Bull. Korean Math. Soc. **43** (2007), No. 3, pp. 475–482

COMPOSITION OPERATORS BETWEEN HARDY AND BLOCH-TYPE SPACES OF THE UPPER HALF-PLANE

S. D. Sharma, Ajay K. Sharma, and Shabir Ahmed

Abstract. In this paper, we study composition operators $C_\varphi f = f \circ \varphi$, induced by a fixed analytic self-map of the of the upper half-plane, acting between Hardy and Bloch-type spaces of the upper half-plane.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and φ be a holomorphic self-map of $\mathbb{D}$. Then the equation $C_\varphi f = f \circ \varphi$, for f analytic in $\mathbb{D}$ defines a composition operator C_φ with inducing map φ. During the past few decades, composition operators have been studied extensively on spaces of functions analytic on the open unit disk $\mathbb{D}$. As a consquence of the Littlewood Subordination principle [3] it is known that every analytic self-map φ of the open unit disk $\mathbb{D}$ induces a bounded composition operator on Hardy and weighted Bergman spaces of the open unit disk $\mathbb{D}$. However, if we move to Hardy and weighted Bergman spaces of the upper half-plane

$$\pi^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\},$$

the situation is entirely different. There do exist analytic self-maps of the upper half-plane, which do not induce composition operators on the Hardy spaces and weighted Bergman spaces of the upper half-plane (see [4], [10] and [12]. Interesting work on composition operators on Hardy spaces of the upper half-plane have been done by Singh [11], Singh and Sharma [12], [13], Sharma [9] and Matache [4] and [5]. Recently, several authors have studied composition operators and weighted composition operators on Bloch-type spaces of functions analytic in the open unit disk $\mathbb{D}$. For example, one can refer to [6] and [7] and the references therein for the study of these operators on Bloch-type spaces. However, composition operators on the Bloch-type spaces of the upper half-plane remain untouched so far. The main theme of this paper is to study composition operators between Hardy and Bloch type spaces of the upper half-plane. The plan of the rest of the paper is as follows. In the next section

Received November 1, 2006; Revised June 8, 2007.
2000 *Mathematics Subject Classification.* Primary 47B33, 46E10; Secondary 30D55.
Key words and phrases. composition operator, Bloch-type spaces, Hardy spaces.

we introduce Hardy and Bloch-type spaces of the upper half-plane. Section 3 is devoted to characterize boundedness of composition operators on the Bloch space of the upper half-plane whereas boundedness of composition operators on Gowth spaces is tackled in section 4. Sections 5 and 6 deals with the boundedness of composition operators between Hardy and Bloch-type spaces of the upper half-plane.

2. Preliminaries

In this section we review the basic concepts and collect some essential facts that will be needed throughout the paper.

2.1. Hardy spaces of the upper half-plane.

For $1 \leq p < \infty$, the Hardy space of the upper half-plane is defined as

$$H^p(\pi^+) = \{f : \pi^+ \to \mathbb{C} | f \text{ is analytic and } ||f||_p^p = \sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^p dx < \infty\}.$$

With this norm $H^p(\pi^+)$ becomes a Banach space and for $p = 2$, it is a Hilbert space. To know more about these spaces, we refer to [1] and [2].

The growth of functions in the Hardy space is essential in our study. To this end the following estimate will be useful. For $f \in H^p(\pi^+)$, we have

$$|f(x+iy)|^p \leq \frac{||f||_p^p}{2\pi y}. \tag{2.1}$$

2.2. Bloch space of the upper half-plane.

The Bloch space of the upper half-plane π^+, denoted by $\mathcal{B}_\infty(\pi^+)$, is defined to be the space of analytic functions f on π^+ such that

$$||f||_{\mathcal{B}_\infty} = \sup_{z \in \pi^+} \{\operatorname{Im} z |f'(z)|\} < \infty.$$

It is easy to check that $||f||_{\mathcal{B}_\infty}$ is a complete semi-norm on $\mathcal{B}_\infty(\pi^+)$.

2.3. Growth space of the upper half-plane.

The Growth space of the upper half-plane π^+, denoted by $\mathcal{A}_\infty(\pi^+)$, is defined to be the space of analytic functions f on π^+ such that

$$||f||_{\mathcal{A}_\infty} = \sup_{z \in \pi^+} \{\operatorname{Im} z |f(z)|\} < \infty.$$

It is easy to check that $\mathcal{A}_\infty(\pi^+)$ is a (non separable) Banach space with the norm defined above.

