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# HIGH-DEGREE INTERPOLATION RULES GENERATED BY A LINEAR FUNCTIONAL

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ABSTRACT. We construct high-degree interpolation rules using not only pointwise values of a function but also of its derivatives up to the $p$-th order at equally spaced nodes on a closed and bounded interval of interest by introducing a linear functional from which we produce systems of linear equations. The linear systems will lead to a conclusion that the rules are uniquely determined for the nodes. An example is provided to compare the rules with the classical interpolating polynomials.

## 1. Introduction

Given any function, defined and continuous on a closed and bounded interval, there exists a polynomial that is as close to the given function as desired. This result is obtained from the Weierstrass approximation theorem [8]. This is an important reason for considering the class of polynomials in the approximation of functions. First we may consider the Taylor polynomials as the interpolating polynomials. But the Taylor polynomials have the property that all the information used in the approximation is concentrated at a single point. This fact limits Taylor polynomial approximation to the situation in which the approximation is needed only at points close to the single point. A good interpolation polynomial requires to provide a relatively accurate approximation over an entire interval of interest. The Hermite interpolating polynomials are among very useful and well-known classes of functions for such a requirement [1, 2, 6]. The fundamental concept of the Hermite interpolation theory was given in [6] and the divided difference method was considered to make it easier to construct the Hermite interpolating polynomials [7]. The core in generating the Hermite interpolating polynomials is to use the information of a function and its derivatives up to a certain order at nodes on the interval of interest. In the paper, we introduce a new method to construct interpolation rules which use the information of a function and its derivatives up to the $p$-th order at equally spaced nodes on a closed and bounded interval. Restricting the method to the case of $p = 1$, we will have the results of [5]. Recently,

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integration formulas constructed by linear functionals using a function or its first derivative were considered for oscillatory functions [3, 4]. In Section 2, we construct high-degree interpolation rules from the system of linear equations which are generated by a linear functional. In the process of the construction, a matrix is investigated whose determinant is not zero and its property leads us to be able to provide the interpolation rules for the equidistant nodes. In Section 3, a classical interpolating polynomial, based on the general Hermite interpolation theory, is presented and compared with the rule to be constructed in Section 2.

## 2. High-degree interpolation rules

Consider a function $f$ and its interpolation rule, denoted by $I$, which involves not only pointwise values of the function but also of its derivatives up to the p-th order at equidistant nodes, viz.:

$$\begin{aligned} & f(x_0+Nht) \approx I(t) \\ (1) \qquad = & \sum_{k=-N}^{N} \alpha_k f(x_0+kh) \\ & + h \sum_{k=-N}^{N} \alpha_k^{(1)} f^{(1)}(x_0+kh) + \cdots + h^p \sum_{k=-N}^{N} \alpha_k^{(p)} f^{(p)}(x_0+kh) \end{aligned}$$

where $N$ is a positive integer, $x_0$ is the middle node on the interval of interest, the other nodes on the interval are equally spaced by $h$ and $-1 \le t \le 1$. The rule $I(t)$, defined on $[-1,1]$, will approximate the function $f$ on the interval $[x_0 - Nh, x_0 + Nh]$ by using the values of the function and its derivatives up to the p-th order at nodes $x_0 - Nh, \ldots, x_0 - h, x_0, x_0 + h, \ldots, x_0 + Nh$. For convenience, keep taking the notations $\alpha_k$ and $\alpha_k^{(j)}$ instead of $\alpha_k(t)$ and $\alpha_k^{(j)}(t)$ indicating that $\alpha_k$ and $\alpha_k^{(j)}$ depend on $t$, where $j = 1, 2, \ldots, p$. From the definition of $I(t)$, we consider a linear functional $L(f(x), h, \mathcal{C})$,

$$\begin{aligned} L(f(x),h,\mathcal{C}) \;=\; & f(x+Nht) \\ & - \sum_{k=-N}^{N} \alpha_k f(x+kh) \\ (2) \qquad & -h \sum_{k=-N}^{N} \alpha_k^{(1)} f^{(1)}(x+kh) \\ & \vdots \\ & -h^p \sum_{k=-N}^{N} \alpha_k^{(p)} f^{(p)}(x+kh), \end{aligned}$$

