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NOTE ABOUT THE ISOCHRONICITY OF HAMILTONIAN SYSTEMS AND THE CURVATURE OF THE ENERGY

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Abstract. In this note we show the relationship between the period of a isochronous center of a planar Hamiltonian system and the Gauss curvature of the surface $S = (x, y, H(x, y))$ where H is the energy function of the system.

1. Introduction

It is well known that the planar analytic Hamiltonian systems are the planar differential systems of the form

$$\left\{ \begin{array}{l} \dot{x} = -\frac{\partial H}{\partial y}(x,y) \\ \dot{y} = \frac{\partial H}{\partial x}(x,y), \end{array} \right. \tag{1.1}$$

where H is an analytic function on $\mathbb{R}^2$. The solutions of these systems are contained in the level curves $\{H(x,y) = h,\ h \in \mathbb{R}\}$. A point p is called center if it has a neighborhood formed by periodic orbits. The largest neighborhood of p which is entirely covered by periodic orbits is called the *period annulus* of p and we will denote it by $\mathcal{P}$. The function which associates to any periodic orbit γ in $\mathcal{P}$ its period is called the *period function*. The center is called *isochronous center* when the period function is a constant ω. It is well known that only nondegenerate centers can be isochronous. And from now on we will assume that $H(0,0) = 0$ and the system (1.1) has a nondegenerate center at the origin.

Cima, Mañosas and Villadelprat in [1] study the isochronous centers of Hamiltonians systems of the form

$$H(x,y) = A(x) + B(x)y + C(x)y^2 \tag{1.2}$$

with A, B, C analytical functions. They prove that if the center is isochronous of period ω then

$$\frac{d^2\left(4AC - B^2\right)}{dx^2}(0) = \frac{8\pi^2}{\omega^2}. \tag{1.3}$$

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Clearly, this allows to calculate ω in an analytic way and without carrying out any integration.

However, in fact, this is a particular case of the following general result.

2. Result

Theorem 1. *Let*

$$S \equiv \overrightarrow{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$$
$$(x,y) \to \overrightarrow{x}(x,y) = (x,y,H(x,y))$$

be the surface generated by the energy function H. Let $K = K_{(0,0,0)}(S)$ be the Gauss curvature of S at the point $(0,0,0)$. Then:

If the center is isochronous of period ω we have

$$K\omega^2 = 4\pi^2.$$

Proof. We know that exist two analytical functions f, g such that

$$H = \frac{f^2+g^2}{2},$$

with $f(0,0) = g(0,0) = 0$ and

$$\det\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \frac{2\pi}{\omega} \tag{2.1}$$

in every point of a neighborhood U of the center point $(0,0)$, (see [2], [3]). We consider the surface:

$$S \equiv \overrightarrow{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$$
$$(x,y) \to \overrightarrow{x}(x,y) = (x,y,H(x,y)) = (x,y,\frac{f^2(x,y)+g^2(x,y)}{2}).$$

Then, the Gauss application is

$$N : S \to S^2$$
$$\overrightarrow{x}(x,y) \to N(x,y) = \frac{\frac{\partial \overrightarrow{x}}{\partial x} \times \frac{\partial \overrightarrow{x}}{\partial y}}{\left\| \frac{\partial \overrightarrow{x}}{\partial x} \times \frac{\partial \overrightarrow{x}}{\partial y} \right\|} = (N_1(x,y), N_2(x,y), N_3(x,y))$$

with

$$N_1(x,y) = -\frac{f\frac{\partial f}{\partial x}+g\frac{\partial g}{\partial x}}{\sqrt{(f\frac{\partial f}{\partial x}+g\frac{\partial g}{\partial x})^2+(f\frac{\partial f}{\partial y}+g\frac{\partial g}{\partial y})^2+1}}$$
$$N_2(x,y) = -\frac{f\frac{\partial f}{\partial y}+g\frac{\partial g}{\partial y}}{\sqrt{(f\frac{\partial f}{\partial x}+g\frac{\partial g}{\partial x})^2+(f\frac{\partial f}{\partial y}+g\frac{\partial g}{\partial y})^2+1}}$$
$$N_3(x,y) = \frac{1}{\sqrt{(f\frac{\partial f}{\partial x}+g\frac{\partial g}{\partial x})^2+(f\frac{\partial f}{\partial y}+g\frac{\partial g}{\partial y})^2+1}}.$$

We calculate and obtain that the Weingarten endomorphism

$$dN_{(0,0,0)}(S) : T_{(0,0,0)}(S) \to T_{(0,0,0)}(S)$$

at the point $(0,0,0)$ in the basis $\left\{\frac{\partial \vec{x}}{\partial x} = \vec{x_1}, \frac{\partial \vec{x}}{\partial y} = \vec{x_2}\right\}$ has the associated matrix

$$dN_{(0,0,0)}(S) \mid_{\vec{x_i}} = \begin{pmatrix} -\left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial g}{\partial x}\right)^2 & -\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial g}{\partial y} \\ -\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial g}{\partial y} & -\left(\frac{\partial f}{\partial y}\right)^2 - \left(\frac{\partial g}{\partial y}\right)^2 \end{pmatrix} \mid_{(0,0)} .$$

We also calculate and obtain:

$$K = K_{(0,0,0)}(S) = \det(dN_{(0,0,0)}(S) \mid_{\vec{x_i}}) = \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)^2 \mid_{(0,0)} .$$

Put in other way

$$K = \left(\det\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \mid_{(0,0)}\right)^2,$$

and, using (2.1), this implies

$$K\omega^2 = 4\pi^2.$$

This complete the proof. □

3. Consequence

We have said in the introduction that (1.3) is a particular case because: if the system has the form (1.2) then

$$\frac{\partial \vec{x}}{\partial x} = \vec{x_1} = (1, 0, \frac{dA}{dx}(x) + \frac{dB}{dx}(x)y + \frac{dC}{dx}(x)y^2)$$
$$\frac{\partial \vec{x}}{\partial y} = \vec{x_2} = (0, 1, B(x) + 2C(x)y).$$

We calculate and obtain:

$$dN_{(0,0,0)}(S) : T_{(0,0,0)}(S) \to T_{(0,0,0)}(S)$$
$$\vec{x_1} \mid_{(0,0)} \to dN_{(0,0,0)}(\vec{x_1} \mid_{(0,0)}) = -\frac{d^2A}{dx^2}(0)\vec{x_1} \mid_{(0,0)} - \frac{dB}{dx}(0)\vec{x_2} \mid_{(0,0)}$$
$$\vec{x_2} \mid_{(0,0)} \to dN_{(0,0,0)}(\vec{x_2} \mid_{(0,0)}) = -\frac{dB}{dx}(0)\vec{x_1} \mid_{(0,0)} -2C(0)\vec{x_2} \mid_{(0,0)} .$$

and using the Theorem 1 we have

$$K = 2C(0)\frac{d^2A}{dx^2}(0) - \frac{dB}{dx}(0)^2 = \frac{4\pi^2}{\omega^2}.$$

But if the center is isochronous then $\frac{dA}{dx}(0) = A(0) = B(0) = 0$ (see [1]); and in consequence

$$\frac{d^2\left(4AC - B^2\right)}{dx^2}(0) = 4C(0)\frac{d^2A}{dx^2}(0) - 2\frac{dB}{dx}(0)^2 = \frac{8\pi^2}{\omega^2}.$$

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