Commun. Korean Math. Soc. **22** (2007), No. 3, pp. 429–440

COMMON FIXED POINT THEOREM FOR WEAKLY COMPATIBLE OF FOUR MAPPINGS

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Abstract. In this paper, a common fixed point theorem for weak compatible maps in complete fuzzy metric spaces is proved.

1. Introduction and preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [22] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application. George and Veeramani [5] and Kramosil and Michalek [9] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics particularly in connections with both string and $\epsilon^{(\infty)}$ theory which were given and studied by El Naschie [1, 2, 3, 4, 19]. Many authors [7, 11, 16, 13, 14, 15] have proved fixed point theorem in fuzzy (probabilistic) metric spaces. Vasuki [20] obtained the fuzzy version of common fixed point theorem which had extra conditions. In fact, Vasuki proved fuzzy common fixed point theorem by a strong definition of Cauchy sequence (see Note 3.13 and Definition 3.15 of [5] also [18, 21]). In this paper, we prove a common fixed point theorem in fuzzy metric spaces for arbitrary t-norms and modified definition of Cauchy sequence in George and Veeramani's sense.

Definition 1.1. A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t-norm if it satisfies the following conditions

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0,1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norm are $a*b = ab$ and $a*b = \min(a,b)$.

Received August 24, 2006.
2000 *Mathematics Subject Classification.* 54E40, 54E35, 54H25.
Key words and phrases. fuzzy contractive mapping, complete fuzzy metric space.

Definition 1.2. A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$ if and only if $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(5) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Let $(X, M, *)$ be a fuzzy metric space . For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

Let $(X, M, *)$ be a fuzzy metric space. Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then τ is a topology on X (induced by the fuzzy metric M). This topology is Hausdorff and first countable. A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exits $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$. The fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent. A subset A of X is said to be F-bounded if there exists $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

Lemma 1.3 ([5]). *Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, t)$ is nondecreasing with respect to t, for all x, y in X.*

Definition 1.4. Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous function on $X^2 \times (0, \infty)$ if

$$\lim_{n \to \infty} M(x_n, y_n, t_n) = M(x, y, t).$$

Whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X^2 \times (0, \infty)$ converges to a point $(x, y, t) \in X^2 \times (0, \infty)$ i.e.

$$\lim_{n \to \infty} M(x_n, x, t) = \lim_{n \to \infty} M(y_n, y, t) = 1 \text{ and } \lim_{n \to \infty} M(x, y, t_n) = M(x, y, t).$$

Lemma 1.5. *Let $(X, M, *)$ be a fuzzy metric space. Then M is continuous function on $X^2 \times (0, \infty)$.*

Proof. See Proposition 1 of [10]. □

Example 1.6. Let $X = \mathbb{R}$. Denote $a * b = a.b$ for all $a, b \in [0, 1]$. For each $t \in]0, \infty[$, define

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

for all $x, y \in X$.

Definition 1.7. Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.

Definition 1.8. Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then the mappings are said to be compatible if

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1, \forall t > 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X.$$

Proposition 1.9 ([17]). *Self-mappings A and S of a fuzzy metric space (X,M,*) are compatible, then they are weak compatible.*

Lemma 1.10. *Let $(X, M, *)$ be a fuzzy metric space. If we define $E_{\lambda,M} : X^2 \to \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\lambda,M}(x, y) = \inf\{t > 0 \ : \ M(x, y, t) > 1 - \lambda\}$$

for $\lambda \in (0, 1)$, then

(i) *for each $\mu \in (0, 1)$ there exists $\lambda \in (0, 1)$ such that*

$$E_{\mu,M}(x_1, x_n) \le E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \cdots + E_{\lambda,M}(x_{n-1}, x_n)$$

for any $x_1, x_2, \ldots, x_n \in X$

(ii) *The sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent in fuzzy metric space $(X, M, *)$ if and only if $E_{\lambda,M}(x_n, x) \to 0$. Also the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy sequence if and only if it is Cauchy with $E_{\lambda,M}$.*

