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Fuzzy Relations and Meet Preserving Maps

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Abstract

We investigate the properties of fuzzy relations and meet preserving maps on strictly two-sided, commutative quantales. Moreover, we study the relations between them.

Key words : stsc-quantales, meet preserving maps, (left) right adjointness

1. Introduction

Quantales were introduced by Mulvey [9] as the non-commutative generalization of the lattice of open sets in topological spaces. Recently, quantales have arisen in an analysis of the semantics of linear logic systems developed by Girard [1], which supports part of foundation of theoretic computer science. Höhle *et al.* [4,5] introduced the notion of L-fuzzy relation on a complete quasi-monoidal lattice (including GL-monoid [2]) L instead of a completely distributive lattice or the unit interval[8,11]. The notion of L-fuzzy relation facilitated to study fuzzy equivalence relations, fuzzy rough sets, L-fuzzy topological structures [8,11].

In this paper, we investigate the properties of fuzzy relations and meet preserving maps on a strictly two-sided, commutative quantale lattice L. Moreover, we study the relations between meet preserving maps and fuzzy relations.

2. Preliminaries

Definition 2.1. [6,9-11] A triple $(L, \leq, \odot)$ is called a *strictly two-sided, commutative quantale* (stsc-quantale, for short) iff it satisfies the following properties:

(L1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a completely distributive lattice where 1 is the universal upper bound and 0 is the universal lower bound;

(L2) $(L, \odot)$ is a commutative semigroup;

(L3) $a = a \odot 1$, for each $a \in L$;

(L4) $\odot$ is distributive over arbitrary joins, i.e.,

$$(\bigvee_{i\in\Gamma} a_i) \odot b = \bigvee_{i\in\Gamma} (a_i \odot b).$$

Remark 2.2. [4,5,7-12](1) A completely distributive lattice (ref. [11]) is a stsc-quantale. In particular, the unit interval $([0,1], \leq, \vee, \wedge, 0, 1)$ is a stsc-quantale.

(2) The unit interval with a continuous t-norm t, $([0,1], \leq, t)$, is a stsc-quantale.

(3) Let $(L, \leq, \odot)$ be a stsc-quantale. For each $x, y \in L$, we define

$$x \to y = \bigvee\{z \in L \mid x \odot z \leq y\}.$$

Then it satisfies Galois correspondence, that is,

$$(x \odot y) \leq z \Leftrightarrow x \leq (y \to z).$$

In this paper, we always assume that $(L, \leq, \odot, ^*)$ is a stsc-quantale with strong negation * where $a^* = a \to 0$. We denote 1_x a characteristic function of $\{x\}$.

Let X be a nonempty set. All algebraic operations on L can be extended pointwisely to the set L^X as follows: for all $x \in X$, $\lambda, \mu \in L^X$ and $\alpha \in L$,

(1) $\lambda \leq \mu$ iff $\lambda(x) \leq \mu(x)$;

(2) $(\lambda \odot \mu)(x) = \lambda(x) \odot \mu(x)$;

(3) $1_X(x) = 1, \quad \alpha \odot 1_X(x) = \alpha$ and $1_\emptyset(x) = 0$;

(4) $(\alpha \to \lambda)(x) = \alpha \to \lambda(x)$ and $(\lambda \to \alpha)(x) = \lambda(x) \to \alpha$;

(5) $(\alpha \odot \lambda)(x) = \alpha \odot \lambda(x)$.

Lemma 2.3. [6,12] For each $x, y, z, x_i, y_i \in L$, we have the following properties.

(1) If $y \leq z$, then $(x \odot y) \leq (x \odot z)$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$.

(2) $x \odot y \leq x \wedge y \leq x \vee y$.

(3) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$.

(4) $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.

(5) $x \to (\bigvee_{i\in\Gamma} y_i) \geq \bigvee_{i\in\Gamma}(x \to y_i)$

(6) $(\bigwedge_{i\in\Gamma} x_i) \to y \geq \bigvee_{i\in\Gamma}(x_i \to y)$.

(7) $\bigwedge_{i\in\Gamma} y_i^* = (\bigvee_{i\in\Gamma} y_i)^*$ and $\bigvee_{i\in\Gamma} y_i^* = (\bigwedge_{i\in\Gamma} y_i)^*$.

(8) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.

(9) $x \odot y = (x \to y^*)^*$.

