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HESSIAN GEOMETRY OF THE HOMOGENEOUS GRAPH DOMAIN

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Abstract. In this paper, we will investigate the Hessian geometry of the homogeneous domain over the hypersurface given by a function $F : \mathbb{R}^n \to \mathbb{R}$ with $|\det DdF| = 1$.

1. Introduction

Let Σ be an affine homogeneous hypersurface, which is a graph of a function $F : \mathbb{R}^n \to \mathbb{R}$ with $|\det DdF| = 1$. Since the affine normals of the graph Σ are parallel, Σ is, in fact, an improper affine hypersphere. If an unimodular equiaffine Lie group $\mathbb{A}$ acts on Σ simply transitively, the torsion free and flat affine connection D on $\mathbb{A}$ is induced from the Blaschke connection on Σ. The connection gives a multiplication $a * b := D_a b \, (a, b \in \mathfrak{a})$ on the Lie algebra $\mathfrak{a}$ of $\mathbb{A}$ which is compatible with the Lie bracket. From the flat condition of D, the multiplication satisfies the following identity:

$$(a * b) * c - a * (b * c) = (b * a) * c - b * (a * c), \qquad \text{for } a, b, c \in \mathfrak{a}.$$

This algebra $\mathcal{A} = (\mathfrak{a}, *)$ is called a left symmetric algebra(or LSA). The first author showed the following in [3]:

Theorem A. *The set of homogeneous hypersurfaces $(\mathbb{A}, \Sigma)$ as above is in one to one correspondence with the set of the n-dimensional complete unimodular LSA $\mathcal{A}$ with a nondegenerate Hessian type inner product.*

Here an inner product on an LSA $\mathcal{A}$ is called *Hessian type* if it satisfies

$$(1) \qquad \langle a * b,\, c\rangle - \langle a,\, b * c\rangle = \langle b * a,\, c\rangle - \langle b,\, a * c\rangle \quad \text{for } a, b, c \in \mathcal{A}.$$

In what follows, we call an LSA with a nondegenerate Hessian type inner product *Hessian algebra.* A Lie group $\mathbb{A}$ is unimodular if and only if $|\det \mathrm{Ad}\, x| = 1$ $(x \in \mathbb{A})$, where Ad is the adjoint representation. For a connected Lie group $\mathbb{A}$,

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this is equivalent to requiring that $\operatorname{tr}\operatorname{ad} a = 0$ $(a \in \mathfrak{a})$, where ad is the adjoint representation of the Lie algebra $\mathfrak{a}$ of $\mathbb{A}$. An LSA $\mathcal{A} = (\mathfrak{a}, *)$ is called *unimodular* if its associated Lie algebra given by $[a,b] = a*b - b*a$ is unimodular, and an LSA $\mathcal{A} = (\mathfrak{a}, *)$ is called *complete* if $\operatorname{tr}\rho_a = 0 (a \in \mathfrak{a})$ where ρ is the operator representing the right multiplication in $\mathcal{A}$. It is well-known that the completeness of $\mathcal{A} = (\mathfrak{a}, *)$ is equivalent to that the developing image of the Lie group $\mathbb{A}$ whose Lie algebra is $\mathfrak{a}$ is the whole $\mathbb{R}^n$.

Let Ω be the domain over the graph Σ. Suppose that a Lie subgroup $\mathbb{G} \subset \operatorname{Aut}(\Omega)$ acts simply transitively on Ω and it contains an unimodular equiaffine subgroup $\mathbb{A}$ which acts on Σ simply transitively. Then the following is shown in [3].