Proof. (i). For every $\mu \in (0, 1)$, we can find a $\lambda \in (0, 1)$ such that

$$\overbrace{(1-\lambda) * (1-\lambda) * \cdots * (1-\lambda)}^{n} \ge 1 - \mu$$

by triangular inequality we have

$$\begin{aligned} & M(x_1, x_n, E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \cdots + E_{\lambda,M}(x_{n-1}, x_n) + n\delta) \\ \ge\ & M(x_1, x_2, E_{\lambda,M}(x_1, x_2) + \delta) * \cdots * M(x_{n-1}, x_n, E_{\lambda,M}(x_{n-1}, x_n) + \delta) \\ \ge\ & \overbrace{(1-\lambda) * (1-\lambda) * \cdots * (1-\lambda)}^{n} \ge 1 - \mu \end{aligned}$$

for very $\delta > 0$, which implies that

$$E_{\mu,M}(x_1, x_n) \le E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \cdots + E_{\lambda,M}(x_{n-1}, x_n) + n\delta.$$

Since $\delta > 0$ is arbitrary, we have

$$E_{\mu,M}(x_1, x_n) \le E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \cdots + E_{\lambda,M}(x_{n-1}, x_n)$$

(ii). Note that since M is continuous in its third place and

$$E_{\lambda,M}(x, y) = \inf\{t > 0 \ : \ M(x, y, t) > 1 - \lambda\}.$$

Hence, we have

$$M(x_n, x, \eta) > 1 - \lambda \Longleftrightarrow E_{\lambda,M}(x_n, x) < \eta$$

for every $\eta > 0$. □

Lemma 1.11. *Let $(X, M, *)$ be a fuzzy metric space. If*

$$M(x_n, x_{n+1}, t) \geq M(x_0, x_1, k^n t)$$

for some $k > 1$ and for every $n \in \mathbb{N}$. Then sequence $\{x_n\}$ is a Cauchy sequence.

Proof. For every $\lambda \in (0,1)$ and $x_n, x_{n+1} \in X$, we have

$$\begin{aligned} E_{\lambda,M}(x_{n+1}, x_n) &= \inf\{t > 0 \ : \ M(x_{n+1}, x_n, t) > 1 - \lambda\} \\ &\leq \inf\{t > 0 \ : \ M(x_0, x_1, k^n t) > 1 - \lambda\} \\ &= \inf\{\frac{t}{k^n} \ : \ M(x_0, x_1, t) > 1 - \lambda\} \\ &= \frac{1}{k^n} \inf\{t > 0 \ : \ M(x_0, x_1, t) > 1 - \lambda\} \\ &= \frac{1}{k^n} E_{\lambda,M}(x_0, x_1). \end{aligned}$$

By Lemma 1.10, for every $\mu \in (0,1)$ there exists $\lambda \in (0,1)$ such that

$$\begin{aligned} & E_{\mu,M}(x_n, x_m) \\ & \leq E_{\lambda,M}(x_n, x_{n+1}) + E_{\lambda,M}(x_{n+1}, x_{n+2}) + \cdots + E_{\lambda,M}(x_{m-1}, x_m) \\ & \leq \frac{1}{k^n} E_{\lambda,M}(x_0, x_1) + \frac{1}{k^{n+1}} E_{\lambda,M}(x_0, x_1) + \cdots + \frac{1}{k^{m-1}} E_{\lambda,M}(x_0, x_1) \\ & = E_{\lambda,M}(x_0, x_1) \sum_{j=n}^{m-1} \frac{1}{k^j} \longrightarrow 0. \end{aligned}$$

Hence sequence $\{x_n\}$ is Cauchy sequence. □

2. The main results

A class of implicit relation

Let Φ denotes a family of mappings such that each $\phi \in \Phi$, $\phi : [0,1]^3 \longrightarrow [0,1]$, and ϕ is continuous and increasing in each co-ordinate variable. Also $\phi(s,s,s) > s$ for every $s \in [0,1)$.