(10) $x \leq (x \to y) \to y$.

(11) $x \leq y \to z$ iff $y \leq x \to z$.

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Definition 2.4. [6,8,11] Let $\phi : M \to N$ and $\psi : N \to M$ be order-preserving maps between partially ordered sets M, N. ϕ is *left adjoint* of ψ, $\phi \dashv \psi$, iff $\phi(a) \leq b \Leftrightarrow a \leq \psi(b)$. Equivalently, $\phi \dashv \psi$ iff $id_M \leq \psi\circ\phi$ and $\phi\circ\psi \leq id_N$.

Definition 2.5. [7] A map $\psi : L^X \to L^Y$ is a meet-preserving map if $\psi(\bigwedge_{i\in\Gamma} \lambda_i) = \bigwedge_{i\in\Gamma} \psi(\lambda_i)$, for $\{\lambda_i\}_{i\in\Gamma} \subset L^X$. We denote $L(X,Y)$ a family of meet-preserving maps.

Theorem 2.6. [7] For $\psi, \psi_1 \in L(X,Y)$ and $\psi_2 \in L(Y,Z)$, we define, for all $\lambda \in L^X, \rho \in L^Y$,

$$\psi^{-1}(\rho) = \bigvee\{\lambda \in L^X \mid \psi(\lambda^*) \geq \rho^*\},$$

$$\psi_1 \circ \psi_2(\lambda) = \psi_1(\psi_2(\lambda)).$$

Then the following properties hold:

(1) $\psi^{\to}(\rho) = \bigwedge\{\lambda \in L^X \mid \psi(\lambda) \geq \rho\}$ such that $\psi^{\to}$ is a left adjoint of ψ with $\rho \leq \psi\circ\psi^{\to}(\rho)$ and $\psi^{\to}\circ\psi(\lambda) \leq \lambda$.

(2) $\psi^{-1}(\rho) = (\psi^{\to}(\rho^*))^*$ and $\psi^{-1} \in L(Y,X)$ such that

$$\psi(\lambda) \geq \rho \Leftrightarrow \lambda \geq \psi^{\to}(\rho) \Leftrightarrow \psi^{-1}(\rho^*) \geq \lambda^*$$

(3) $(\psi^{-1})^{-1} = \psi$.

(4) If $\psi_1 \leq \psi_2$, then $\psi_1^{-1} \leq \psi_2^{-1}$.

(5) If $\phi \in L(Y,Z)$, then $\phi \circ \psi \in L(X,Z)$ and $(\phi\circ\psi)^{-1} = \psi^{-1}\circ\phi^{-1} \in L(Z,X)$.

(6) If $\psi(1_x \to \lambda(x)) = \rho_x$ for all $x \in X$, then $\psi(\lambda) = \bigwedge_{z\in X} \rho_z$.

(7) If $\psi_1(1_x \to \alpha) = \psi_2(1_x \to \alpha)$ for all $x \in X$, then $\psi_1 = \psi_2$.

3. Fuzzy relations and Meet preserving maps

In this section, we investigate the relationships between fuzzy relations and meet preserving maps.

Theorem 3.1. For each $u \in L^{X\times Y}$, we define mappings $\Phi_1(u) : L^X \to L^Y$ and $\Phi_2(u) : L^Y \to L^X$ as follows:

$$\Phi_1(u)(\lambda)(y) = \bigwedge_{x\in X} (u(x,y) \to \lambda(x)),$$

$$\Phi_2(u)(\rho)(x) = \bigwedge_{y\in Y} (u(x,y) \to \rho(y)).$$

Then we have the following properties:

(1) $\Phi_1(u) \in L(X,Y)$ and $\Phi_2(u) \in L(Y,X)$. For each $i = 1,2$, $\Phi_i(u)$ has a left adjoint mapping $\Phi_i^{\to}(u)$, respectively, defined by

