon his reasoning and confusions about the concept.

3. Knowledge of Pedagogy

To motivate the students' understanding of a certain topic, Mr. Cochran seemed to try to select and use examples of an application in a real world context. When introducing a Pythagorean Theorem, he posed a problem: A teenager gets a long pole of 5ft at a store and has to take a bus to get home. But it is allowed to carry a maximum of 4ft long on the bus. How could he or she resolve the problem? Although no students got the answer, the students seemed to really be engaged in working the problem. He appeared to expect them to get into the Pythagorean Theorem by being interested and excited, working the example and seeing how mathematics would be applied in the real world. Mr. Cochran liked the problem, and it was the first time he used it. He was asked the problem by a person at a school party and could not figure it out until he was given a hint to use the Pythagorean Theorem.

To teach the topics, Mr. Cochran followed the organization of the textbook. The chapter Radicals, Functions, and Coordinate Geometry consisted of 8 topics: (1) Operations with Radicals; (2) Square-root Functions and Radical Equations; (3) Pythagorean Theorem; (4) Distance Formula; (5) Geometric Properties; (6) Tangent Function; (7) Sine and Cosine Functions; and (8)

Introduction to Matrices. He spent one class periods to cover Theoretical Probability.

However, the organization of each lesson did not correspond to the one suggested in the textbook. Although he used example and practice problems from the textbook, he reorganized the concepts in a topic or made a connection between the concepts and the students' previous knowledge. For example, Mr. Cochran used a handwritten handout for the lesson on operations with radicals that included a brief explanation of simplifying "rules" and lists of problems. Some of the problems were from the textbook, and the rest were similar. He demonstrated his solution to a couple of problems on the handout.

Discussing simplifying, the textbook says the following:

Because the radical sign designates the principle square root, the value of $\sqrt{x^2}$ must be positive. Use the absolute value sign to indicate this when the exponent of a variable in the radical is even and the simplified exponent outside of theradical is odd.

Mr. Cochran, in contrast, extended the examples in the textbook, and illustrated them on the board:

$$\sqrt{x^2} = x$$
$$\sqrt{x^3} = x\sqrt{x}$$
$$\sqrt{x^4} = x^2$$
$$\sqrt{x^5} = x^2\sqrt{x}$$
$$\sqrt{x^6} = x^3$$
$$\sqrt{x^7} = x^3\sqrt{x}$$
$$\sqrt{x^8} = x^4$$
$$\sqrt{x^9} = x^4\sqrt{x}$$
$$\sqrt{x^{10}} = x^5$$

The textbook did not go beyond the explanation; he presented the examples above on the board and kept them there for the whole lesson. He did not, however, pay attention to problems in the textbook such as $\sqrt{a^2b^{10}}$ and did not make a connection with the problems $\sqrt{400}$ and $\sqrt{72m^2n^5}$ in an example in the textbook. During the instruction on the Pythagorean Theorem and the distance formula, Mr. Cochran tried to get his students to find relationships, rules, or formulas by working many cases or problems. Although the textbook did not treat the topic, he used equilateral triangles of different sizes to let them see how the Pythagorean Theorem could be applied to find the height of each one. Although it tended to be step-by-step procedure, he let the students try to discover the distance formula from the Pythagorean Theorem as well. In introducing probability in the pre-algebra course, Mr. Cochran's lesson was based on an activity in which the students rolled a die to experiment with theoretical Probability.

In contrast, Mr. Cochran sometimes appeared to disregard an important concept or a fact that ought to be addressed to the students and was explained in the textbook. He neither mentioned $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$ nor explained it until a student raised a related question. The textbook clearly presented the concept with problems showing counterexamples. Among the problems used in lessons or the review materials, the problems on simplifying radicals mostly focused on numbers, rather than variable. On the topic of square-root functions, neither the textbook nor Mr. Cochran did enough not only to explain graphing the square-root functions but also to connect graphing and solving radical equations. He clearly covered the topics of how to graph with calculators, but the explanation on graphing the functions was very rough. The graphs of examples of radical functions that Mr. Cochran represented were too vague to differentiate. Those graphs looked alike and he did not make clear how they differ from each other.