Example 2.1. Let $\phi : [0,1]^3 \longrightarrow$ is define by

(i) $\phi(x_1, x_2, x_3) = (\min\{x_i\})^h$ for some $0 < h < 1$.
(ii) $\phi(x_1, x_2, x_3) = x_1^h$ for some $0 < h < 1$.
(iii) $\phi(x_1, x_2, x_3) = \max\{x_1^{\alpha_1}, x_2^{\alpha_2}, x_3^{\alpha_3}\}$, where $0 < \alpha_i < 1$ for $i = 1, 2, 3$.

In this paper p is a positive real number and $\phi^{2p}(s,s,s) = [\phi(s,s,s)]^{2p}$ for every $s \in [0,1)$. Also

$$M(Sx, By, t) \vee M(Ty, Ax, t) = \max\{M(Sx, By, t), M(Ty, Ax, t)\}.$$

Our main result, for a complete fuzzy metric space X, reads follows:

Theorem 2.2. *Let A, B, S and T be a self-mapping of complete fuzzy metric space $(X, M, *)$, satisfying the following conditions:*

(i) *(A, S) and (B, T) are weakly compatible pairs such that $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ also $A(X)$ or $B(X)$ is a closed subset of X;*

(ii) *there exist $\psi, \phi \in \Phi$ such that for all $x, y \in X$,*

$$\begin{aligned} & M^{2p}(Ax, By, t) \\ \geq\ & a(s)\phi^{2p}\begin{pmatrix} M(Sx, Ty, kt), & M(Ax, Sx, kt) \\ M(By, Ty, kt) & \end{pmatrix} \\ & + b(s)\psi^{p}\begin{pmatrix} M^2(Sx, Ty, kt), & M(Sx, Ax, kt)M(Ty, By, kt) \\ M(Sx, By, kt) \vee M(Ty, Ax, kt) & \end{pmatrix}, \end{aligned}$$

for some $k > 1$, where $a, b : [0, 1] \longmapsto [0, 1]$ are two continuous functions such that $a(s) + b(s) = 1$ for every $s = M(x, y, t)$.

Then A, B and S, T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point as $A(X) \subseteq T(X), B(X) \subseteq S(X)$, there exist $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$, $Bx_1 = Sx_2$. Inductively, construct sequence $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, for $n = 0, 1, 2, \ldots$.

Now, we prove $\{y_n\}$ is a Cauchy sequence. For simplicity, we set

$$d_n(t) = M(y_n, y_{n+1}, t),\ n = 0, 1, 2, \ldots.$$

Then we have

$$\begin{aligned} & d_{2n}^{2p}(t) \\ =\ & M^{2p}(y_{2n}, y_{2n+1}, t) \\ =\ & M^{2p}(Ax_{2n}, Bx_{2n+1}, t) \\ \geq\ & a(s)\phi^{2p}\begin{pmatrix} M(Sx_{2n}, Tx_{2n+1}, kt), & M(Ax_{2n}, Sx_{2n}, kt) \\ M(Bx_{2n+1}, Tx_{2n+1}, kt) & \end{pmatrix} \\ & + b(s)\psi^{p} \\ & \cdot \begin{pmatrix} M^2(Sx_{2n}, Tx_{2n+1}, kt), M(Sx_{2n}, Ax_{2n}, kt)M(Tx_{2n+1}, Bx_{2n+1}, kt) \\ M(Sx_{2n}, Bx_{2n+1}, kt) \vee M(Tx_{2n+1}, Ax_{2n}, kt) \end{pmatrix}. \end{aligned}$$