3. Composition operators on $\mathcal{B}_\infty(\pi^+)$

In [4], Matache proved that a linear fractional map

$$\varphi(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0, \tag{3.1}$$

induces a bounded composition operator on Hardy spaces $H^p(\pi^+)$ of the upper half plane if and only if $c = 0$. However, by a simple application of the Schwarz-Pick Theorem in the upper half-plane, we can show that every holomorphic map φ of π^+ such that $\varphi(\pi^+) \subset \pi^+$ induces a bounded composition operator on the Bloch space $\mathcal{B}_\infty(\pi^+)$. Let us first state the Schwarz-Pick Theorem in the upper half-plane.

Schwarz-Pick Theorem in the upper half-plane. Let φ be a holomorphic map of π^+ such that $\varphi(\pi^+) \subset \pi^+$. Then for all $z_1, z_2 \in \pi^+$,

$$\left|\frac{\varphi(z_1) - \varphi(z_2)}{\overline{\varphi(z_1)} - \varphi(z_2)}\right| \leq \left|\frac{z_1 - z_2}{\overline{z_1} - z_2}\right|.$$

Also for all $z \in \pi^+$,

$$\frac{|\varphi'(z)|}{\operatorname{Im}\varphi(z)} \leq \frac{1}{\operatorname{Im} z}.$$

Moreover, if equality holds in one of the two inequalities above, then φ must be a Mobius transformation with real coefficients. That is, if equality holds, then φ is given by (3.1).

Theorem 3.1. *For any holomorphic map φ of π^+ such that $\varphi(\pi^+) \subset \pi^+$, the composition operator $C_\varphi : \mathcal{B}_\infty(\pi^+) \to \mathcal{B}_\infty(\pi^+)$ is bounded.*

Proof. For arbitrary $z \in \pi^+$ and $f \in \mathcal{B}_\infty(\pi^+)$

$$\begin{aligned} \operatorname{Im} z|(C_\varphi f)'(z)| &= \operatorname{Im} z|f'(\varphi(z))||\varphi'(z)| \\ &\leq \frac{\operatorname{Im} z}{\operatorname{Im}\varphi(z)}|\varphi'(z)|\|f\|_{\mathcal{B}_\infty}, \end{aligned}$$

and, consequently, by a simple application of the Schwarz-Pick Theorem on the upper half-plane,

$$\sup_{z\in\pi^+} \frac{\operatorname{Im} z}{\operatorname{Im}\varphi(z)}|\varphi'(z)| < 1,$$

we have $C_\varphi f \in \mathcal{B}_\infty(\pi^+)$. Hence by an analogue of the Closed Graph Theorem C_φ is bounded. □

4. Composition operators on $\mathcal{A}_\infty(\pi^+)$

Theorem 4.1. *Let φ be a holomorphic map of π^+ such that $\varphi(\pi^+) \subset \pi^+$. Then $C_\varphi : \mathcal{A}_\infty(\pi^+) \to \mathcal{A}_\infty(\pi^+)$ is bounded if and only if*

$$\sup_{z\in\pi^+} \frac{\operatorname{Im} z}{\operatorname{Im}\varphi(z)} < \infty. \tag{4.1}$$

Proof. First suppose that (4.1) holds. Then boundedness of C_φ on $\mathcal{A}_\infty(\pi^+)$ can be proved on similar lines as in the proof of Theorem 3.1.

Conversely, suppose C_φ is bounded. Fix a point $z_0 \in \pi^+$ and let $w = \varphi(z_0)$. Consider the function $f_w(z) = 1/(z - \overline{w})$. Then $f \in \mathcal{A}_\infty(\pi^+)$ and $||f_w||_{\mathcal{A}_\infty} \leq 1$. Boundedness of $C_\varphi : \mathcal{A}_\infty(\pi^+) \to \mathcal{A}_\infty(\pi^+)$ implies that there is a positive constant C such that, for each $z \in \pi^+$ we have $\operatorname{Im} z|f_w(\varphi(z))| \leq C$. In particular take $z = z_0$, we get

$$\frac{\operatorname{Im} z_0}{\operatorname{Im} \varphi(z_0)} \leq 2C.$$

Since $z_0 \in \pi$ is arbitrary, the result follows. □

Note. If $c = a + ib \in \pi^+$ and $\varphi(z) = c$ for all $z \in \pi^+$, then φ does not induce a bounded composition operator on $\mathcal{A}_\infty(\pi^+)$.