Theorem B. *The set of homogeneous domains $(\Omega, \mathbb{G})$ over Σ is in one to one correspondence with the set of graph extensions $\mathcal{G} = (\mathfrak{g}, \cdot)$ of a complete unimodular Hessian algebra $\mathcal{A} = (\mathfrak{a}, *, \langle , \rangle)$, where $\mathfrak{g} = \mathfrak{a} + \operatorname{span}\{e\}$ as a vector space and the algebra structure is given by the followings:*

(a) $e \cdot e = e$, $e \neq 0$, *that is, e is an idempotent.*
(b) $e \cdot a \in \mathcal{A}$, $\quad a \cdot e = 0$ *for all* $a \in \mathcal{A}$.
(c) $a \cdot b = a * b + \langle a, b\rangle e$ *for all* $a, b \in \mathcal{A}$.

In this case, there is a nondegenerate Hessian metric on Ω which is $\mathbb{G}$-invariant, equivalently, $\mathcal{G}$ is an Hessian algebra.

If we define an inner product $\langle\langle , \rangle\rangle$ on $\mathcal{G}$ by

$$\langle\langle e, e\rangle\rangle = 1, \quad \langle\langle e, a\rangle\rangle = 0, \quad \langle\langle a, b\rangle\rangle = \langle a, b\rangle \quad (a, b \in \mathcal{A}), \tag{2}$$

which is a natural extension of the inner product $\langle , \rangle$ on $\mathcal{A}$, then it is easy to check that $\langle\langle , \rangle\rangle$ is of Hessian type. Moreover it is equal to the inner product defined by $\operatorname{tr}\rho$, that is, $\langle\langle x, y\rangle\rangle = \operatorname{tr}\rho_{xy}$ where ρ is the right multiplication operator on $\mathcal{G}$.

Definition 1.1. Let $\mathcal{G} = (\mathfrak{g}, \cdot)$ be an LSA. A nondegenerate inner product $\langle\langle , \rangle\rangle$ on $\mathcal{G}$ is called *Koszul type*, if there exists a Lie algebra homomorphism $s : \mathfrak{g} \to \mathbb{R}$ such that $\langle\langle x, y\rangle\rangle = s(x \cdot y)$ for $x, y \in \mathcal{G}$. In this case, the LSA $\mathcal{G}$ will be called simply a *Koszul algebra.*

We note that the graph extension is a Koszul algebra with $s = \operatorname{tr}\rho$. A Koszul algebra $\mathcal{G}$ is of course a Hessian algebra because $s((xy)z) = \langle\langle xy, z\rangle\rangle$ for $x, y, z \in \mathcal{G}$ and satisfies (1) trivially.

As a simply connected pseudo-Riemannian manifold, the geometry of Ω is interesting, and this is studied by some authors through the Hessian algebra and the Koszul algebra. For some properties of Koszul algebra, we will treat them in Section 2.

In [12], Shima studied the geometry of an affine homogeneous convex domain Ω containing no full straight line, whose counterpart is a clan in the algebra side(*see* [15]). He defined *elementary clan* $\mathcal{G}$ for a direct sum $\operatorname{span}\{e\} + \mathcal{A}$ satisfying the followings:

(i) $e \cdot e = e,\ e \neq 0$.
(ii) $e \cdot x = \frac{1}{2}x,\ x \cdot e = 0$ for $x \in \mathcal{A}$.
(iii) $x \cdot y = \langle x, y \rangle e$ for $x, y \in \mathcal{A}$, where $\langle , \rangle$ is a positive definite symmetric bilinear form on $\mathcal{A}$.

The domain Ω corresponding to an elementary clan is the interior of a paraboloid which is the graph of a polynomial $F(x) = \frac{1}{2}\langle x,\ x \rangle$ on $\mathbb{R}^n (\cong \mathcal{A})$. He also expressed the curvature tensor and the sectional curvature κ of Ω using the symmetric part of the left multiplication λ_x in $\mathcal{G}$ and proved the following:

Theorem C. *Let $\mathcal{G}$ be a clan. Then the following conditions are equivalent:*
(a) *the sectional curvature $\kappa < 0$.*
(b) *$\mathcal{G}$ is an elementary clan.*

In [8], Mizuhara studied some algebra structures on the underlying Lie algebra of an LSA $\mathcal{G} = (\mathfrak{g}, \cdot)$ satisfying
(i) $\mathfrak{g} = \text{span}\{e\} + P$ where e is an idempotent and $P = \{a \in \mathfrak{g} \mid a \cdot e = 0\}$,
(ii) $P = \sum_{i=1}^{k} P_{\alpha_i}$ where $P_{\alpha_i} = \{a \in P \mid e \cdot a = \alpha_i a\}$.

