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구간치 퍼지집합 상에서 쇼케이적분에 의해 정의된 거리측도와 유사측도에 관한 연구

A note on distance measure and similarity measure defined by Choquet integral on interval-valued fuzzy sets

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Abstract

Interval-valued fuzzy sets were suggested for the first time by Gorzafczany(1983) and Turksen(1986). Based on this, Zeng and Li(2006) introduced concepts of similarity measure and entropy on interval-valued fuzzy sets which are different from Bustince and Burillo(1996). In this paper, by using Choquet integral with respect to a fuzzy measure, we introduce distance measure and similarity measure defined by Choquet integral on interval-valued fuzzy sets and discuss some properties of them. Choquet integral is a generalization concept of Lebesgue inetgral, because the two definitions of Choquet integral and Lebesgue integral are equal if a fuzzy measure is a classical measure.

Key words : interval-valued fuzzy set, fuzzy measure, distance measure, similarity measure, Choquet integral.

1. Introduction

Distance measure, similarity measure, and entropy are three basic concepts in fuzzy set theory. In the papers[4,7], Fan, Ma, and Liu discussed the relationships between distance measure and similarity measure and entropy. Distance measure can be considered as a dual concept of similarity measure.

Many researchers, such as Gorzalczany [5], Turksen[12], Burillo and Bustince [2], Wang and Li [13], Hong and Lee [6], and Zeng and Li[15] gave the axiom definition of distance measure and similarity measure on interval-valued fuzzy sets and have been applied to the fields of approximate inference, information and uncertainty theory.

Interval-valued fuzzy sets were suggested for the first time by Gorzafcancy[5] and Turksen[12]. Based on this, Zeng and Li [15] introduced concepts of similarity measure and entropy on interval-valued fuzzy sets which are different from Bustine and Burillo [2]. We remark that distance measure and similarity measure in [1,2,5,6,13,15] were defined by Lebesgue integral with respect to a classical measure.

In this paper, by using Choquet integral with respect to a fuzzy measure instead of Lebesgue integral with respect to a classical measure, we define distance measure and similarity measure on interval-valued fuzzy sets. Choquet integral is a generalization concept of Lebesgue inetgral, because the two definitions of Choquet integral and Lebesgue integral are equal if a fuzzy measure is a classical measure. And also Choquet integral is often used in information nonlinear aggregation tool(see[8,9,10,11]).

In section 2, the elementary concepts and some results of interval-valued fuzzy sets are introduced. In section 3, we introduce Choquet integral with respect to a fuzzy measure and their properties which are used in the next section. In section 4, we introduce distance measure and similarity measure interval-valued fuzzy sets and discuss some properties of them.

2. Interval-valued fuzzy sets and Choquet integrals.

Throughout this paper, we will denote the unit interval $[0,1]$.

$$[I] = \{ \bar{a} = [a^-, a^+] \mid a^-, a^+ \in I \text{ and } a^- \le a^+ \}.$$

Then, according to Zadeh's extension principle [14], we can popularize these operations such as maximum $(\vee)$, minimum $(\wedge)$ and complement (c) to $[I]$ defined by

$$\bar{a} \vee \bar{b} = [a^- \vee b^-, a^+ \vee b^+],$$
$$\bar{a} \wedge \bar{b} = [a^- \wedge b^-, a^+ \wedge b^+],$$
$$\bar{a}^c = [1 - a^+, 1 - a^-],$$

thus $([I], \vee, \wedge, c)$ is a complete lattice with a mini-

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mal element $\bar{0}=[0,0]$ and a maximal element $\bar{1}=[1,1]$.

Definition 2.1 Let $\bar{a},\bar{b}\in[I]$. Then, we define the following equal, inequality operations:

$\bar{a}=\bar{b}$ and $a^-=b^-$ and $a^+=b^+$,

$\bar{a}\le\bar{b}$ if and only if $a^-\le b^-$ and $a^+\le b^+$,

$\bar{a}<\bar{b}$ if and only if $\bar{a}\le\bar{b}$ and $a\ne b$.

Let X be a set, $IF(X)$ stands for the set of all interval-valued fuzzy sets in X, $F(X)$ and $\wp(X)$ stands for the set of all fuzzy sets and crisp subsets in X, respectively.