where $\mathcal{C}$ is the vector of coefficients $\alpha_k$ and $\alpha_k^{(j)}$ which have to be expressed in terms of the variable $t$, $\mathcal{C} = (\alpha_{-N}, \ldots, \alpha_N, \alpha_{-N}^{(1)}, \ldots, \alpha_N^{(1)}, \ldots, \alpha_{-N}^{(p)}, \ldots, \alpha_N^{(p)})$. When the values of the function $f$ and its derivatives up to the $p$-th order at the nodes are assumed to be known, our aim is to determine the values of the coefficients $\alpha_k$ and $\alpha_k^{(j)}$ from the conditions

$$L(x^{n-1}, h, \mathcal{C}) = 0 \quad (n = 1, 2, \ldots). \tag{3}$$

By inserting each monomial $f(x) = 1, x, x^2, \ldots$ into (2), we get

(4)
$$\begin{aligned}
L(1, h, \mathcal{C}) &= 1 - \sum_{k=-N}^{N} \alpha_k, \\
L(x, h, \mathcal{C}) &= x(1 - \sum_{k=-N}^{N} \alpha_k) + h(Nt - \sum_{k=-N}^{N} \alpha_k k - \sum_{k=-N}^{N} \alpha_k^{(1)}), \\
L(x^2, h, \mathcal{C}) &= x^2(1 - \sum_{k=-N}^{N} \alpha_k) + 2xh(Nt - \sum_{k=-N}^{N} \alpha_k k - \sum_{k=-N}^{N} \alpha_k^{(1)}) \\
&\quad + h^2((Nt)^2 - \sum_{k=-N}^{N} \alpha_k k^2 - 2 \sum_{k=-N}^{N} \alpha_k^{(1)} k - 2 \cdot 1 \sum_{k=-N}^{N} \alpha_k^{(2)}), \\
&\;\;\vdots
\end{aligned}$$

The values of $L(x^m, h, \mathcal{C})$ at $x = 0$ for each $m = 0, 1, 2, \ldots$, will be denoted by $L_m(h, \mathcal{C})$. Then we have

$$\begin{aligned}
L_0(h, \mathcal{C}) &= 1 - \sum_{k=0}^{N} \alpha_k^+, \\
L_1(h, \mathcal{C}) &= h\left(Nt - \sum_{k=0}^{N} \alpha_k^- k - \sum_{k=0}^{N} \alpha_k^{(1)^+}\right), \\
L_2(h, \mathcal{C}) &= h^2\left((Nt)^2 - \sum_{k=0}^{N} \alpha_k^+ k^2 - 2\sum_{k=0}^{N} \alpha_k^{(1)^-} k - 2 \cdot 1 \sum_{k=0}^{N} \alpha_k^{(2)^+}\right), \\
&\;\;\vdots
\end{aligned} \tag{5}$$

in general, for odd $m \geq 3$

$$\begin{aligned}
L_m(h, \mathcal{C}) \;=\; & h^m \Big((Nt)^m - \textstyle\sum_{k=0}^{N} \alpha_k^- k^m - m \sum_{k=0}^{N} \alpha_k^{(1)^+} k^{m-1} \\
& - m(m-1) \textstyle\sum_{k=0}^{N} \alpha_k^{(2)^-} k^{m-2} - \cdots \\
& - m(m-1) \cdots (m-(p-1)) \textstyle\sum_{k=0}^{N} \triangle \times k^{m-p}\Big),
\end{aligned} \tag{6}$$

and for even $m \geq 4$

$$(7)\quad \begin{aligned} L_m(h,\mathcal{C}) \;=\; & h^m \Big( (Nt)^m - \textstyle\sum_{k=0}^{N} \alpha_k^{+} k^m - m \sum_{k=0}^{N} \alpha_k^{(1)^-} k^{m-1} \\ & - m(m-1) \textstyle\sum_{k=0}^{N} \alpha_k^{(2)^+} k^{m-2} - \cdots \\ & - m(m-1)\cdots(m-(p-1)) \textstyle\sum_{k=0}^{N} \square \times k^{m-p} \Big), \end{aligned}$$

