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HYPONORMAL TOEPLITZ OPERATORS ON THE BERGMAN SPACE. II.

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ABSTRACT. In this paper we consider the hyponormality of Toeplitz operators T_φ on the Bergman space $L_a^2(\mathbb{D})$ with symbol in the case of function $f+\overline{g}$ with polynomials f and g. We present some necessary conditions for the hyponormality of T_φ under certain assumptions about the coefficients of φ.

1. Introduction

A bounded linear operator A on a Hilbert space is said to be hyponormal if its selfcommutator $[A^*, A] := A^*A - AA^*$ is positive semidefinite. The purpose of this paper is to study hyponormality for Toeplitz operators acting on the *Bergman space* $L_a^2(\mathbb{D})$ *of the unit disc* $\mathbb{D}$. *In particular, our interest is* Toeplitz operators with polynomial symbols which satisfy certain constraints.

If P denotes the orthogonal projection of $L^2(\mathbb{D})$ onto $L_a^2(\mathbb{D})$, the Toeplitz operator T_φ on $L_a^2(\mathbb{D})$ is defined by

$$T_\varphi f = P(\varphi \cdot f),$$

where φ is measurable and f is in $L_a^2(\mathbb{D})$. It is clear that those operators are bounded if φ is in $L^\infty(\mathbb{D})$. The Hankel operator $H_\varphi : L_a^2 \longrightarrow {L_a^2}^\perp$ is defined by $H_\varphi(f) = (I-P)(\varphi \cdot f)$. Let $H^2(\mathbb{T})$ denote the Hardy space of the unit circle $\mathbb{T} = \partial\mathbb{D}$. Recall that given $\psi \in L^\infty(\mathbb{T})$, the Toeplitz operator on $H^2(\mathbb{T})$ is the operator T_ψ on $H^2(\mathbb{T})$ defined by $T_\psi f = P_+(\psi \cdot f)$, where f is in $H^2(\mathbb{T})$ and P_+ denotes the orthogonal projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$.

Basic properties of the Bergman space and the Hardy space can be found in [1], [3] and [4]. The hyponormality of Toeplitz operators on the Hardy space has been studied by C. Cowen [2], T. Nakazi and K. Takahashi [8], W. Y. Lee [5], [6] and others. In [2], Cowen characterized the hyponormality of Toeplitz operator T_φ on $H^2(\mathbb{T})$ by properties of the symbol $\varphi \in L^\infty(\mathbb{T})$. The solution

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is based on a dilation theorem of Sarason [10]. It also exploited the fact that functions in ${H^2}^{\perp}$ are conjugates of functions in zH^2. For the Bergman space, ${L_a^2}^{\perp}$ is much larger than the conjugates of functions in zL_a^2, and no dilation theorem (similar to Sarason's theorem) is available. Indeed it is quite difficult to determine the hyponormality of T_φ on $L_a^2(\mathbb{D})$. In fact the study of hyponormal Toeplitz operators on the Bergman space seems to be scarce from the literature.

In this paper we study the hyponormality of Toeplitz operators T_φ on the Bergman space $L_a^2(\mathbb{D})$ with symbols in the case of function $\varphi = \overline{g}+f$ with polynomials f and g. Since the hyponormality of operators is translation invariant we may assume that $f(0) = g(0) = 0$. We shall list the well-known properties of Toeplitz operators T_φ on the Bergman space.

If f, g are in $L^\infty(\mathbb{D})$ then we can easily check that

$$\begin{aligned}
&\text{(i) } T_{f+g} = T_f + T_g \\
&\text{(ii) } {T_f}^* = T_{\overline{f}} \\
&\text{(iii) } T_{\overline{f}}T_g = T_{\overline{f}g} \text{ if } f \text{ or } g \text{ is analytic.}
\end{aligned}$$

These properties enable us to establish several consequences of hyponormality.

Proposition 1.1 ([7], [9]). *Let f, g be bounded and analytic. Then the followings are equivalent.*

(i) $T_{\overline{g}+f}$ *is hyponormal.*
(ii) $H_{\overline{g}}^* H_{\overline{g}} \leq H_{\overline{f}}^* H_{\overline{f}}$.
(iii) $||(I-P)(\overline{g}k)|| \leq ||(I-P)(\overline{f}k)||$ *for any k in L_a^2.*

Very recently, in [7], it was shown that if $\varphi(z) = a_{-m}\overline{z}^m + a_{-N}\overline{z}^N + a_m z^m + a_N z^N \quad (0 < m < N)$ and if $a_m\overline{a_N} = a_{-m}\overline{a_{-N}}$, then

$$T_\varphi \text{ is hyponormal}$$
$$\Longleftrightarrow \begin{cases} \frac{1}{N+1}(|a_N|^2 - |a_{-N}|^2) \geq \frac{1}{m+1}(|a_{-m}|^2 - |a_m|^2) & \text{if } |a_{-N}| \leq |a_N| \\ N^2(|a_{-N}|^2 - |a_N|^2) \leq m^2(|a_m|^2 - |a_{-m}|^2) & \text{if } |a_N| \leq |a_{-N}|. \end{cases}$$

In this paper we continue to examine the hyponormality of T_φ in the cases where φ is a trigonometric polynomial.