Corollary 4.2. *Let* $\varphi(z) = \dfrac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{R}$ *and* $ad - bc > 0$. *Then necessary and sufficient condition that* C_φ *is bounded on* $\mathcal{A}_\infty(\pi^+)$ *is that* $c = 0$.

Proof. First suppose that C_φ is bounded. Then for $z = x + iy \in \pi^+$,

$$\sup_{z\in\pi^+} \frac{\operatorname{Im} z}{\operatorname{Im} \varphi(z)} = \sup_{z\in\pi^+} \frac{(cx+d)^2 + c^2y^2}{(ad - bc)},$$

which is finite only if $c = 0$. Conversely, if $c = 0$, then $\varphi(z) = (a/d)z + (b/d)$, where $ad > 0$ and so

$$\sup_{z\in\pi^+} \frac{\operatorname{Im} z}{\operatorname{Im} \varphi(z)} = \frac{d}{a} < \infty.$$

Thus C_φ is bounded on $\mathcal{A}_\infty(\pi^+)$. □

5. Composition operators from $H^p(\pi^+)$ into $\mathcal{A}_\infty(\pi^+)$

Theorem 5.1. *Let* $1 \leq p < \infty$ *and* φ *be a holomorphic map of* π^+ *such that* $\varphi(\pi^+) \subset \pi^+$. *Then* $C_\varphi : H^p(\pi^+) \to \mathcal{A}_\infty(\pi^+)$ *is bounded if and only if*

$$\sup_{z\in\pi^+} \frac{\operatorname{Im} z}{(\operatorname{Im} \varphi(z))^{1/p}} < \infty.$$

Proof. First suppose that

$$M = \sup_{z\in\pi^+} \frac{\operatorname{Im} z}{(\operatorname{Im} \varphi(z))^{1/p}} < \infty.$$

By (2.1), $|f(z)|^p \leq ||f||_p^p/2\pi y$, for all $z = x + iy \in \pi^+$ and $f \in H^p(\pi^+)$. Thus, for $f \in H^p(\pi^+)$

$$\begin{aligned} ||C_\varphi f||_{\mathcal{A}_\infty} &= \sup_{z\in\pi^+} \operatorname{Im} z|C_\varphi f(z)| \\ &\leq \sup_{z\in\pi^+} \operatorname{Im} z/(2\pi \operatorname{Im} \varphi(z))^{1/p}||f||_p \\ &= (M/(2\pi)^{1/p})||f||_p. \end{aligned}$$

Hence $C_\varphi : H^p(\pi^+) \to \mathcal{A}_\infty(\pi^+)$ is bounded. Conversely, suppose that $C_\varphi : H^p(\pi^+) \to \mathcal{A}_\infty(\pi^+)$ is bounded. Fix a point $z_0 \in \pi^+$ and let $w = \varphi(z_0)$. Consider the function

$$f_w(z) = \frac{(\operatorname{Im} w)^{2-1/p}}{\pi^{1/p}(z-\overline{w})^2}.$$

Then

$$\begin{aligned} ||f_w||_p^p &= \sup_{y>0} \int_{-\infty}^{\infty} |f_w(x+iy)|^p dx \\ &= \frac{(\operatorname{Im} w)^{2p-1}}{\pi} \sup_{y>0} \int_{-\infty}^{\infty} \frac{1}{|z-\overline{w}|^{2p}} dx. \end{aligned}$$

Writing $w = u + iv$ and $z = x + iy$, we get

$$|z-\overline{w}|^{2p} \geq (v+y)^{2p-2}((x-u)^2 + (y+v)^2)$$

and so

$$\begin{aligned} ||f_w||_p^p &\leq \frac{v^{2p-1}}{\pi} \sup_{y>0} \frac{1}{(y+v)^{2p-1}} \int_{-\infty}^{\infty} \frac{y+v}{(x-u)^2+(y+v)^2} dx \\ &= \frac{v^{2p-1}}{\pi} \sup_{y>0} \frac{1}{(y+v)^{2p-1}} \pi \\ &= 1. \end{aligned}$$