where

(i) for $1 \leq j \leq p$ and $1 \leq k \leq N$

$$(8)\quad \begin{aligned} & \alpha_0^{\pm} = \alpha_0, \quad \alpha_k^{+} = \alpha_{-k} + \alpha_k, \quad \alpha_k^{-} = -\alpha_{-k} + \alpha_k, \\ & \alpha_0^{(j)^{\pm}} = \alpha_0^{(j)}, \quad \alpha_k^{(j)^+} = \alpha_{-k}^{(j)} + \alpha_k^{(j)}, \quad \alpha_k^{(j)^-} = -\alpha_{-k}^{(j)} + \alpha_k^{(j)}, \end{aligned}$$

(ii)

$$(9)\quad \Delta = \begin{cases} \alpha_k^{(p)^-} & \text{if} \quad p = \text{even} \\ \alpha_k^{(p)^+} & \text{if} \quad p = \text{odd} \end{cases}$$

and

$$(10)\quad \square = \begin{cases} \alpha_k^{(p)^+} & \text{if} \quad p = \text{even} \\ \alpha_k^{(p)^-} & \text{if} \quad p = \text{odd}. \end{cases}$$

Therefore (4) can be rearranged as follows:

$$(11)\quad \begin{aligned} L(1,h,\mathcal{C}) &= L_0(h,\mathcal{C}), \\ L(x,h,\mathcal{C}) &= xL_0(h,\mathcal{C}) + L_1(h,\mathcal{C}), \\ L(x^2,h,\mathcal{C}) &= x^2 L_0(h,\mathcal{C}) + 2xL_1(h,\mathcal{C}) + L_2(h,\mathcal{C}), \\ &\vdots \end{aligned}$$

Since $L$ in (2) is a linear functional, it follows that, upon taking $f(x)$ as an expansion of power functions, $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, we have

(12)

$$\begin{aligned} L(f(x),h,\mathcal{C}) &= \sum_{m=0}^{\infty} a_m L(x^m,h,\mathcal{C}) \\ &= L_0(h,\mathcal{C})(a_0 + a_1 x + a_2 x^2 + \cdots) \\ &\quad + L_1(h,\mathcal{C})(a_1 + 2a_2 x + 3a_3 x^2 + \cdots) \\ &\quad + L_2(h,\mathcal{C})(a_2 + 3a_3 x + 6a_4 x^2 + \cdots) + \cdots \\ &= L_0(h,\mathcal{C}) f(x) + \frac{1}{1!} L_1(h,\mathcal{C}) f^{(1)}(x) + \frac{1}{2!} L_2(h,\mathcal{C}) f^{(2)}(x) + \cdots \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} L_m(h,\mathcal{C}) f^{(m)}(x). \end{aligned}$$

Next, let us determine the values of the coefficients $\alpha_k$ and $\alpha_k^{(j)}$ such that the functional $L$ is identically vanishing at any $x$ and $h \neq 0$ for as many successive terms as the number of the coefficients. For such purpose it is natural to impose that

(13) $$L_m(h,\mathcal{C}) = 0, \quad m = 0, 1, \ldots, (p+1)(2N+1) - 1,$$

since the number of the coefficients of the rule $I$ in (1) is $(p+1)(2N+1)$. We now obtain a system of $(p+1)(2N+1)$ linear equations in $\alpha_k$ and $\alpha_k^{(j)}$ (or $\alpha_k^{\pm}$ and $\alpha_k^{(j)^{\pm}}$). But, instead of handling the system directly to find its solution $\alpha_k$ and $\alpha_k^{(j)}$, we break the linear system into two types of smaller linear systems,

(14) $$M_o X = T_o \text{ and } M_e Y = T_e,$$

which are easier to handle individually. The former is generated from the odd equations, that is $L_m(h,\mathcal{C}) = 0$ for odd $m$, while the latter from the even equations, $L_m(h,\mathcal{C}) = 0$ for even $m$. Therefore the former governs coefficients $\alpha_k^-, \alpha_k^{(1)^+}, \alpha_k^{(2)^-}, \ldots$, while the latter does $\alpha_k^+, \alpha_k^{(1)^-}, \alpha_k^{(2)^+}, \ldots$. In detail, we have that