2. Main result

In this section we present some necessary conditions for hyponormality of T_φ. First of all, observe that for any s, t nonnegative integers,

$$P(\overline{z}^t z^s) = \begin{cases} \frac{s-t+1}{s+1} z^{s-t} & \text{if } s \geq t \\ 0 & \text{if } s < t. \end{cases}$$

For $0 \leq i \leq N-1$, write

$$k_i(z) := \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i}.$$

The following lemmas will be used for proving the main result of this section.

Lemma 2.1 ([7]). *For $0 \leq m \leq N$, we have*

(i) $||\overline{z}^m k_i(z)||^2 = \sum_{n=0}^{\infty} \frac{1}{Nn+i+m+1} |c_{Nn+i}|^2.$

(ii) $$||P(\overline{z}^m k_i(z))||^2 = \begin{cases} \sum_{n=0}^{\infty} \frac{Nn+i-m+1}{(Nn+i+1)^2} |c_{Nn+i}|^2 & \text{if } m \leq i \\ \sum_{n=1}^{\infty} \frac{Nn+i-m+1}{(Nn+i+1)^2} |c_{Nn+i}|^2 & \text{if } m > i. \end{cases}$$

Lemma 2.2. *Let $f(z) = a_m z^m + a_N z^N, g(z) = a_{-m} z^m + a_{-N} z^N \quad (0 < m < N)$. If $T_{\overline{g}+f}$ is hyponormal, then*

(i) $\frac{1}{N+1}(|a_N|^2 - |a_{-N}|^2) \geq \frac{1}{m+1}(|a_{-m}|^2 - |a_m|^2)$.

(ii) $|a_m| < |a_{-m}|$ *implies* $|a_N| > |a_{-N}|$.

(iii) $|a_N| < |a_{-N}|$ *implies* $|a_m| > |a_{-m}|$.

Proof. Let $T_{f+\overline{g}}$ be a hyponormal operator. By proposition 1.1, we have $||f|| \geq ||g||$. Observe that

$$||f||^2 = \frac{1}{m+1}|a_m|^2 + \frac{1}{N+1}|a_N|^2 \text{ and } ||g||^2 = \frac{1}{m+1}|a_{-m}|^2 + \frac{1}{N+1}|a_{-N}|^2.$$

This proves the equation (i). The equation (ii) and (iii) are immediate from (i). $\square$

Our main result now follows:

Theorem 2.3. *Let $\varphi(z) = \overline{g(z)} + f(z)$, where*

$$f(z) = a_m z^m + a_N z^N \quad \text{and} \quad g(z) = a_{-m} z^m + a_{-N} z^N \quad (0 < m < N).$$

If T_φ is hyponormal and $|a_N| \leq |a_{-N}|$, then we have

$$N^2(|a_{-N}|^2 - |a_N|^2) \leq m^2(|a_m|^2 - |a_{-m}|^2). \tag{1}$$

Proof. Put $k_i(z) := \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i}$ for $i = 0, 1, 2, \ldots, N-1$. Then we have

$$\left\langle k_i(z)\overline{z}^m, k_i(z)\overline{z}^N \right\rangle = 0.$$

Thus by Lemma 2.1, we have
(2)

$$\begin{aligned}&\langle M_{\overline{f}}k_i(z), M_{\overline{f}}k_i(z)\rangle\\ &= |a_m|^2\sum_{n=0}^{\infty}\frac{1}{Nn+m+i+1}|c_{Nn+i}|^2 + |a_N|^2\sum_{n=0}^{\infty}\frac{1}{Nn+N+i+1}|c_{Nn+i}|^2,\end{aligned}$$

and
(3)

$$\begin{aligned}&\langle M_{\overline{g}}k_i(z), M_{\overline{g}}k_i(z)\rangle\\ &= |a_{-m}|^2\sum_{n=0}^{\infty}\frac{1}{Nn+m+i+1}|c_{Nn+i}|^2 + |a_{-N}|^2\sum_{n=0}^{\infty}\frac{1}{Nn+N+i+1}|c_{Nn+i}|^2.\end{aligned}$$