V. Conclusions

The middle school mathematics teacher's pedagogical content knowledge was characterized by knowledge of mathematics, knowledge of students' understanding, and knowledge of pedagogy. Mr. Cochran's knowledge of mathematics included his purposes of teaching mathematics, understanding concepts to teach, connection among topics, various ways of solving problems, and textbook knowledge. His goals for teaching mathematics were for students both to understand abstract concepts of mathematics and to apply them to realistic applications. Mr. Cochran seemed to apply his knowledge about fractions to teaching radicals and radical equations. He, however, saw himself not only as developing his knowledge about mathematics but also as having limited knowledge about mathematics. Mr. Cochran especially considered himself having limited knowledge of radicals and graphing radical equations because he did not take any courses in relation to those topics in college. In contrast, his college professor of statistics definitely influenced the way he taught probability and increased his confidence in his knowledge of probability.

Next, Mr. Cochran's knowledge of students' understanding involved particular students' understanding and students' misconceptions, common errors, difficulties, and confusions about a topic or concept. Although he gained and used his understanding about the students' understanding in his instruction, his knowledge of assessment was not explicitly revealed. He frequently used his understanding about students' common mistakes and confusions in simplifying fractions and, further, connected such mistakes in working on fractions to mistakes with radicals. Students were always encouraged to try various approaches to solving a problem; Mr. Cochran incorporated students' methods into his instruction.

Finally, Mr. Cochran's knowledge of pedagogy was revealed in the form of both his efforts to motivate his students by providing realistic applications of mathematical topics and his use of the textbook and materials. He tried to find problems and stories to motivate his students. He took problems from the textbook and used them in his instruction and for students' homework. He did not use many materials for teaching radicals and radical equations except using a calculator for graphing radical equations.

VI. Implications

The findings of this study show that the teacher's knowledge of mathematics enabled them to teach mathematics (Ball, 1991; Ball et al., 2001; Even & Tirosh, 1995; Fernandez, 1997; Leinhardt et al., 1991; Ma, 1999). The teacher not only realized his strengths, weaknesses, and limitations in his knowledge of mathematics but also saw him as developing it through teaching experiences (Hiebert & Carpenter, 1992).

The findings of the study suggest that focusing on students' understanding and using realistic examples and applications of mathematical topics play key roles in mathematics teaching for understanding. The teacher appeared to use his understanding of students' common misconceptions of multiplication and division of fractions, which hindered the students in understanding new concepts and in solving problems. He also used his knowledge of students' understanding to refine his explanations and to elicit the students' thinking.

The results of the study also suggest that the pedagogical content knowledge of the middle school mathematics teacher appeared to be relatively dependent on his knowledge of students' understanding and his knowledge of mathematics, rather than on his knowledge of pedagogy. In addition, this study supports that the teacher's pedagogical content knowledge is dynamic (Cochran et al., 1993); Pedagogical content knowledge neither static nor unchangeable. The teacher appeared developing his pedagogical content knowledge through his own learning and experience of teaching.

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교수법적 내용 지식: 미국 중학교 수학 교사 사례 연구

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이 연구는 중학교 수학 교사의 수업 활동에 나타나는 교수법적 내용 지식이 어떻게 구성되어 있는지 알아보기 위한 것으로, 미국의 8학년 수학 교사의 수업 활동과 면담을 수집하고 분석한 사례 연구이다. 참여자인 수학 교사의 교수법적 내용 지식은 첫째, 각 개념과 주제들 간의 연계성, 다양한 형태의 문제 해결력, 그리고 교과서에 대한 이해 등을 포함한 수학 교과 내용 지식, 둘째, 수학의 특정 개념과 주제에 대한 학생들의 잘못된 생각과 일반적인 오류, 학생들이 느끼는 어려움과 혼동을 이해하는 학생들의 학습 과정에 대한 지식, 그리고 마지막으로 학생들의 동기 유발과 실제 상황에의 적용을 중시하는 교수학적 지식으로 이루어진 것으로 나타났다.

* **Key words** : mathematics instruction(수학 수업), pedagogical content knowledge(교수법적 내용 지식), teacher knowledge(교사 지식), middle school mathematics teaching(중학교 수학 수업)

논문접수: 2007. 5. 31
심사완료: 2007. 8. 18