(15)
$$M_o = \begin{pmatrix} 1 & \ldots & N & 1 & 1 & \ldots & 1 & \ldots & 0 & \ldots & 0 \\ 1^3 & \ldots & N^3 & 0 & 3\cdot 1^2 & \ldots & 3\cdot N^2 & \ldots & 0 & \ldots & 0 \\ & & & & & & & & & & \\ & & & & & & & & & & \\ V_1^{(0)} & \ldots & V_N^{(0)} & V_0^{(1)} & V_1^{(1)} & \ldots & V_N^{(1)} & \ldots & V_1^{(p)} & \ldots & V_N^{(p)} \end{pmatrix}$$

where, for $k = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, p$,

(i) $p =$ odd

$$\begin{aligned} V_k^{(0)} &= k^{2N(p+1)+p}, \\ V_k^{(j)} &= (2N(p+1)+p)(2N(p+1)+p-1) \\ &\quad \cdots (2N(p+1)+p-(j-1))k^{2N(p+1)+p-j}, \end{aligned}$$

(ii) $p =$ even

$$\begin{aligned} V_k^{(0)} &= k^{2N(p+1)+p-1}, \\ V_k^{(j)} &= (2N(p+1)+p-1)(2N(p+1)+p-2) \\ &\quad \cdots (2N(p+1)+p-j)k^{2N(p+1)+p-1-j}. \end{aligned}$$

In particular,

$$V_0^{(j)} = \begin{cases} 0 \quad (j = 1, 3, \cdots, p) & \text{if } p = \text{odd} \\ 0 \quad (j = 1, 3, \cdots, p-1) & \text{if } p = \text{even}. \end{cases}$$

$X$ has two different forms in the first equation of (14), depending on $p$, that is, for odd $p$,

(16)
$$X = (\alpha_1^- \quad \ldots \quad \alpha_N^- \quad \alpha_0^{(1)^+} \quad \alpha_1^{(1)^+} \quad \ldots \quad \alpha_N^{(1)^+} \quad \ldots \quad \alpha_1^{(p)^+} \quad \ldots \quad \alpha_N^{(p)^+})^T,$$

and for even $p$,
(17)
$$X = (\alpha_1^- \;\dots\; \alpha_N^- \;\; \alpha_0^{(1)^+} \;\; \alpha_1^{(1)^+} \;\dots\; \alpha_N^{(1)^+} \;\dots\; \alpha_1^{(p)^-} \;\dots\; \alpha_N^{(p)^-})^T$$

and $T_o$ is given as follows:

$$(18)\qquad T_o = \begin{cases} (Nt \;\; (Nt)^3 \;\; \dots \;\; (Nt)^{2N(p+1)+p})^T & \text{if } p = \text{odd}, \\ (Nt \;\; (Nt)^3 \;\; \dots \;\; (Nt)^{2N(p+1)+p-1})^T & \text{if } p = \text{even}. \end{cases}$$

Note that, for even $j$, $\alpha_0^{(j)^-}$ (in fact, $\alpha_0^{(j)}$) does not appear in $X$. As a result, there is not any column in the matrix $M_o$ in the first equation of (14) corresponding to the component $\alpha_0^{(j)^-}$ for even $j$. But, there exist columns in $M_o$ corresponding to $\alpha_0^{(j)^+}$ for odd $j$.

Let us investigate the existence of the unique solution of the linear system, $M_oX = T_o$. This will be done by constructing a matrix whose determinant is not zero. For distinct real number $w_k$, let $W_k$ denote a column vector by

$$(w_k^{p+q}, w_k^{p+q+2}, \dots, w_k^{p+q+2j}, \dots, w_k^{p+q+2(N(p+1)-1)})^T.$$