We prove that $d_{2n}(t) \geq d_{2n-1}(t)$.Now, if $d_{2n}(t) < d_{2n-1}(t)$ for some $n \in \mathbb{N}$, since ϕ and ψ are increasing functions, then

$$\begin{aligned} & d_{2n}^{2p}(t) \\ \geq\ & a(s)\phi^{2p}(d_{2n-1}(kt), d_{2n-1}(kt), d_{2n}(kt)) \\ & \quad + b(s)\psi^{p}(d_{2n-1}^2(kt), d_{2n-1}(kt)d_{2n}(kt), 1) \\ \geq\ & a(s)\phi^{2p}(d_{2n}(kt), d_{2n}(kt), d_{2n}(kt)) + b(s)\psi^{p}(d_{2n}^2(kt), d_{2n}^2(kt), 1) \\ >\ & a(s)d_{2n}^{2p}(kt) + b(s)d_{2n}^{2p}(kt) = d_{2n}^{2p}(kt), \end{aligned}$$

hence we have $d_{2n}(t) > d_{2n}(kt)$ is a contradiction. Therefore $d_{2n}(t) \geq d_{2n-1}(t)$. Similarly, one can prove that $d_{2n+1}(t) \geq d_{2n}(t)$ for $n = 0, 1, 2, \ldots$. Consequently, $\{d_n(t)\}$ is a increasing sequence of non-negative real. Thus

$$\begin{aligned} & d_{2n}^{2p}(t) \\ & \geq a(s)\phi^{2p}(d_{2n-1}(kt), d_{2n-1}(kt), d_{2n-1}(kt)) + b(s)\psi^p(d_{2n-1}^2(kt), d_{2n-1}^2(kt), 1) \\ & \geq a(s)d_{2n-1}^{2p}(kt) + b(s)d_{2n-1}^{2p}(kt) = d_{2n-1}^{2p}(kt). \end{aligned}$$

That is $d_{2n}(t) \geq d_{2n-1}(kt)$, similarly, we have $d_{2n+1}(t) \geq d_{2n}(kt)$. Thus

$$d_n(t) \geq d_{n-1}(kt).$$

That is

$$M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, kt).$$

So

$$M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, kt) \geq \cdots \geq M(y_0, y_1, k^n t).$$

By Lemma 1.11 sequence $\{y_n\}$ is a Cauchy sequence, then it is converges to $y \in X$. That is

$$\begin{aligned} \lim_{n \to \infty} y_n &= \lim_{n \to \infty} y_{2n} = \lim_{n \to \infty} y_{2n+1} \\ &= \lim_{n \to \infty} Ax_{2n} = \lim_{n \to \infty} Bx_{2n+1} = \lim_{n \to \infty} Sx_{2n} = \lim_{n \to \infty} Tx_{2n+1} = y. \end{aligned}$$

As $B(X) \subseteq S(X)$, there exist $u \in X$ such that $Su = y$. So, we have

$$\begin{aligned} & M^{2p}(Au, Bx_{2n+1}, t) \\ & \geq a(s)\phi^{2p}\begin{pmatrix} M(Su, Tx_{2n+1}, kt), & M(Su, Au, kt) \\ M(Tx_{2n+1}, Bx_{2n+1}, kt) & \end{pmatrix} \\ & \quad + b(s)\psi^p\begin{pmatrix} M^2(Su, Tx_{2n+1}, kt), M(Su, Au, kt)M(Tx_{2n+1}, Bx_{2n+1}, kt) \\ M(Su, Bx_{2n+1}, kt) \vee M(Tx_{2n+1}, Au, kt) \end{pmatrix}. \end{aligned}$$

By continuous M and ϕ, on making $n \longrightarrow \infty$ the above inequality, we get

$$\begin{aligned} M^{2p}(Au, y, t) \geq\; & a(s)\phi^{2p}\left(M(y, y, kt), \quad M(Au, y, kt), M(y, y, kt) \right) \\ & + b(s)\psi^p\begin{pmatrix} M^2(y, y, kt), \quad M(Au, y, kt)M(y, y, kt) \\ M(y, y, kt) \vee M(y, Au, kt) \end{pmatrix}, \end{aligned}$$

hence we have

$$\begin{aligned} M^{2p}(Au, y, t) \geq\; & a(s)\phi^{2p}(M(Au, y, kt), M(Au, y, kt), M(Au, y, kt)) \\ & + b(s)\psi^p(M^2(Au, y, kt), M(Au, y, kt)M(Au, y, kt), 1). \end{aligned}$$