Boundedness of $C_\varphi : H^p(\pi^+) \to \mathcal{A}_\infty(\pi^+)$ implies that there is a positive constant C such that, for each $z \in \pi^+$ we have $\operatorname{Im} z |f_w(\varphi(z))| \leq C$. In particular take $z = z_0$, we get

$$\frac{\operatorname{Im} z_0}{(\operatorname{Im} \varphi(z_0))^{1/p}} \leq 4\pi^{1/p} C.$$

Since $z_0 \in \pi^+$ is arbitrary, the result follows. □

Corollary 5.2. *Let* $\varphi(z) = \dfrac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{R}$ *and* $ad - bc > 0$. *Then* $C_\varphi : H^p(\pi^+) \to \mathcal{B}_\infty(\pi^+)$ *is bounded if and only if* $c = 0$ *and* $p = 1$.

Proof. First suppose that C_φ is bounded. Then for $z = x + iy \in \pi^+$,

$$\sup_{z\in\pi^+} \frac{\operatorname{Im} z}{(\operatorname{Im} \varphi(z))^{1/p}} = \sup_{z\in\pi^+} \frac{(cx+d)^2 + c^2y^2)^{1/p} y}{(ad-bc)^{1/p} y^{1/p}},$$

which is finite only if $c = 0$ and $p = 1$. Conversely, if $c = 0$ and $p = 1$, then

$$\sup_{z\in\pi^+} \frac{\operatorname{Im} z}{\operatorname{Im} \varphi(z)} = \frac{d}{a} < \infty.$$

Hence $C_\varphi : H^p(\pi^+) \to \mathcal{B}_\infty(\pi^+)$ is bounded. □

We next characterize boundedness of composition operators from $H^p(\pi^+)$ into $\mathcal{B}_\infty(\pi^+)$.

6. Composition operators from $H^p(\pi^+)$ into $\mathcal{B}_\infty(\pi^+)$

Theorem 6.1. *Let $1 \le p < \infty$ and φ be a holomorphic map of π^+ such that $\varphi(\pi^+) \subset \pi^+$. Then $C_\varphi : H^p(\pi^+) \to \mathcal{B}_\infty(\pi^+)$ is bounded if and only if*

$$\sup_{z\in\pi^+} \frac{\operatorname{Im} z}{(\operatorname{Im}\varphi(z))^{(p+1)/p}} |\varphi'(z)| < \infty. \tag{6.1}$$

Proof. First suppose that (6.1) holds. Let $f \in H^p(\pi^+)$. Then by Cauchy integral formula in π^+ [**1**], we have

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{(t-z)}dt,\ z = x+iy \in \pi^+.$$

Thus

$$\begin{aligned} |f'(z)| &\le \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|f(t)|}{|t-z|^2}dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|f(t)|}{(t-x)^2+y^2}dt. \end{aligned}$$

Since

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{((t-x)^2+y^2)}dt = 1,$$

x^p is convex, we have by Jensen's inequality [8], p 62,

$$\begin{aligned} |f'(z)|^p &\le \int_{-\infty}^{\infty} \frac{|f(t)|^p}{2^p y^p}\frac{y}{((t-x)^2+y^2)}dt \\ &= \frac{1}{2^p y^{p-1}}\int_{-\infty}^{\infty} \frac{|f(t)|^p}{((t-x)^2+y^2)}dt \\ &\le \frac{1}{2^p y^{p+1}}\int_{-\infty}^{\infty} |f(t)|^p dt. \end{aligned}$$

Thus

$$|f'(z)|^p \le \frac{||f||_p^p}{2^p y^{p+1}}.$$

Thus, for $f \in H^p(\pi^+)$

$$\begin{aligned} ||C_\varphi f||_{\mathcal{B}_\infty} &= \sup_{z\in\pi^+} \operatorname{Im} z |(C_\varphi f)'(z)| \\ &\le \sup_{z\in\pi^+} \operatorname{Im} z/(2^p \operatorname{Im}\varphi(z))^{(p+1)/p}|\varphi'(z)|\ ||f||_p \\ &= M||f||_p. \end{aligned}$$