Define a $N(p+1) \times N(p+1)$ matrix $W$ as

$$(19)\qquad W = (W_1, \dots, W_N, W_1^{(1)}, \dots, W_N^{(1)}, \dots, W_1^{(p)}, \dots, W_N^{(p)}),$$

where the superscript on $W_k$ means the order of the derivative of $W_k$ with respect to $w_k$, that is $W_k^{(j)} = d^jW_k/dw_k^j$. Consider the determinant of $W$, $\det(W)$, as a polynomial $P(w_1)$ in $w_1$ and expand $\det(W)$ using the $(Nk+1)$th columns of $W$ ($k = 0, 1, \dots, p$). Then, the lowest degree term in $P(w_1)$ has degree

$$(20)\qquad \frac{1}{2}(p+1)(3p+2q).$$

Thus, we have

$$(21)\qquad P(w_1) = w_1^{\frac{1}{2}(p+1)(3p+2q)}\tilde{P}(w_1),$$

where $\tilde{P}(w_1)$ is a polynomial in $w_1$ whose coefficients consist of polynomials in $w_2, \dots, w_N$. Moreover, $w_k$ and $-w_k$ ($k = 2, 3, \dots, N$) are zeros of $P(w_1)$ with multiplicity $(p+1)^2$, respectively. The multiplicity comes from the fact that the determinant of a matrix with two equal columns is zero. Note that the derivative of $\det(W)$ with respect to $w_1$, $\frac{d}{dw_1}\det(W)$, is the sum of $p+1$ determinants obtained by replacing the elements of the $(Nk+1)$th column of $W$ ($k = 0, 1, \dots, p$) by their derivatives with respect to $w_1$, the second derivative of $\det(W)$ with respect to $w_1$ is obtained from applying the replacement to each determinant in the derivative of $\det(W)$ and so on. Therefore

$$(22)\qquad w_1^{\frac{1}{2}(p+1)(3p+2q)}\prod_{i=2}^{N}(w_1^2 - w_i^2)^{(p+1)^2}$$

is a factor of $\det(W)$. Repeat the above procedure to $\det(W)$ for each $w_k$ and then get other factors of it,

$$w_k^{\frac{1}{2}(p+1)(3p+2q)} \prod_{i=k+1}^{N} (w_k^2 - w_i^2)^{(p+1)^2} \quad \text{for } k = 2, 3, \ldots, N. \tag{23}$$

Thus, the determinant, $\det(W)$, has a factor

$$\prod_{k=1}^{N} w_k^{\frac{1}{2}(p+1)(3p+2q)} \prod_{i=k+1}^{N} (w_k^2 - w_i^2)^{(p+1)^2} \tag{24}$$

so that its degree in all $w_k$ is at least

$$N^2(p+1)^2 + N(p+1)(\frac{1}{2}p + q - 1) \tag{25}$$

which is calculated from

$$\textstyle\sum_{k=1}^{N} \left(\frac{1}{2}(p+1)(3p+2q) + 2(p+1)^2(N-k)\right). \tag{26}$$

On the other hand, a direct calculation of the determinant of $W$ shows that the degree of $\det(W)$ in all $w_k$ is exactly the same as (25). Let us explain such a fact. We first take $w_k^p$ out of each column $W_k$ between the first column and the $N$th column of the $\det(W)$ and at the same time $w_k^{p-j}$ out of each column $W_k^{(j)}$ between the $(Nj+1)$th column and the $N(j+1)$th column of the $\det(W)$ for each $j = 1, 2, \ldots, p-1$. This step brings that the exponents, depending on rows of $W$, become all same. That is, all components of the first row of $W$ have an exponent $q$, the second row's components do all $q+2$, and so on. So the degree of $\det(W)$ in all $w_k$ becomes

$$N\sum_{k=1}^{p} k + \sum_{k=0}^{N(p+1)-1} (q+2k), \tag{27}$$

resulting in (25). Therefore, we finally have

$$\det(W) = \beta \prod_{k=1}^{N} w_k^{\frac{1}{2}(p+1)(3p+2q)} \prod_{i=k+1}^{N} (w_k^2 - w_i^2)^{(p+1)^2}, \tag{28}$$