Definition 2.2 For each $A\in IF(X)$ and $x\in X$, $A(x)=[A^-(x),A^+(x)]$ is called the degree of membership of an element x to A, then fuzzy sets $A^-:X\to I$ and $A^+:X\to I$ are called a lower fuzzy set of A and an upper fuzzy set of A, respectively.

For simplicity, we denote $A=[A^-,A^+]$. Then, some operations such as $\vee,\wedge,c$ can be introduced into $IF(X)$ as the followings: for each $A,B\in IF(X)$,

$$(A\vee B)(x)=A(x)\vee B(x),$$
$$(A\wedge B)(x)=A(x)\wedge B(x),$$
$$(A^c)(x)=A(x)^c=[1-A^+(x),1-A^-(x)].$$

Thus, $(IF(X),\vee,\wedge,c)$ is a complete lattice. If $A,B\in IF(X)$, then the following operations can be introduced in the papers [15,16]:

$A\le B$ if and only if $A^-(x)\le B^-(x)$ and $A^+(x)\le B^+(x)$ for all $x\in X$,

$A=B$ if and only if $A^-(x)=B^-(x)$ and $A^+(x)=B^+(x)$ for all $x\in X$.

Now, we introduce Choquet integrals and their basic properties which are used in the next sections (see [8,9,10,11]).

Definition 2.3 (1) Let (X,Ω) be a measurable space. A fuzzy measure on X is a real-valued set function $\mu:\Omega\to I$ satisfying

(i) $\mu(\varnothing)=0$ and $\mu(X)=1$,

(ii) $\mu(A)\le\mu(B)$,

whenever $A,B\in\Omega$, $A\subset B$.

(2) A fuzzy measure μ is said to be continuous from above[below, resp.] if for each increasing[decreasing, resp.] sequence $\{E_n\}\subset\Omega$,

$$\mu(\cup_{n=1}^{\infty}E_n)[\mu(\cap_{n=1}^{\infty}E_n),\ resp.]=\lim_{n\to\infty}\mu(E_n).$$

(3) If μ is continuous both from above and from below, it is said to be continuous.

We recall that the concept of Choquet integral with respect to a classical measure was first introduced in capacity theory by Choquet([3]). Its use as a Choquet integral with respect to a fuzzy measure was then proposed by Sugeno and et. al(see [8,9,10]).

Definition 2.4 (1) The Choquet integral of a measurable function f with respect to a fuzzy measure μ is defined by

$$(C)\int f\,d\mu=\int_0^{\infty}\mu_f(r)\,dr$$

where $\mu_f(r)=\mu(\{x\in X|f(x)>r\}$ and the integral on the right-hand side is an ordinary one.

(2) If X is a finite set, that is, $X=\{x_1,\cdots,x_n\}$, then the Choquet integral of f on X is defined by

$$(C)\int f\,d\mu=\sum_{i=1}^{n}x_{(i)}[\mu(A_{(i)})-\mu(A_{(i+1)})]$$

where $(\cdot)$ indicates a permutation on $\{1,2,\cdots,n\}$ such that $x_{(1)}\le\cdots\le x_{(n)}$. Also $A_{(i)}=\{(i),(i+1),\cdots,(n)\}$ and $A_{(n+1)}=\varnothing$.

We note that Choquet integral with respect to a fuzzy measure is often used in information fusion and data mining as an aggregation tool. We also introduce the concept of comonotonic between two functions and some characterizations of Choquet integral which are used to define distance measure and similarity measure.

Definition 2.5 Let $f,g:X\to I$ be measurable functions. We say that f and g are comonotonic, the symbol $f\sim g$ if and only if

$$f(x)<f(x')\to g(x)\le g(x')$$

for all $x,x'\in X$.

Theorem 2.6 Let $f,g,h:X\to I$ be measurable functions. Then we have the followings:

(i) $f\sim f$,

(ii) $f\sim g\to g\sim f$,

(iii) $f\sim a$ for all $a\in I$,

(iv) $f\sim g$ and $f\sim h\to f\sim g+h$.

Theorem 2.7 Let $f,g:X\to I$ be measurable functions.

(i) If $f\le g$, then $(C)\int f d\mu\le(C)\int g d\mu$.