He called the idempotent e a *principal idempotent of type* $(\alpha_1, \ldots, \alpha_k)$ and induced a Hessian type inner product $\langle , \rangle$ by $\langle x, y \rangle =$ e-component of $x \cdot y$ for $x, y \in \mathcal{G}$. With this Hessian type inner product $\langle , \rangle$, he defined an algebra $\hat{\mathcal{G}} = (\mathfrak{g}, \wedge)$ with left multiplication operator $\hat{\lambda}_x = \frac{1}{2}(\lambda_x + \lambda'_x)$ where λ'_x is the adjoint of λ_x with respect to $\langle , \rangle$ and an algebra $\bar{P} = (P, \triangle)$ constructed by $a \triangle b = a \wedge b - \frac{1}{2}\langle a, b \rangle e$ for $a, b \in P$. Then he showed the following:

Theorem D. *Let $\mathcal{G} = (\mathfrak{g}, \cdot)$ be an LSA with a principal idempotent e of type $(\alpha_1, \ldots, \alpha_k)$ satisfying $1 > \alpha_1 > \cdots > \alpha_k > 0$. Then the following two conditions are equivalent:*
(a) *The algebra $\hat{\mathcal{G}} = (\mathfrak{g}, \wedge)$ is of constant sectional curvature $-\frac{1}{4}$.*
(b) *The algebra $\bar{P} = (P, \triangle)$ is left symmetric.*

As an application, he gave an equivariant affine immersion of the simply connected Lie group $\mathbb{G}$, whose Lie algebra is $\mathfrak{g}$, as the interior Ω of a nondegenerate, generalized paraboloid such that there exists on Ω an invariant metric of negative constant sectional curvature.

We note that the elementary clan in [12] is a graph extension of $\mathbb{R}^n$(trivial algebra) with positive definite inner product, and the LSA with principal idempotent of type $(\alpha_1, \ldots, \alpha_k)$ in [8] is also a graph extension of some unimodular complete Hessian algebra $\mathcal{A} = (P, *)$ induced by $a * b := a \cdot b - \langle a,\ b \rangle e$ for $a, b \in P$: Since $P = \cup P_{\alpha_i}$ and $P_{\alpha_i} \cdot P_{\alpha_j} \subset P_{\alpha_{i+j}}$ (*cf.* [8]), $\mathcal{A} = (P, *)$ is unimodular and complete. However, it does not clear that a graph extension of some unimodular complete Hessian algebra is an LSA with a principal idempotent of type $(\alpha_1, \ldots, \alpha_k)$.

The elementary clan is the LSA with principal idempotent of type $(\frac{1}{2})$ and the geometric properties of the convex domain are generalized to the interior of a nondegenerate, generalized paraboloid. Some of the geometric properties of these domains are expected to be generalized to the graph domains. In this paper, we will translate their works to the terminology of graph extension and generalize their results on curvature properties to the graph domains.