where $\beta$ is a constant which is independent of $w_1, w_2, \ldots, w_N$. The result given in (28) reveals that the value of $\det(W)$ is not zero as long as all $w_k (\neq 0)$ are distinct. Returning to the first equation of (14), let us discuss whether or not the value of the determinant of $M_o$ can be zero. When $p$ is odd, that is $p = 2\mu - 1$ ($\mu = 1, 2, \ldots$), expand the determinant of $M_o$ according to the $(N+1)$, $(3N+2)$, $\ldots$, $(pN+\mu)$th column of the matrix $M_o$, respectively, and then only one minor whose size is $(p+1)N \times (p+1)N$ survives while the other minors all vanish. The survived minor is exactly the same, up to nonzero constant, when $w_k = k$ and $q = 2$ in the determinant of $W$. When $p$ is even, $p = 2\mu$ ($\mu = 1, 2, \ldots$), expand the determinant of $M_o$ according to the $(N+1)$, $(3N+2)$, $\ldots$, $((p-1)N+\mu)$th column of the matrix $M_o$, respectively. The same arguments as the case of odd $p$ come from when $w_k = k$ and $q = 1$ in the determinant of $W$. So far the fact that the determinant of $M_o$ in (15) is not zero, irrespective of $p$, has been established.

On the other hand, in the second equation of (14), we have

(29)

$$M_e = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 & \dots & 1 & \dots & 0 & \dots & 0 \\ 0 & 1^2 & \dots & N^2 & 2\cdot 1 & \dots & 2\cdot N & \dots & 0 & \dots & 0 \\ & & & & & & & & & & \\ \tilde{V}_0^{(0)} & \tilde{V}_1^{(0)} & \dots & \tilde{V}_N^{(0)} & \tilde{V}_1^{(1)} & \dots & \tilde{V}_N^{(1)} & \dots & \tilde{V}_1^{(p)} & \dots & \tilde{V}_N^{(p)} \end{pmatrix}$$

where, for $k = 1, 2, \dots, N$ and $j = 1, 2, \dots, p$,

(i) $p =$ odd

$$\begin{aligned} \tilde{V}_k^{(0)} &= k^{2N(p+1)+p-1}, \\ \tilde{V}_k^{(j)} &= (2N(p+1)+p-1)(2N(p+1)+p-2) \\ &\quad \cdots (2N(p+1)+p-j)k^{2N(p+1)+p-1-j}, \end{aligned}$$

(ii) $p =$ even

$$\begin{aligned} \tilde{V}_k^{(0)} &= k^{2N(p+1)+p}, \\ \tilde{V}_k^{(j)} &= (2N(p+1)+p)(2N(p+1)+p-1) \\ &\quad \cdots (2N(p+1)+p-(j-1))k^{2N(p+1)+p-j}. \end{aligned}$$

As might be expected,

$$\tilde{V}_0^{(j)} = \begin{cases} 0 & (j = 0, 2, \cdots, p-1) \quad \text{if } p = \text{odd} \\ 0 & (j = 0, 2, \cdots, p) \quad\quad \text{if } p = \text{even}. \end{cases}$$

Compared with the forms of $X$, $Y$ also has two different forms in the second equation of (14), depending on $p$, that is, for odd $p$,

(30)

$$Y = (\alpha_0^+ \quad \alpha_1^+ \quad \dots \quad \alpha_N^+ \quad \alpha_1^{(1)^-} \quad \dots \quad \alpha_N^{(1)^-} \quad \dots \quad \alpha_1^{(p)^-} \quad \dots \quad \alpha_N^{(p)^-})^T,$$

and for even $p$,

(31)

$$Y = (\alpha_0^+ \quad \alpha_1^+ \quad \dots \quad \alpha_N^+ \quad \alpha_1^{(1)^-} \quad \dots \quad \alpha_N^{(1)^-} \quad \dots \quad \alpha_1^{(p)^+} \quad \dots \quad \alpha_N^{(p)^+})^T.$$

In turn, $T_e$ is given by

$$T_e = \begin{cases} (1 \quad (Nt)^2 \quad \dots \quad (Nt)^{2N(p+1)+p-1})^T & \text{if } p = \text{odd}, \\ (1 \quad (Nt)^2 \quad \dots \quad (Nt)^{2N(p+1)+p})^T & \text{if } p = \text{even}. \end{cases} \tag{32}$$