If $Au \neq y$, by above inequality we get

$$M^{2p}(Au, y, t) > a(s)M^{2p}(Au, y, kt) + b(s)M^{2p}(Au, y, kt) = M^{2p}(Au, y, kt)$$

which is contradiction. Hence $M(Au, y, t) = 1$, i.e $Au = y$. Thus $Au = Su = y$.

As $A(X) \subseteq T(X)$ there exist $v \in X$, such that $Tv = y$. So,

$$\begin{aligned} M^{2p}(y, Bv, t) &= M^{2p}(Au, Bv, t) \\ &\geq a(s)\phi^{2p}(M(Su, Tv, kt), M(Au, Su, kt), M(Bv, Tv, kt)) \\ &\quad + b(s)\psi^p(M^2(Su, Tv, kt), M(Su, Au, kt)M(Tv, Bv, kt), \\ &\qquad M(Su, Bv, kt) \vee M(Tv, Au, kt)) \\ &= a(s)\phi^{2p}(1, 1, M(Bv, y, kt)) + b(s)\psi^p(1, 1, 1). \end{aligned}$$

We claim that $Bv = y$. For if $Bv \neq y$, then $M(Bv, y, t) < 1$.

On the above inequality we get

$$\begin{aligned} M^{2p}(y, Bv, t) &\geq a(s)\phi^{2p}(M(y, Bv, kt), M(y, Bv, kt), M(y, Bv, kt)) \\ &\quad + b(s)\psi^p(M^2(y, Bv, kt), M^2(y, Bv, kt), M^2(y, Bv, kt)) \\ &> a(s)M^{2p}(y, Bv, kt) + b(s)M^{2p}(y, Bv, kt) = M^{2p}(y, Bv, kt), \end{aligned}$$

a contradiction. Hence $Tv = Bv = Au = Su = y$. Since (A, S) is weak compatible, we get that $ASu = SAu$, that is $Ay = Sy$. Since (B, T) is weak compatible, we get that $TBv = BTv$, that is, $Ty = By$. If $Ay \neq y$, then $M(Ay, y, t) < 1$. However

$$\begin{aligned} & M^{2p}(Ay, y, t) \\ = \; & M^{2p}(Ay, Bv, t) \\ \geq \; & a(s)\phi^{2p}(M(Sy, Tv, kt), M(Ay, Sy, kt), M(Bv, Tv, kt)) \\ & + b(s)\psi^p(M^2(Sy, Ty, kt), M(Sy, Ay, kt)M(Tv, Bv, kt), \\ & \qquad M(Sy, Bv, kt) \vee M(Tv, Ay, kt)) \\ = \; & a(s)\phi^{2p}(M(Ay, y, kt), 1, 1) + b(s)\psi^p(M^2(Ay, y, kt), 1, M(Ay, y, kt)) \\ \geq \; & a(s)\phi^{2p}(M(Ay, y, kt), M(Ay, y, kt), M(Ay, y, kt)) \\ & + b(s)\psi^p(M^2(Ay, y, kt), M^2(Ay, y, kt), M^2(Ay, y, kt)) \\ > \; & a(s)M^{2p}(Ay, y, kt) + b(s)M^{2p}(Ay, y, kt) = M^{2p}(Ay, y, kt) \end{aligned}$$

a contradiction. Thus $Ay = y$, hence $Ay = Sy = y$. Similarly we prove that $By = y$. For if $By \neq y$, then $M(By, y, t) < 1$, however