$$\Phi_1^{\to}(u)(\rho)(x) = \bigvee_{y\in Y} (u(x,y) \odot \rho(y)),$$

$$\Phi_2^{\to}(u)(\lambda)(y) = \bigvee_{x\in X} (u(x,y) \odot \lambda(x)).$$

(2) For each $x \in X, \alpha \in L$ and $y \in Y$,

$$\Phi_1(u)(1_x \to \alpha)(y) = u(x,y) \to \alpha,$$

$$\Phi_2(u)(1_y \to \alpha)(x) = u(x,y) \to \alpha,$$

(3) For each $\lambda \in L^X$ and $\rho \in L^Y$,

$$\Phi_1(u)(\lambda)(y) = \bigwedge_{x\in X} \Phi_1(u)(1_x \to \lambda(x))(y),$$

$$\Phi_2(u)(\rho)(x) = \bigwedge_{y\in Y} \Phi_2(u)(1_y \to \rho(y))(x).$$

(4) For each $u \in L^{X\times Y}$ and each $i = 1,2$, we define

$$\Phi_1(u)^{-1}(\rho) = (\Phi_1^{\to}(u)(\rho^*))^*,$$

$$\Phi_2(u)^{-1}(\lambda) = (\Phi_2^{\to}(u)(\lambda^*))^*.$$

Then $\Phi_1(u)^{-1} = \Phi_2(u)$ and $\Phi_2(u)^{-1} = \Phi_1(u)$.

Proof. (1) $\Phi_1(u) \in L(X,Y)$ from:

$$\begin{aligned}&\Phi_1(u)(\textstyle\bigwedge_{i\in\Gamma} \lambda_i)(y)\\ &= \textstyle\bigwedge_{x\in X}(u(x,y) \to \bigwedge_{i\in\Gamma} \lambda_i(x))\\ &\text{(by Lemma 2.3(3))}\\ &= \textstyle\bigwedge_{i\in\Gamma}\left(\bigwedge_{x\in X}(u(x,y) \to \lambda_i(x)\right)\\ &= \textstyle\bigwedge_{i\in\Gamma} \Phi_1(u)(\lambda_i)(y).\end{aligned}$$

Since $\Phi_1(u) \in L(X,Y)$, by Theorem 2.6(1), we obtain $\Phi_1^{\to}(u)$ as follows:

$$\begin{aligned}&\Phi_1^{\to}(u)(\rho)(x)\\ &= \textstyle\bigwedge\{\lambda(x) \mid \rho(y) \leq \Phi_1(u)(\lambda)(y)\}\\ &= \textstyle\bigwedge\{\lambda(x) \mid \rho(y) \leq \bigwedge(u(x,y) \to \lambda(x))\}\\ &= \textstyle\bigwedge\{\lambda(x) \mid \bigvee_{y\in Y}(\rho(y) \odot u(x,y)) \leq \lambda(x))\}\\ &= \textstyle\bigvee_{y\in Y}(\rho(y) \odot u(x,y))\end{aligned}$$

It follows

$$\begin{aligned}&\Phi_1(u)(\Phi_1^{\to}(u)(\rho))(y)\\ &= \textstyle\bigwedge_{x\in X}\{u(x,y) \to \Phi_1^{\to}(u)(\rho)(x)\}\\ &= \textstyle\bigwedge_{x\in X}\{u(x,y) \to \bigvee_{y\in Y}(\rho(y) \odot u(x,y))\}\\ &\text{(by Lemma 2.3(5))}\\ &\geq \textstyle\bigwedge_{x\in X} \bigvee_{y\in Y}\{u(x,y) \to (\rho(y) \odot u(x,y))\}\\ &\geq \rho(y).\end{aligned}$$

$$\begin{aligned}&\Phi_1^{\to}(u)(\Phi_1(u)(\lambda))(x)\\ &= \textstyle\bigvee_{y\in Y}\{u(x,y) \odot \Phi_1(u)(\lambda)(y)\}\\ &= \textstyle\bigvee_{y\in Y}\{u(x,y) \odot \bigwedge_{x\in X}(u(x,y) \to \lambda(x))\}\\ &\leq \textstyle\bigvee_{y\in Y}\{u(x,y) \odot (u(x,y) \to \lambda(x))\}\\ &\leq \lambda(x).\end{aligned}$$

Hence $\Phi_1(u)$ has a left adjoint mapping $\Phi_1^{\to}(u)$.

(2) It follows from:

$$\begin{aligned}\Phi_1(u)(1_x \to \alpha)(y) &= \bigwedge_{z\in X}(u(z,y) \to (1_x \to \alpha)(z)) \\ &= u(x,y) \to \alpha.\end{aligned}$$

Other case is similarly proved.