If $i \geq m$, it follows from Lemma 2.1 that
(4)

$$\begin{aligned}&\langle T_{\overline{f}}k_i(z), T_{\overline{f}}k_i(z)\rangle\\ &= |a_m|^2\sum_{n=0}^{\infty}\frac{Nn+i-m+1}{(Nn+i+1)^2}|c_{Nn+i}|^2 + |a_N|^2\sum_{n=1}^{\infty}\frac{Nn+i-N+1}{(Nn+i+1)^2}|c_{Nn+i}|^2,\end{aligned}$$

and
(5)

$$\begin{aligned}&\langle T_{\overline{g}}k_i(z), T_{\overline{g}}k_i(z)\rangle\\ &= |a_{-m}|^2\sum_{n=0}^{\infty}\frac{Nn+i-m+1}{(Nn+i+1)^2}|c_{Nn+i}|^2 + |a_{-N}|^2\sum_{n=1}^{\infty}\frac{Nn+i-N+1}{(Nn+i+1)^2}|c_{Nn+i}|^2.\end{aligned}$$

Combining (2) and (4), we see that

$$\begin{aligned}&\langle H_{\overline{f}}^*H_{\overline{f}}k_i(z), k_i(z)\rangle\\ &= |a_m|^2\sum_{n=0}^{\infty}\Big(\frac{1}{Nn+m+i+1} - \frac{Nn+i-m+1}{(Nn+i+1)^2}\Big)|c_{Nn+i}|^2\\ &\quad + |a_N|^2\Bigg(\frac{1}{N+i+1}|c_i|^2 + \sum_{n=1}^{\infty}\Big(\frac{1}{Nn+N+i+1} - \frac{Nn+i-N+1}{(Nn+i+1)^2}\Big)|c_{Nn+i}|^2\Bigg).\end{aligned}$$

Combining (3) and (5), we see that

$$\begin{aligned}&\langle H_{\overline{g}}^*H_{\overline{g}}k_i(z), k_i(z)\rangle\\ &= |a_{-m}|^2\sum_{n=0}^{\infty}\Big(\frac{1}{Nn+m+i+1} - \frac{Nn+i-m+1}{(Nn+i+1)^2}\Big)|c_{Nn+i}|^2\\ &\quad + |a_{-N}|^2\Bigg(\frac{1}{N+i+1}|c_i|^2 + \sum_{n=1}^{\infty}\Big(\frac{1}{Nn+N+i+1} - \frac{Nn+i-N+1}{(Nn+i+1)^2}\Big)|c_{Nn+i}|^2\Bigg).\end{aligned}$$

Therefore applying Proposition 1.1 gives that if T_φ is hyponormal then

$$\begin{aligned}
&\langle (H_{\bar f}^* H_{\bar f} - H_{\bar g}^* H_{\bar g}) k_i(z), k_i(z)\rangle \\
&= (|a_m|^2 - |a_{-m}|^2)\sum_{n=0}^{\infty}\Big(\frac{1}{Nn+m+i+1} - \frac{Nn+i-m+1}{(Nn+i+1)^2}\Big)|c_{Nn+i}|^2 \\
&\quad + (|a_N|^2 - |a_{-N}|^2)\Bigg(\frac{1}{N+i+1}|c_i|^2 + \sum_{n=1}^{\infty}\Big(\frac{1}{Nn+N+i+1} \\
&\qquad - \frac{Nn+i-N+1}{(Nn+i+1)^2}\Big)|c_{Nn+i}|^2\Bigg) \qquad (6)\\
&\geq 0.
\end{aligned}$$

If $|a_N| \leq |a_{-N}|$, it follows from Lemma 2.2 that $|a_m| > |a_{-m}|$. Define ξ by

$$\xi(n) := \frac{\frac{1}{Nn+m+i+1} - \frac{Nn+i-m+1}{(Nn+i+1)^2}}{\frac{1}{Nn+N+i+1} - \frac{Nn+i-N+1}{(Nn+i+1)^2}} \qquad (n \geq 1).$$

Then ξ is a strictly decreasing function and

$$\lim_{n\to\infty} \xi(n) = \frac{m^2}{N^2}.$$

Since $\xi(n) \geq \frac{m^2}{N^2}$, it follows from (6) that T_φ is hyponormal, then we have

$$N^2(|a_{-N}|^2 - |a_N|^2) \leq m^2(|a_m|^2 - |a_{-m}|^2).$$

This completes the proof. □

The following example shows that the converse of Theorem 3.2 is not true.

Example. Consider the trigonometric polynomial

$$\varphi(z) = 2\bar z^2 + 2\bar z - 4z + z^2.$$

Then φ satisfies the inequality (1). But a straightforward calculation shows that

$$\Big\langle (H_{\bar f}^* H_{\bar f} - H_{\bar g}^* H_{\bar g})(1+2z), (1+2z)\Big\rangle = 9\frac{1}{3} - 14 < 0.$$

Therefore T_φ is not hyponormal.

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