$$\begin{aligned} M^{2p}(y, By, t) &= M^{2p}(Ay, By, t) \\ &\geq a(s)\phi^{2p}(M(Sy, Ty, kt), M(Ay, Sy, kt), M(By, Ty, kt)) \\ &\quad + b(s)\psi^p(M^2(Sy, Ty, kt), M(Sy, Ay, kt)M(Ty, By, kt), \\ &\qquad M(Sy, By, kt) \vee M(Ty, Ay, kt)) \\ &= a(s)\phi^{2p}(M(y, By, kt), M(y, y, kt), M(By, By, kt)) \\ &\quad + b(s)\psi^p(M^2(y, By, kt), 1, M(y, By, kt)) \\ &\geq a(s)\phi^{2p}(M(y, By, kt), M(y, By, kt), M(y, By, kt)) \\ &\quad + b(s)\psi^p(M^2(y, By, kt), M^2(y, By, kt), M^2(y, By, kt)) \\ &> a(s)M^{2p}(y, By, kt) + b(s)M^{2p}(y, By, kt) = M^{2p}(y, By, kt), \end{aligned}$$

a contradiction. Therefore, $Ay = By = Sy = Ty = y$, that is, y is a common fixed of A, B, S and T. Uniqueness, let x be another common fixed point of A, B, S and T. That is, $x = Ax = Bx = Sx = Tx$. If $M(x, y, t) < 1$, then

$$\begin{aligned} M^{2p}(y,x,t) &= M^{2p}(Ay,Bx,t) \\ &\geq a(s)\phi^{2p}(M(Sy,Tx,kt),M(Ay,Sy,kt),M(Bx,Tx,kt)) \\ &\quad +b(s)\psi^{p}(M^2(Sy,Tx,kt),M(Sy,Ay,kt)M(Tx,Bx,kt), \\ &\qquad M(Sy,Bx,kt)\vee M(Tx,Ay,kt)) \\ &= a(s)\phi^{2p}(M(y,x,kt),1,1)+b(s)\psi^{p}(M^2(y,x,kt),1,M(y,x,kt)) \\ &\geq a(s)\phi^{2p}(M(y,x,kt),M(y,x,kt),M(y,x,kt)) \\ &\quad +b(s)\psi^{p}(M^2(y,x,kt),M^2(y,x,kt),M^2(y,x,kt)) \\ &> a(s)M^{2p}(y,x,kt)+b(s)M^{2p}(y,x,kt) = M^{2p}(y,x,kt), \end{aligned}$$

a contradiction. Therefore, y is the unique common fixed point of self-maps A, B, S and T. □

In the following Theorem, function $\phi : [0,1]^4 \longrightarrow [0,1]$, is continuous and increasing in each co-ordinate variable. Also $\phi(s,s,s,s) > s$ for every $s \in [0,1)$.

Theorem 2.3. *Let A, B, S and T be self-mappings of a complete fuzzy metric space $(X, M, *)$, satisfying that*

(i) *$A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ and $A(X)$ or $B(X)$ is a complete subset of X,*

(ii) $M(Ax,By,t) \geq \phi\begin{pmatrix} M(Sx,Ty,kt), M(Ax,Sx,kt), \\ M(By,Ty,kt), M(Ax,Ty,kt)\vee M(By,Sx,kt) \end{pmatrix}$ *for every x, y in X, $k > 1$ and $\phi \in \Phi$,*

(iii) *the pairs (A,S) and (B,T) are be weak compatible.*

Then A, B, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point as $A(X) \subseteq T(X), B(X) \subseteq S(X)$, there exist $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$, $Bx_1 = Sx_2$. Inductively, construct sequence $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, for $n = 0, 1, 2, \ldots$.