(3) Since $\Phi_1(u) \in L(X,Y)$ and $\lambda = \bigwedge_{x\in X}(1_x \to \lambda(x))$, we have

$$\begin{aligned}\Phi_1(u)(\lambda)(y) &= \Phi_1(u)(\bigwedge_{x\in X}(1_x \to \lambda(x)))(y) \\ &= \bigwedge_{x\in X}\Phi_1(u)(1_x \to \lambda(x))(y).\end{aligned}$$

Other case is similarly proved.

(4)

$$\begin{aligned}\Phi_1(u)^{-1}(\rho)(x) &= (\Phi_1^{\rightarrow}(u)(\rho^*)(x))^* \\ &= \Big(\bigvee_{y\in Y} u(x,y)\odot \rho^*(y)\Big)^* \\ &\text{(by Lemma 2.3(7,9))} \\ &= \bigwedge_{y\in Y}(u(x,y) \to \rho(y)) \\ &= \Phi_2(u)(\rho)(x).\end{aligned}$$

□

Theorem 3.2. We define mappings $\Phi_1 : L^{X\times Y} \to L(X,Y)$ and $\Phi_2 : L^{X\times Y} \to L(Y,X)$ as follows:

$$\Phi_1(u)(\lambda)(y) = \bigwedge_{x\in X}(u(x,y) \to \lambda(x)),$$

$$\Phi_2(u)(\rho)(x) = \bigwedge_{y\in Y}(u(x,y) \to \rho(y)).$$

Then we have the following properties:

(1) We define a mapping $\Psi_1 : L(X,Y) \to L^{X\times Y}$ as follows:

$$\Psi_1(\phi)(x,y) = \bigvee\{u(x,y) \mid \Phi_1(u) \geq \phi\}.$$

Then $\Psi_1(\phi)(x,y) = \bigwedge_\alpha \Big(\phi(1_x \to \alpha)(y) \to \alpha\Big)$. Moreover, if $\phi(1_x \to \alpha) = \phi(1_x) \to \alpha$ for $\phi \in L(X,Y)$, then $\Psi_1(\phi)(x,y) = \phi(1_x)(y)$.

(2) We define a mapping $\Psi_2 : L(Y,X) \to L^{X\times Y}$ as follows:

$$\Psi_2(\psi)(x,y) = \bigvee\{u(x,y) \mid \Phi_2(u) \geq \psi\}.$$

Then $\Psi_2(\psi)(x,y) = \bigwedge_\alpha \Big(\psi(1_y \to \alpha)(x) \to \alpha\Big)$. Moreover, if $\psi(1_y \to \alpha) = \psi(1_y) \to \alpha$ for $\psi \in L(Y,X)$, then $\Psi_2(\psi)(x,y) = \psi(1_y)(x)$.

(3) $\Phi_1 \circ \Psi_1 \geq 1_{L(X,Y)}$. If $\phi(1_x \to \alpha) = \phi(1_x) \to \alpha$ for $\phi \in L(X,Y)$, the equality holds.

(4) $\Phi_2 \circ \Psi_2 \geq 1_{L(Y,X)}$. If $\psi(1_y \to \alpha) = \psi(1_y) \to \alpha$ for $\psi \in L(Y,X)$, the equalities hold.

(5) $\Psi_1 \circ \Phi_1 = 1_{L^{X\times Y}}$ and $\Psi_2 \circ \Phi_2 = 1_{L^{X\times Y}}$.

(6) Let $\phi \in L(X,Y)$. Then $\phi \in \Phi_1(L^{X\times Y})$ if $\phi(1_x \to \alpha) = \phi(1_x) \to \alpha$.

(7) Let $\psi \in L(Y,X)$. Then $\psi \in \Phi_2(L^{X\times Y})$ if $\psi(1_y \to \alpha) = \psi(1_y) \to \alpha$.

Proof. (1) Since $\Phi_1(\bigvee_{i\in\Gamma} u_i)(\lambda)(y) = \bigwedge_{i\in\Gamma}\Phi_1(u_i)(\lambda)(y)$ from Lemma 2.3(4) and $\lambda = \bigwedge_{z\in X}(1_{\{z\}} \to \lambda(z))$, we have:

$$\begin{aligned}&\Psi_1(\phi)(x,y) \\ &= \bigvee\{u(x,y) \mid \Phi_1(u) \geq \phi\} \\ &\text{((by Theorem 2.6(7))} \\ &= \bigvee\{u(x,y) \mid \phi(1_x \to \lambda(x))(y) \\ &\leq \Phi_1(u)(1_x \to \lambda(x))(y)\} \\ &= \bigvee\{u(x,y) \mid \phi(1_x \to \lambda(x))(y) \\ &\leq \bigwedge_{z\in X}(u(z,y) \to (1_x \to \lambda(x))(z))\} \\ &= \bigvee\{u(x,y) \mid \phi(1_x \to \lambda(x))(y) \\ &\leq u(x,y) \to \lambda(x)\} \\ &= \bigvee\{u(x,y) \mid u(x,y) \\ &\leq \bigwedge_{\alpha\in L}\Big(\phi(1_x \to \alpha)(y) \to \alpha\Big)\} \\ &= \bigwedge_{\alpha\in L}\Big(\phi(1_x \to \alpha)(y) \to \alpha\Big)\end{aligned}$$

If $\phi(1_x \to \alpha) = \phi(1_x) \to \alpha$ for $\phi \in L(X,Y)$, then

$$\Psi_1(\phi)(x,y) = \bigwedge_{\alpha\in L}\Big((\phi(1_x)(y) \to \alpha) \to \alpha\Big).$$

Since $(\phi(1_x)(y) \leq (\phi(1_x)(y) \to \alpha) \to \alpha$, $\Psi_1(\phi)(x,y) \geq \phi(1_x)(y)$.

Since $\Psi_1(\phi)(x,y) \leq (\phi(1_x)(y) \to 0) \to 0) = \phi(1_x)(y)$ from Lemma 2.3(10), we have $\Psi_1(\phi)(x,y) = \phi(1_x)(y)$.

(2) It is similarly proved as in (1).

(3) We have $\Phi_1 \circ \Psi_1 \geq 1_{L(X,Y)}$ from

$$\begin{aligned}&\Phi_1(\Psi_1(\phi))(\lambda)(y) \\ &= \bigwedge_{x\in X}(\Psi_1(\phi)(x,y) \to \lambda(x)) \\ &= \bigwedge_{x\in X}\Big((\bigwedge_{\alpha\in L}(\phi(1_x \to \alpha)(y) \to \alpha)) \to \lambda(x)\Big) \\ &\geq \bigwedge_{x\in X}\Big((\phi(1_x \to \lambda(x))(y) \to \lambda(x)) \to \lambda(x)\Big) \\ &\geq \bigwedge_{x\in X}(\phi(1_x \to \lambda(x))(y) \\ &= \phi(\bigwedge_{x\in X}(1_x \to \lambda(x)))(y) \\ &= \phi(\lambda)(y).\end{aligned}$$

Let $\phi(1_x \to \alpha) = \phi(1_x) \to \alpha$ for $\phi \in L(X,Y)$. Since $\bigwedge_{\alpha\in L}\Big((\phi(1_x)(y) \to \alpha) \to \alpha\Big) = \phi(1_x)(y)$, we have

$$\begin{aligned}&\Phi_1(\Psi_1(\phi))(\lambda)(y) \\ &= \bigwedge_{x\in X}\Big((\bigwedge_{\alpha\in L}(\phi(1_x \to \alpha)(y) \to \alpha)) \to \lambda(x)\Big) \\ &= \bigwedge_{x\in X}\Big((\bigwedge_{\alpha\in L}(\phi(1_x)(y) \to \alpha) \to \alpha)) \to \lambda(x)\Big) \\ &= \bigwedge_{x\in X}\Big((\phi(1_x)(y) \to \lambda(x)\Big) \\ &= \phi(\lambda)(y).\end{aligned}$$

(4) It is similarly proved as in (3).

(5) We have $\Psi_1 \circ \Phi_1 = 1_{L^{X\times Y}}$ from

$$\begin{aligned}&\Psi_1(\Phi_1(u))(x,y)\\&= \bigwedge_\alpha \Big(\Phi_1(u)(1_x \to \alpha)(y) \to \alpha\Big)\\&= \bigwedge_\alpha \Big(\bigwedge_{z\in X}(u(z,y) \to (1_x \to \alpha)(z)) \to \alpha\Big)\\&= \bigwedge_\alpha \Big((u(x,y) \to \alpha) \to \alpha\Big)\\&= u(x,y).\end{aligned}$$

Other case is similarly proved.
(6) It follows from:

$$\begin{aligned}\phi(\lambda)(y) &= \phi\Big(\bigwedge_{x\in X}(1_x \to \lambda(x))\Big)(y)\\&= \bigwedge_{x\in X} \phi(1_x \to \lambda(x))(y)\\&= \bigwedge_{x\in X} \Big(\phi(1_x)(y) \to \lambda(x)\Big)\\&\quad(\text{put } u(x,y) = \phi(1_x)(y))\\&= \bigwedge_{x\in X}(u(x,y) \to \lambda(x))\\&= \Phi_1(u)(\lambda)(y).\end{aligned}$$

(7) It is similar to (6).