(ii) If $f\sim g$ and $a,b\in I$, then

$$(C)\int(af+bg)d\mu=a(C)\int fd\mu+b(C)\int gd\mu.$$

(iii) If we define $(f\vee g)(x)=f(x)\vee g(x)$ for all $x\in X$, then

$$(C)\int f\vee g d\mu \geq (C)\int f d\mu \vee (C)\int g d\mu.$$

(iv) If we define $(f\wedge g)(x)=f(x)\wedge g(x)$ for all $x\in X$, then

$$(C)\int f\wedge g d\mu \geq (C)\int f d\mu \wedge (C)\int g d\mu.$$

3. Distance measure and similarity measure.

In this section, we first state definitions and give some formulas to calculate distance measure and similarity measure on $IF(X)$. In 2006, Zeng and Li [15] introduced formulas of similarity measure and entropy defined by Lebsgue integral on $IF(X)$. Based on this, we introduce concepts of distance measure and similarity measure defined by Choquet integral with respect to a fuzzy measure instead of Lebesgue integral with respect to a classical measure on $IF(X)$. We recall that for each $A,B\in IF(X)$,

$$A=B \text{ if and only if } \mu(\{x|A(x)\neq B(x)\})=0,$$

that is, A is equal to B on almost everywhere X.

Definition 3.1 A real function $D: IF(X)\times IF(X)\to I$ is called a distance measure on $IF(X)$ if D satisfies the following properties:

(D_1) $D(A,A)=0$ if A is a crisp set,

(D_2) $D(A,B)=0 \quad \to A=B$,

(D_3) $D(A,B)=D(B,A)$,

(D_4) for all $A,B,C\in IF(X)$, if $A\leq B\leq C$, then $D(A,C)\geq D(A,B)$ and $D(A,C)\geq D(B,C)$.

We recall that A is a crisp set if and only if $A(x)=A=[A^-,A^+]$, for all $x\in X$. Then we can give the following formulas to calculate distance measure on interval-valued fuzzy sets A,B:

$$D_c(A,B)=(C)\int d_H(A(x),B(x))d\mu(x)$$
$$=\int_0^\infty \mu(\{x|d_H(A(x),B(x))\})dr,$$

where d_H is the Hausdorff metric between $A(x)$ and $B(x)$. Since $A(x)=[A^-(x),A^+(x)]$ and $B(x)=[B^-(x),B^+(x)$, clearly, we obtain

$$d_H(A(x),B(x))$$
$$=max\{sup_{a\in A(x)}inf_{b\in B(x)}|a-b|, sup_{b\in B(x)}inf_{a\in A(x)}|a-b|\}$$
$$=\max\{|A^-(x)-B^-(x)|, |A^+(x)-B^+(x)|\}.$$

Thus,

$$D_c(A,B)$$
$$=(C)\int \max\{|A^-(x)-B^-(x)|, |A^+(x)-B^+(x)|\}d\mu.$$

We note that since $\mu(X)=1$, $D_c(A,B)\leq 1$ for all $A,B\in IF(X)$, that is D_c is a function from $IF(X)\times IF(X)$ to I. From now on, we assume that a fuzzy measure μ is continuous.

Theorem 3.2 The real function $D_c: IF(X)\times IF(X)\to I$ is a distance measure on $IF(X)$, we say that D_c is a Choquet distance measure.

Proof. (D_1) If A is a crisp set, then $A(x)=\{A^-,A^+]$ for all $x\in X$. Thus,

$$d_H(A(x),B(x))$$
$$=(C)\int \max\{|A^-(x)-A^-(x)|, |A^+(x)-A^+(x)|\}d\mu$$
$$=0.$$

(D_2) If $D(A,B)=0$, then

$$(C)\int max\{|A^-(x)-B^-(x)|, |A^+(x)-B^+(x)|\}d\mu=0.$$

Thus,

$$\mu(\{x||A^-(x)-B^-(x)|>r\})=0$$

and

$$\mu(\{x||A^+(x)-B^+(x)|>r\})=0$$

for almost all r. We note that

$$\cap_{n=1}^\infty\{x||A^-(x)-B^-(x)|>\frac{1}{n}\}$$
$$=\{x||A^-(x)\neq B^-(x)\}$$

and

$$\cap_{n=1}^\infty\{x||A^+(x)-B^+(x)|>\frac{1}{n}\}$$
$$=\{x||A^+(x)\neq B^+(x)\}.$$

Then, clearly, by the continuity of μ, we obtain

$$\mu(\{x|A^-(x)\neq B^-(x)\})$$
$$=\lim_{n\to\infty}\mu(\{x||A^-(x)-B^-(x)>\frac{1}{n}\})$$
$$=0.$$

and

$$\begin{aligned} &\mu(\{x|\, A^+(x) \neq B^+(x)\}) \\ &= \lim_{n\to\infty} \mu(\{x||A^+(x) - B^+(x) > \frac{1}{n}\}) \\ &= 0. \end{aligned}$$

That is, $A^- = B^-$ and $A^- = B^-$ $a.e.$on I.