Now, we prove $\{y_n\}$ is a Cauchy sequence. Let $d_m(t) = M(y_m, y_{m+1}, t), t > 0$ we prove $\{d_m(t)\}$ is increasing w.r.t m. Set, $m = 2n$, we have

$$\begin{aligned} (2.1)\ & d_{2n}(t) \\ &= M(y_{2n},y_{2n+1},t) \\ &= M(Ax_{2n},Bx_{2n+1},t) \\ &\geq \phi\begin{pmatrix} M(Sx_{2n},Tx_{2n+1},kt), & M(Ax_{2n},Sx_{2n},kt), \\ M(Bx_{2n+1},Tx_{2n+1},kt), & M(Ax_{2n},Tx_{2n+1},kt)\vee M(Bx_{2n+1},Sx_{2n},kt) \end{pmatrix} \\ &= \phi\begin{pmatrix} M(y_{2n-1},y_{2n},kt), & M(y_{2n},y_{2n-1},kt), \\ M(y_{2n+1},y_{2n},kt), & M(y_{2n},y_{2n},kt)\vee M(y_{2n+1},y_{2n-1},kt) \end{pmatrix} \end{aligned}$$

$$\begin{aligned} &= \phi(d_{2n-1}(kt), d_{2n-1}(kt), d_{2n}(kt), 1) \\ &\geq \phi(d_{2n-1}(kt), d_{2n-1}(kt), d_{2n}(kt), 1). \end{aligned}$$

Since, ϕ is an increasing function we claim that for every $n \in N$, $d_{2n}(kt) \geq d_{2n-1}(kt)$. For if $d_{2n}(kt) < d_{2n-1}(kt)$, then in inequality (2.1), we have

$$d_{2n}(t) \geq \phi(d_{2n}(kt), d_{2n}(kt), d_{2n}(kt), d_{2n}(kt)) > d_{2n}(kt).$$

That is, $d_{2n}(t) > d_{2n}(kt)$, a contradiction. Hence $d_{2n}(kt) \geq d_{2n-1}(kt)$ for every $n \in N$ and $\forall t > 0$. Similarly, we have $d_{2n+1}(kt) \geq d_{2n}(kt)$. Thus $\{d_n(t)\}$ is an increasing sequence in $[0,1]$. By inequality (2.1) and $d_n(t)$ is an increasing sequence, we get

$$d_{2n}(t) \geq \phi(d_{2n-1}(kt), d_{2n-1}(kt), d_{2n-1}(kt), d_{2n-1}(kt)) \geq d_{2n-1}(kt).$$

Similarly, we have $d_{2n+1}(t) \geq d_{2n}(kt)$.Thus $d_n(t) \geq d_{n-1}(kt)$. That is,

$$M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, kt) \geq \cdots \geq M(y_0, y_1, k^n t).$$

Hence by Lemma 1.11 $\{y_n\}$ is Cauchy and the completeness of X, $\{y_n\}$ converges to y in X. That is,

$$\begin{aligned} \lim_{n\to\infty} y_n = y \Rightarrow \lim_{n\to\infty} y_{2n} &= \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} \\ &= \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = y. \end{aligned}$$

As $B(X) \subseteq S(X)$, there exist $u \in X$ such that $Su = y$. So, we have

$$\begin{aligned} &M(Au, Bx_{2n+1}, t) \\ &\geq \phi\left(\begin{array}{ll} M(Su, Tx_{2n+1}, kt), & M(Au, Su, kt), \\ M(Bx_{2n+1}, Tx_{2n+1}, kt), & M(Au, Tx_{2n+1}, kt) \vee M(Bx_{2n+1}, Su, kt) \end{array} \right). \end{aligned}$$

If $Au \neq y$, by continuous M and ϕ, on making $n \longrightarrow \infty$ the above inequality, we get

$$\begin{aligned} M(Au, y, t) &\geq \phi\left(\begin{array}{ll} M(y, y, kt), & M(Au, y, kt), \\ M(y, y, kt), & M(Au, y, kt) \vee M(y, y, kt) \end{array} \right) \\ &\geq \phi\left(\begin{array}{ll} M(Au, y, kt), & M(Au, y, kt), \\ M(Au, y, kt), & M(Au, y, kt) \end{array} \right) \\ &> M(Au, y, kt). \end{aligned}$$

That is, $M(Au, y, t) > M(Au, y, kt)$ which is contradiction. Hence

$$M(Au, y, t) = 1,$$

i.e., $Au = y$. Thus $Au = Su = y$.