Hence $C_\varphi : H^p(\pi^+) \to \mathcal{B}_\infty(\pi^+)$ is bounded. Conversely, suppose that $C_\varphi : H^p(\pi^+) \to \mathcal{B}_\infty(\pi^+)$ is bounded. Fix a point $z_0 \in \pi^+$ and let $w = \varphi(z_0)$. Consider the function

$$f_w(z) = \frac{(\operatorname{Im} w)^{2-1/p}}{\pi^{1/p}(z-\overline{w})^2}.$$

Then

$$f'_w(z) = \frac{(\operatorname{Im} w)^{2-1/p}}{\pi^{1/p}(z-\overline{w})^3}.$$

As in Theorem 5.1, we have $||f_w||_p^p \leq 1$. Boundedness of $C_\varphi : H^p(\pi^+) \to \mathcal{B}_\infty(\pi^+)$ implies that there is a positive constant C such that $||C_\varphi f||_{\mathcal{B}_\infty} \leq C||f_w||_p \leq C$. Hence, for each $z \in \pi^+$

$$\operatorname{Im} z|f'_w(\varphi(z))\varphi'(z)| \leq C.$$

In particular, putting $z = z_0$, we get

$$\frac{\operatorname{Im} z_0|\varphi'(z_0)|}{(\operatorname{Im}\varphi(z_0))^{(p+1)/p}} < 4\pi^{1/p}C.$$

Since $z_0 \in \pi^+$ is arbitrary, the result follows. □

Corollary 6.2. *Let $\varphi(z)$ be a holomorphic self-map of π^+ given by* (3.2). *Then $C_\varphi : H^p(\pi^+) \to \mathcal{B}_\infty(\pi^+)$ is not bounded.*

Acknowledgements. We are extremely thankful to the referee for going through the manuscript and pointing out several typographical and mathematical errors.

References

[1] P. L. Duren, *Theory of H^p spaces*, Pure and Applied Mathematics, Vol. 38 Academic Press, New York-London, 1970.

[2] K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall Series in Modern Analysis Prentice-Hall, Inc., Englewood Cliffs, N. J., 1962.

[3] J. E. Littlewood, *On inequalities in the theory of functions*, Proc. Lond. Math. Soc. (2) **23** (1925), 481–519.

[4] V. Matache, *Composition operators on H^p of the upper half-plane*, An. Univ. Timişoara Ser. Ştiinţ. Mat. **27** (1989), no. 1, 63–66.

[5] ———, *Composition operators on Hardy spaces of a half-plane*, Proc. Amer. Math. Soc. **127** (1999), no. 5, 1483–1491.

[6] S. Ohno and R. Zhao, *Weighted composition operators on the Bloch space*, Bull. Austral. Math. Soc. **63** (2001), no. 2, 177–185.

[7] S. Ohno, K. Stroethoff, and R. Zhao, *Weighted composition operators between Bloch-type spaces*, Rocky Mountain J. Math. **33** (2003), no. 1, 191–215.

[8] W. Rudin, *Real and complex analysis*, Third edition. McGraw-Hill Book Co., New York, 1987.

[9] S. D. Sharma, *Compact and Hilbert-Schmidt composition operators on Hardy spaces of the upper half-plane*, Acta Sci. Math. (Szeged) **46** (1983), no. 1-4, 197–202.

[10] S. D. Sharma, A. K. Sharma, and S. Ahmed, *Carleson measures in a vector-valued Bergman space*, J. Anal. Appl. **4** (2006), no. 1, 65–76.

[11] R. K. Singh, *A relation between composition operators on $H^2(D)$ and $H^2(P^+)$*, Pure Appl. Math. Sci. **1** (1974/75), no. 2, 1–5.

[12] R. K. Singh and S. D. Sharma, *Composition operators on a functional Hilbert space*, Bull. Austral. Math. Soc. **20** (1979), no. 3, 377–384.

[13] ———, *Noncompact composition operators*, Bull. Austral. Math. Soc. **21** (1980), no. 1, 125–130.

S. D. Sharma
Department of Mathematics
University of Jammu
Jammu-180006, India
E-mail address: somdatt_jammu@yahoo.co.in

Ajay K. Sharma
School of Applied Physics and Mathematics
Shri Mata Vashino Devi University
P/O Kakryal, Udhampur-182121, INDIA
E-mail address: aksju_76@yahoo.com

Shabir Ahmed
Department of Mathematics
University of Jammu
Jammu-180006, India