Some other arguments involving the first equation of (14) are similarly stated about the second equation of (14) as follows. For odd $j$, there is not any column in the matrix $M_e$ corresponding to the component $\alpha_0^{(j)^-}$ because $\alpha_0^{(j)^-}$ (in fact, $\alpha_0^{(j)}$) does not appear in $Y$. But, there exist columns in $M_e$ corresponding to $\alpha_0^{(j)^+}$ for even $j$. Likewise, by similarly taking the procedures to expand the determinant of $M_o$, the same conclusion as the case of $M_o$ is obtained for the linear equation associated with $M_e$ from the following substitutions into the matrix $W$:

(i) for odd $p$, $w_k = k$ and $q = 1$,
(ii) for even $p$, $w_k = k$ and $q = 2$.

Hence each linear system of (14) has the unique solution, respectively. It implies that all the coefficients $\alpha_k$ and $\alpha_k^{(j)}$ of the rule (1) can be determined by algebraically manipulating the relations given in (8). Next section, an example case will be given and investigated.

## 3. Discussion

Let us consider the interpolation rule $I$ in (1) for the case of $N = 1$ and $p = 1$. Section 2 says that

$$(33)\qquad M_o = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 0 & 5 \end{pmatrix}, \qquad X = \begin{pmatrix} \alpha_1^- \\ \alpha_0^{(1)^+} \\ \alpha_1^{(1)^+} \end{pmatrix}, \qquad T_o = \begin{pmatrix} t \\ t^3 \\ t^5 \end{pmatrix}$$

and

$$(34)\qquad M_e = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix}, \qquad Y = \begin{pmatrix} \alpha_0^+ \\ \alpha_1^+ \\ \alpha_1^{(1)^-} \end{pmatrix}, \qquad T_e = \begin{pmatrix} 1 \\ t^2 \\ t^4 \end{pmatrix}.$$

From solving the two linear systems,

$$M_o X = T_o \text{ and } M_e Y = T_e,$$

respectively, the interpolation rule (1) becomes

$$\begin{aligned} I(t) \;&=\; \frac{1}{4}t^2(4+3t)(t-1)^2 f(x_0-h) \;+\; (t+1)^2(t-1)^2 f(x_0) \\ (35)\qquad &+\; \frac{1}{4}t^2(4-3t)(t+1)^2 f(x_0+h) \;+\; \frac{1}{4}ht^2(t+1)(t-1)^2 f^{(1)}(x_0-h) \\ &+\; ht(t+1)^2(t-1)^2 f^{(1)}(x_0) \;+\; \frac{1}{4}ht^2(t-1)(t+1)^2 f^{(1)}(x_0+h), \end{aligned}$$

where all the coefficients of the rule are computed from Eqs. (8). In fact, the classical interpolating polynomial, say $P_5$, of degree at most five agreeing with $f$ and $f^{(1)}$ at three nodes $x_0 - h, x_0$ and $x_0 + h$, can be constructed by the general interpolation theories introduced in Chapter 3 of [6] which was due to Hermite. In order to construct the classical interpolating polynomial, the Hermite theories guide us to follow two steps: one is to express the remainder of the interpolating polynomial for a given function by a line integral and the other is to evaluate the line integral by residues. Through the steps,

$$(36)\quad P_5(x) = \sum_{k=1}^{3}\sum_{j=0}^{1} \frac{f^{(j)}(z_k)(x-z_1)^2(x-z_2)^2(x-z_3)^2}{j!(x-z_k)^2} \sum_{s=0}^{1-j} c_s^{(k)}(x-z_k)^{j+s},$$

where

(37) $$z_1 = x_0 - h, \quad z_2 = x_0, \quad z_3 = x_0 + h,$$

(38) $$\begin{aligned} c_0^{(1)} &= \frac{1}{(z_1-z_2)^2(z_1-z_3)^2}, & c_1^{(1)} &= -\frac{2(z_1-z_3)+(z_1-z_2)}{(z_1-z_2)^3(z_1-z_3)^3}, \\ c_0^{(2)} &= \frac{1}{(z_2-z_1)^2(z_2-z_3)^2}, & c_1^{(2)} &= -\frac{2(z_2-z_3)+(z_2-z_1)}{(z_2-z_1)^3(z_2-z_3)^3}, \\ c_0^{(3)} &= \frac{1}{(z_3-z_1)^2(z_3-z_2)^2}, & c_1^{(3)} &= -\frac{2(z_3-z_2)+(z_3-z_1)}{(z_3-z_1)^3(z_3-z_2)^3}. \end{aligned}$$