□

Example 3.3. Let $([0,1],\odot)$ be a quantale defined as $x \odot y = (x+y-1) \vee 0$. We obtain

$$x \to y = (1-x+y) \wedge 1, \quad x \oplus y = (x+y) \wedge 1.$$

Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$ be sets and $u \in L^{X\times Y}$ as follows

$$u(x_1,y_1) = 0.8, u(x_1,y_2) = 0.7,$$

$$u(x_2,y_1) = 0.3, u(x_2,y_2) = 0.9.$$

We obtain $\Phi_1(u)$ as follows:

$$\begin{aligned}&\Phi_1(u)(\lambda)(y_1)\\&= \bigwedge_{x\in X}(u(x,y_1) \to \lambda(x))\\&= (u(x_1,y_1) \to \lambda(x_1)) \wedge (u(x_2,y_1) \to \lambda(x_2))\\&= (0.2+\lambda(x_1)) \wedge (0.7+\lambda(x_2)) \wedge 1\end{aligned}$$

$$\begin{aligned}&\Phi_1(u)(\lambda)(y_2)\\&= \bigwedge_{x\in X}(u(x,y_2) \to \lambda(x))\\&= (u(x_1,y_2) \to \lambda(x_1)) \wedge (u(x_2,y_2) \to \lambda(x_2))\\&= (0.3+\lambda(x_1)) \wedge (0.1+\lambda(x_2)) \wedge 1\end{aligned}$$

$$\begin{aligned}&\Phi_1^{\to}(u)(\Phi_1(u))(\lambda)(x_1)\\&= \bigvee_{y\in Y} \Big(u(x_1,y) \odot \Phi_1(u)(\lambda)(y)\Big)\\&= (u(x_1,y_1) \odot \Phi_1(u)(\lambda)(y_1)) \vee (u(x_1,y_2) \odot \Phi_1(u)(\lambda)(y_2))\\&= (0.8 \odot \Phi_1(u)(\lambda)(y_1)) \vee (0.7 \odot \Phi_1(u)(\lambda)(y_2))\\&= \Big(\lambda(x_1) \wedge (0.5+\lambda(x_2)\Big) \vee \Big(\lambda(x_1) \wedge (-0.2+\lambda(x_2)\Big)\\&= \lambda(x_1) \wedge (0.5+\lambda(x_2)).\end{aligned}$$

For each $\rho \in L^Y$,

$$\Phi_1^{\to}(u)(\rho)(x_1) = (-0.2+\rho(y_1)) \vee (-0.3+\rho(y_2)) \vee 0$$

$$\Phi_1^{\to}(u)(\rho)(x_2) = (-0.7+\rho(y_1)) \vee (-0.1+\rho(y_2)) \vee 0$$

$$\Phi_1(u)(\Phi_1^{\to}(u))(\rho)(y_1) = \rho(y_1) \vee (\rho(y_2)+0.6).$$

For each $u \in L^{X\times Y}$,

$$\begin{aligned}&\Psi_1(\Phi_1(u))(x_1,y_1)\\&= \bigwedge_{\alpha\in L} \Big(\Phi_1(u)(1_{\{x_1\}} \to \alpha)(y_1) \to \alpha\Big)\\&= \bigwedge_{\alpha\in L} \Big((0.2+\alpha) \wedge (0.7+1) \wedge 1) \to \alpha\Big)\\&= 0.8.\end{aligned}$$

By a similar method, $\Psi_1 \circ \Phi_1 = 1_{L^{X\times Y}}$.