On the other hand, in [4], Dillen and Vrancken studied the affine hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ whose difference tensor $K = D - \nabla$ is parallel with respect to the induced affine connection D where ∇ is the Levi-Civita connection of the affine metric on Σ. They showed the followings:

Theorem E. *Let Σ be an n-dimensional affine hypersurface in $\mathbb{R}^{n+1}$.*

(a) *DK is totally symmetric if and only if $S = \lambda I$ and $[K_X, K_Y] = 0$ for each X and Y, where S is the shape operator of Σ.*
(b) *If $DK = 0$ and $K \neq 0$, then $S = 0$ and the affine metric h is flat.*
(c) *If $[K_Y, K_Z] = 0$ for all Y and Z, then K_X is nilpotent for each X, that is, $(K_X)^n = 0$ for all X.*

Moreover they proved four theorems, classifying hypersurfaces with parallel difference tensor for some cases, that is, the case when $K \neq 0,\ K^2 = 0$, the Lorentzian case, the case when $K^{n-1} \neq 0$ and the case when $K^{n-2} \neq 0,\ K^{n-1} = 0$.

We note that some of the polynomials in their classifying theorems in [4], which define the improper hyperspheres, are equal to the polynomials obtained from the complete unimodular Hessian algebra.(*cf.* [3]) In authors's thought, all the polynomials in [4] might be obtained from the complete unimodular Hessian algebra. Furthermore, we conjecture that the affine hypersurface with $DK = 0$ is homogeneous.

2. Geometry of homogeneous Hessian domain

Let (Ω, D, g) be an affine domain in $\mathbb{R}^n$, where D is a torsion free affine flat connection and $g = Dd\phi$ denotes a Hessian metric with a potential function $\phi : \Omega \to \mathbb{R}$. In this case, (Ω, D, g) is called a *Hessian domain*[14]. Let's assume that an affine Lie subgroup $\mathbb{G} \subset \mathrm{Aut}(\Omega)$ acts simply transitively and isometrically on the Hessian domain Ω. Then the affine connection D and the Hessian metric g are pulled back to the Lie group $\mathbb{G}$. We will use the same notation for the induced connection and the induced Hessian metric on $\mathbb{G}$. We note that the affine connection D on $\mathbb{G}$ gives an LSA $\mathcal{G} = (\mathfrak{g}, \cdot)$ where $\mathfrak{g}$ is the Lie algebra of $\mathbb{G}$, and the Hessian metric g on $\mathbb{G}$ gives a Hessian type inner product $\langle , \rangle$ on $\mathcal{G}$. Therefore from an homogeneous Hessian domain $(\Omega, D, g, \mathbb{G})$, we could obtain an Hessian algebra $\mathcal{G} = (\mathfrak{g}, \cdot, \langle , \rangle)$.

Let ∇ denote the Livi-Civita connection of the Hessian metric g on Ω. Then ∇ is characterized by the Koszul formula(*cf.* [10]): for $X, Y, Z \in \mathfrak{X}(\Omega)$,

$$\begin{aligned} & 2g(\nabla_X Y, Z) \\ = \ & Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ & -g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]). \end{aligned} \tag{3}$$

The induced connection on $\mathbb{G}$ from the Levi-Civita connection will be denoted by the same notation ∇. Since the Hessian metric is left invariant under the $\mathbb{G}$-action, ∇ is also left invariant, so it gives a multiplication on the Lie algebra $\mathfrak{g} = \mathrm{Lie}\,\mathbb{G}$ which is compatible with the Lie bracket. But the algebra structure determined by ∇ need not be left symmetric since ∇ is not flat in general. In an LSA $\mathcal{G} = (\mathfrak{g}, \cdot)$, we will often use λ_x for the left multiplication by $x \in \mathcal{G}$ instead of D_x. The following three Propositions can be generalized from the cases of clan in [12] to the cases of Hessian algebra.

Proposition 2.1. *For $x \in \mathfrak{g}$, $\nabla_x = \dfrac{1}{2}(\lambda_x - \lambda'_x)$, where λ'_x is the adjoint linear transformation of λ_x with respect to the Hessian type inner product.*