After first substituting all $z_k$ and $c_s^{(k)}$ of (37) and (38) into the right hand side of (36) and then rearranging it, $P_5(x)$ becomes

(39) $$\begin{aligned} &(1 + \frac{3}{h}(x - (x_0 - h))) \left(\frac{(x - x_0)(x - (x_0 + h))}{2h^2}\right)^2 f(x_0 - h) \\ &+ \left(\frac{(x - (x_0 - h))(x - (x_0 + h))}{h^2}\right)^2 f(x_0) \\ &+(1 - \frac{3}{h}(x - (x_0 + h))) \left(\frac{(x - (x_0 - h))(x - x_0)}{2h^2}\right)^2 f(x_0 + h) \\ &+(x - (x_0 - h)) \left(\frac{(x - x_0)(x - (x_0 + h))}{2h^2}\right)^2 f^{(1)}(x_0 - h) \\ &+(x - x_0) \left(\frac{(x - (x_0 - h))(x - (x_0 + h))}{h^2}\right)^2 f^{(1)}(x_0) \\ &+(x - (x_0 + h)) \left(\frac{(x - (x_0 - h))(x - x_0)}{2h^2}\right)^2 f^{(1)}(x_0 + h). \end{aligned}$$

By using the change of variables,

(40) $$x = x_0 + ht,$$

the interpolating polynomial $P_5(x)$, defined on $[x_0 - h, x_0 + h]$, is transformed into the $t$-dependent rule $I(t)$ given in (35). Also, the rule $I(t)$ can be linearly transformed into the classical interpolating polynomial. Such transformations make sense from the existence and uniqueness of the Hermite interpolating polynomial because we have

(41)
$$\begin{aligned} &P_5(x_0 - h) = f(x_0 - h), \qquad P_5(x_0) = f(x_0), \qquad P_5(x_0 + h) = f(x_0 + h), \\ &P_5^{(1)}(x_0 - h) = f^{(1)}(x_0 - h), P_5^{(1)}(x_0) = f^{(1)}(x_0), P_5^{(1)}(x_0 + h) = f^{(1)}(x_0 + h), \end{aligned}$$

from the relation $I(t) = P_5(x_0 + ht)$.

The general Hermite interpolation theories were established using Cauchy integral formulas and residue theorems while we only use matrix computations and simple algebraic calculations with Eqs.(8) to get the $t$-dependent interpolation rule $I$. Once the rule $I$ is obtained, in other words, the coefficients $\alpha_k$ and $\alpha_k^{(j)}$ of the rule are determined and expressed in terms of the parameter $t$, it is itself an interpolation rule and simultaneously it is expected to be seen that the rule is linearly transformed into the classical Hermite interpolating polynomial through the relation of the change of the variables given in (40) as we test the example case this section.

## References

[1] K. E. Atkinson, *An Introduction to Numerical Analysis*, John Wiley and Sons, 1989.
[2] R. L. Burden and J. D. Faires, *Numerical Analysis*, Brooks/Cole, 2001.
[3] K. J. Kim, *Quadrature rules for the integration of the product of two oscillatory functions with different frequencies*, Comput. Phys. Commun. **153** (2003), 135–144.
[4] ———, *Two-frequency-dependent Gauss quadrature rules*, J. Comput. Appl. Math. **174** (2005), 43–55.
[5] ———, *Polynomial-fitting interpolation rules generated by a linear functional*, Commun. Korean Math. Soc. **21** (2006), no. 2, 397–407.
[6] V. I. Krylov, *Approximate Calculation of Integrals*, Macmillan, New York, 1962.
[7] M. J. D. Powell, *Approximation Theory and Methods*, Cambridge University Press, Cambridge, 1988.
[8] W. Rudin, *Principles of Mathematical Analysis*, McGRAW-Hill, Singapore, 1976.

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