Example 3.4. Let $([0,1],\odot)$ be a quantale defined in Example 3.3. Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$ be sets. For $\rho(y_1) = 0.8, \rho(y_2) = 0.5, \rho(y_3) = 0.6$, $\mu(x_1) = 0.7, \mu(x_2) = 0.5$, we define $\psi_{\mu,\rho} : L^X \to L^Y$ as follows:

$$\psi_{\mu,\rho}(\lambda) = \begin{cases} \overline{1} & \text{if } \lambda = \overline{1}, \\ \rho & \text{if } \overline{1} \neq \lambda \geq \mu, \\ \overline{0} & \text{otherwise} \end{cases}$$

then $\psi_{\mu,\rho} \in L(X,Y)$. We obtain

$$\Psi_1(\psi_{\mu,\rho})(x_1,y_1) = \bigwedge_\alpha \Big(\psi_{\mu,\rho}(1_{\{x_1\}} \to \alpha)(y_1) \to \alpha\Big) = 0.9.$$

$$\Psi_1(\psi_{\mu,\rho})(x_1,y_2) = \Psi_1(\psi_{\mu,\rho})(x_1,y_3) = 1,$$

$$\Psi_1(\psi_{\mu,\rho})(x_2,y_1) = 0.7, \quad \Psi_1(\psi_{\mu,\rho})(x_2,y_2) = 1,$$

$$\Psi_1(\psi_{\mu,\rho})(x_2,y_3) = 0.9.$$

Since

$$\rho = \psi_{\mu,\rho}(1_{\{x_1\}} \to 0.7) \neq \Big(\psi_{\mu,\rho}(1_{\{x_1\}}) \to 0.7\Big) = 1,$$

we have $0.9 = \Psi_1(\psi_{\mu,\rho})(x_1,y_1) \neq \psi_{\mu,\rho}(1_{\{x_1\}})(y_1) = 0$. Furthermore, we have

$$\begin{aligned}&\Phi_1(\Psi_1(\psi_{\mu,\rho})(\lambda)(y_1)\\&= \bigwedge_{x\in X} \Big(\Psi_1(\psi_{\mu,\rho})(x,y_1) \to \lambda(x)\Big)\\&= \Big(\Psi_1(\psi_{\mu,\rho})(x_1,y_1) \to \lambda(x_1)\Big)\\&\quad \wedge\Big(\Psi_1(\psi_{\mu,\rho})(x_2,y_1) \to \lambda(x_2)\Big)\\&= (0.9 \to \lambda(x_1)) \wedge (0.7 \to \lambda(x_2))\\&= (0.1+\lambda(x_1)) \wedge (0.3+\lambda(x_2)) \wedge 1\\&\geq \psi_{\mu,\rho}(\lambda)(y_1).\end{aligned}$$

By a similar method, we have $\Phi_1(\Psi_1(\psi_{\mu,\rho})(\lambda) \geq \psi_{\mu,\rho}(\lambda)$.

Example 3.5. Let $f : X \to Y$ be a function and $f^{\leftarrow} : L^Y \to L^X$ defined by $f^{\leftarrow}(\rho)(x) = \rho(f(x))$. Since $f^{\leftarrow}(\wedge_{i\in\Gamma}\rho_i) = \wedge_{i\in\Gamma} f^{\leftarrow}(\rho_i) \in L(Y,X)$ and

$$\begin{aligned} & f^{\leftarrow}(1_y \to \alpha)(x) = (1_y \to \alpha)(f(x)) \\ & = 1_y(f(x)) \to \alpha = f^{\leftarrow}(1_y)(x) \to \alpha, \end{aligned}$$

we obtain:

$$\begin{aligned} \Psi_2(f^{\leftarrow})(x,y) & = \bigwedge_{\alpha\in L}\Big(f^{\leftarrow}(1_y \to \alpha)(x) \to \alpha\Big) \\ & = \bigwedge_{\alpha\in L}\Big((f^{\leftarrow}(1_y)(x) \to \alpha) \to \alpha\Big) \\ & = f^{\leftarrow}(1_y)(x). \end{aligned}$$

$$\begin{aligned} \Phi_2(\Psi_2(f^{\leftarrow})(\rho))(x) & = \bigwedge_{y\in Y}\Big(\Psi_2(f^{\leftarrow})(x,y) \to \rho(y)\Big) \\ & = \bigwedge_{y\in Y}\Big(f^{\leftarrow}(1_y)(x) \to \rho(y)\Big) \\ & = f^{\leftarrow}(\bigwedge_{y\in Y}(1_y)(x) \to \rho(y)) \\ & = f^{\leftarrow}(\rho)(x). \end{aligned}$$

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