Hence $A = B$.

(D_3) The proof of D_3 is clear!

(D_4) Since$A \le B \le C$, $A^- \le B^- \le C^-$ and $A^+ \le B^+ \le C^+$. Thus,

$A^- - B^- \le A^- - C^-$

and

$A^+ - B^+ \le A^+ - C^+$.

Then,

$$\begin{aligned} d_H(A, C) &= max\,\{|A^- - C^-|, |A^+ - C^+|\} \\ &\ge max\,\{|A^- - B^-|, |A^+ - B^-|, |A^+ - B^+|\} \\ &= d_H(A, B). \end{aligned}$$

Thus,

$D_c(A, C) \ge D_c(A, B)$.

Similarly, we obtain

$D_c(A, C) \ge D_c(B, C)$.

Theorem 3.3 If $A \in IF(X)$ and A is a crisp set, then $D_c(A, A^c) = 1$.

Proof. Since A is a crisp set and $A = [A^-, A^+]$, $A^-(x) = A^+(x) = 1$ for all $x \in A$ and $A^-(x) = A^+(x) = 1$ for all $x \not\in A$. Then

$$\begin{aligned} &d_H(A(x), A^c(x)) \\ &= |A^+(x) + A^-(x) - 1| \\ &= 1 \end{aligned}$$

for all $x \in X$. Thus,

$$\begin{aligned} &D_c(A, A^c) \\ &= (C)\int d_H(A(x), A^c(x))d\mu(x) \\ &= \int_0^1 \mu(\{x|d_H(A(x), A^c(x)) > r\})dr \\ &= \int_0^1 \mu(X)d\mu = 1. \end{aligned}$$

Definition 3.4 A real function S: $IF(X) \times IF(X) \to I$ is called a distance measure on $IF(X)$ if S satisfies the following properties:

(S_1) $S(A, A^c) = 0$ if A is a crisp set,

(S_2) $S(A, B) = 1 \quad \to A = B$,

(S_3) $S(A, B) = S(B, A)$,

(S_4) for all $A, B, C \in IF(X)$, if $A \le B \le C$, then $S(A, C) \le S(A, B)$ and $S(A, C) \le S(B, C)$.

For example, we define

$S_c(A, B) = 1 - D_c(A, B)$,

for all $A, B \in IF(X)$. Then, it is easily to see that $0 \le S_c(A, B) \le 1$. Thus, we have the following theorem.

Theorem 3.5 The real function S_c: $IF(X) \times IF(X) \to I$ is a similarity measure, we say that S_c is a Choquet similarity measure.

Proof. (S_1) Since A is a crisp set, by Theorem 4.3, $D_c(A, A^c) = 1$. Thus,

$$S_c(A, A^c) = 1 - D_c(A, A^c) = 1.$$

(S_2) The proof of (S_2) is similar to the proof of (D_2) in Theorem 4.2.

(S_3) Since $D_c(A, B) = D_c(B, A)$,

$$\begin{aligned} S_c(A, B) &= 1 - D_c(A, B) \\ &= 1 - D_c(B, A) \\ &= S_c(B, A). \end{aligned}$$

(S_4) If $A \le B \le C$, then by Theorem4.2,

$D_c(A, C) \ge D_c(A, B)$

and

$D_c(A, C) \ge D_c(B, C)$.

Thus,

$$\begin{aligned} S_c(A, C) &= 1 - D_c(A, C) \\ &\le 1 - D_c(A, B) \\ &= S_c(A, B) \end{aligned}$$

and

$$\begin{aligned} S_c(A, C) &= 1 - D_c(A, C) \\ &\le 1 - D_c(B, C) \\ &= S_c(B, C). \end{aligned}$$

Therefore, S_c is a similarity measure on $IF(X)$.

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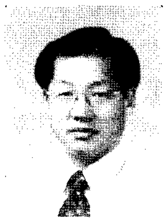

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