Proof. Since the Hessian type inner product $\langle\, ,\, \rangle$ and $x, y, z \in \mathfrak{g}$ are left invariant, $x\langle y, z\rangle = D_x\langle y, z\rangle = 0$. Similarly we have $y\langle z, x\rangle = z\langle x, y\rangle = 0$. Then from the Koszul formula (3),

$$\begin{aligned} & 2\langle \nabla_x y, z\rangle \\ = \; & -\langle x, [y, z]\rangle + \langle y, [x, z]\rangle + \langle z, [x, y]\rangle \\ = \; & -\langle x, yz - zy\rangle + \langle y, xz - zx\rangle + \langle z, xy - yx\rangle \\ = \; & \langle x, zy\rangle - \langle y, zx\rangle \\ = \; & \langle z, xy\rangle - \langle xz, y\rangle \\ = \; & \langle (\lambda_x - \lambda'_x)y, z\rangle. \end{aligned}$$

Because the Hessian type inner product is nondegenerate, we have $\nabla_x y = \dfrac{1}{2}(\lambda_x - \lambda'_x)y$. □

The difference tensor K on Ω is defined by $K(X, Y) = K_X Y = D_X Y - \nabla_X Y$ for $X, Y \in \mathfrak{X}(\Omega)$. Since K is also left-invariant under the $\mathbb{G}$-action, it also gives a multiplication on $\mathfrak{g}$, whose left multiplication by $x \in \mathfrak{g}$ is represented as the following:

$$K_x = \lambda_x - \frac{1}{2}(\lambda_x - \lambda'_x) = \frac{1}{2}(\lambda_x + \lambda'_x).$$

Notice that ∇_x is skew-adjoint and K_x is self-adjoint with respect to the Hessian type inner product, that is,

$$(4) \qquad \langle \nabla_x y, z\rangle = -\langle y, \nabla_x z\rangle, \quad \langle K_x y,\, z\rangle = \langle y,\, K_x z\rangle \quad \text{for } x, y, z \in \mathfrak{g}.$$

Proposition 2.2. *For all $x, y \in \mathfrak{g}$, we have*

(a) $K_x y = K_y x$.
(b) *Let R be the curvature tensor of ∇, then $R(x, y) = -[K_x, K_y]$.*

Proof. (a) $K_x y - K_y x = (D_x y - \nabla_x y) - (D_y x - \nabla_y x) = [x, y] - [x, y] = 0$.

(b) For all $x, y \in \mathfrak{g}$,

$$
\begin{aligned}
& R(x,y) \\
= \;& [\nabla_x, \nabla_y] - \nabla_{[x,y]} \\
= \;& [\frac{1}{2}(\lambda_x - \lambda'_x), \frac{1}{2}(\lambda_y - \lambda'_y)] - \frac{1}{2}(\lambda_{[x,y]} - \lambda'_{[x,y]}) \\
= \;& \frac{1}{4}\{[\lambda_x, \lambda_y] - [\lambda_x, \lambda'_y] - [\lambda'_x, \lambda_y] + [\lambda'_x, \lambda'_y]\} - \frac{2}{4}\lambda_{[x,y]} + \frac{2}{4}\lambda'_{[x,y]} \\
= \;& -\frac{1}{4}[\lambda_x, \lambda_y] - \frac{1}{4}[\lambda_x, \lambda'_y] - \frac{1}{4}[\lambda'_x, \lambda_y] - \frac{1}{4}[\lambda'_x, \lambda'_y] \\
= \;& -[\frac{1}{2}(\lambda_x + \lambda'_x), \frac{1}{2}(\lambda_y + \lambda'_y)] \\
= \;& -[K_x, K_y].
\end{aligned}
$$

□

Let κ be the sectional curvature on Ω and by abusing the notation, also be the sectional curvature on $\mathbb{G}$. Then we have

Proposition 2.3. *For all $x, y \in \mathfrak{g}$ such that* $\mathrm{span}\{x, y\}$ *is a nondegenerate subspace of* $\mathfrak{g}$,

$$\kappa(x,y) = \frac{\langle\langle K_x y, K_y x\rangle\rangle - \langle\langle K_x x, K_y y\rangle\rangle}{Q(x,y)},$$

where $Q(x,y) = \langle\langle x, x\rangle\rangle\langle\langle y, y\rangle\rangle - \langle\langle x, y\rangle\rangle^2$.