As $A(X) \subseteq T(X)$ there exist $v \in X$, such that $Tv = y$. So,

$$\begin{aligned} M(y, Bv, t) &= M(Au, Bv, t) \\ &\geq \phi\left(\begin{array}{ll} M(Su, Tv, kt), & M(Au, Su, kt), \\ M(Bv, Tv, kt), & M(Au, Tv, kt) \vee M(Bv, Su, kt) \end{array} \right) \\ &= \phi\left(\begin{array}{ll} 1, & 1, \\ M(Bv, y, kt), & 1 \end{array} \right). \end{aligned}$$

We claim that $Bv = y$. For if $Bv \neq y$, then $M(Bv, y, t) < 1$.

On the above inequality we get

$$\begin{aligned} M(y, Bv, t) &\geq \phi\begin{pmatrix} M(y, Bv, kt), & M(y, Bv, kt), \\ M(y, Bv, kt), & M(y, Bv, kt) \end{pmatrix} \\ &> M(y, Bv, kt), \end{aligned}$$

a contradiction. Hence $Tv = Bv = Au = Su = y$. Since (A, S) is weak compatible, we get that $ASu = SAu$, that is $Ay = Sy$.

Since (B, T) is weak compatible, we get that $TBv = BTv$, that is $Ty = By$. If $Ay \neq y$, then $M(Ay, y, t) < 1$. However

$$\begin{aligned} M(Ay, y, t) &= M(Ay, Bv, t) \\ &\geq \phi\begin{pmatrix} M(Sy, Tv, kt), & M(Ay, Sy, kt), \\ M(Bv, Tv, kt), & M(Ay, Tv, kt) \vee M(Bv, Sy, kt) \end{pmatrix} \\ &\geq \phi(M(Ay, y, kt), 1, 1, M(Ay, y, kt)) \\ &\geq \phi\begin{pmatrix} M(Ay, y, kt), & M(Ay, y, kt), \\ M(Ay, y, kt), & M(Ay, y, kt) \end{pmatrix} \\ &> M(Ay, y, kt) \end{aligned}$$

a contradiction. Thus $Ay = y$, hence $Ay = Sy = y$. Similarly we prove that $By = y$. For if $By \neq y$, then $M(By, y, t) < 1$, however

$$\begin{aligned} M(y, By, t) &= M(Ay, By, t) \\ &\geq \phi\begin{pmatrix} M(Sy, Ty, kt), & M(Ay, Sy, kt), \\ M(By, Ty, kt), & M(Ay, Ty, kt) \vee M(By, Sy, kt) \end{pmatrix} \\ &\geq \phi(M(y, By, kt), M(y, By, kt), M(y, By, kt), M(y, By, kt)) \\ &> M(y, By, kt) \end{aligned}$$

a contradiction. Therefore, $Ay = By = Sy = Ty = y$, that is, y is a common fixed of A, B, S and T. Uniqueness, let x be another common fixed point of A, B, S and T. That is $x = Ax = Bx = Sx = Tx$. If $M(x, y, t) < 1$, then

$$\begin{aligned} M(y, x, t) &= M(Ay, Bx, t) \\ &\geq \phi\begin{pmatrix} M(Sy, Tx, kt), & M(Ay, Sy, kt), \\ M(Bx, Tx, kt), & M(Ay, Tx, kt) \vee M(Bx, Sy, kt) \end{pmatrix} \\ &= \phi\begin{pmatrix} M(y, x, kt), & 1, \\ 1, & M(y, x, kt) \vee M(x, y, kt) \end{pmatrix} \\ &\geq \phi(M(y, x, kt), M(y, x, kt), M(y, x, kt), M(y, x, kt)) \\ &> M(y, x, kt) \end{aligned}$$

a contradiction . Therefore, y is the unique common fixed point of self-maps A, B, S and T. □

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