Proof. For any $x, y, z \in \mathfrak{g}$, by using Proposition 2.2 (b),

$$
\begin{aligned}
\langle\langle R(x,y)y, x\rangle\rangle &= -\langle\langle [K_x, K_y]y, x\rangle\rangle \\
&= -\langle\langle K_x K_y y, x\rangle\rangle + \langle\langle K_y K_x y, x\rangle\rangle \\
&= -\langle\langle K_y y, K_x x\rangle\rangle + \langle\langle K_x y, K_y x\rangle\rangle.
\end{aligned}
$$

Hence, for $x, y \in \mathfrak{g}$ such that $\mathrm{span}\{x, y\}$ is a nondegenerate subspace of $\mathfrak{g}$,

$$
\begin{aligned}
\kappa(x,y) &= \frac{\langle\langle R(x,y)y, x\rangle\rangle}{Q(x,y)} \\
&= \frac{\langle\langle K_x y, K_y x\rangle\rangle - \langle\langle K_x x, K_y y\rangle\rangle}{Q(x,y)}.
\end{aligned}
$$

□

Remark 2.4. (a) If the Lie group $\mathbb{G}$ is abelian, the condition (1) of Hessian type inner product on the Lie algebra $\mathfrak{g}$ is reduced to

$$\langle xy, z\rangle = \langle x, yz\rangle, \quad \text{for all } x, y, z \in \mathfrak{g}.$$

So we have $\lambda'_x = \lambda_x$ for all $x \in \mathfrak{g}$, and hence $\nabla_x = 0$ and $K_x = \lambda_x$ for all $x \in \mathfrak{g}$. Therefore, in this case, the Levi-Civita connection must be flat.

(b) The multiplication $\circ$ on the Lie algebra $\mathfrak{g}$ defined by

$$x \circ y := K_x y \quad \text{for } x, y \in \mathfrak{g},$$

is not compatible with the Lie algebra structure of $\mathfrak{g}$ if it is not abelian. However from Proposition 2.2 (a) and the equation (4), $(\mathfrak{g}, \circ)$ is a commutative algebra with an invariant inner product. If the Levi-Civita connection on $\mathfrak{g}$ is flat, that is, $[K_x, K_y] = 0$ from Proposition 2.2 (b), then $(\mathfrak{g}, \circ)$ is also an associative algebra, hence it becomes, in fact, a commutative nilpotent Frobenius algebra by Theorem E (c) applied to our Hessian algebra.

The covariant derivative DK with respect to the affine connection is called *Hessian curvature tensor*[14], which is given by $DK(x, y, z) = (D_x K)(y, z)$ for $x, y, z \in \mathfrak{g}$. From Proposition 2.2 (a), $DK(x, y, z)$ is symmetric in y, z. The following Proposition is the homogeneous Hessian version of Theorem E (a).

Proposition 2.5. *The Hessian curvature DK is totally symmetric if and only if $[K_x, K_y] = 0$ for all $x, y \in \mathfrak{g}$.*

Proof. For any $x, y, z \in \mathfrak{g}$,

$$\begin{aligned} DK(x,y,z) &= (D_x K)(y,z) \\ &= D_x K(y,z) - K(D_x y, z) - K(y, D_x z) \\ &= D_x D_y z - D_x \nabla_y z - D_{D_x y} z + \nabla_{D_x y} z - D_y D_x z + \nabla_y D_x z. \end{aligned}$$

Then by using the flat and torsion free condition of D and the torsion free condition of ∇, we have $DK(x, y, z) - DK(y, x, z) = [K_x, K_y]z$. □

With the above Proposition 2.5 and Proposition 2.2 (b), we note that the Hessian curvature is totally symmetric if and only if the curvature of the Levi-Civita connection vanishes.

On a Hessian domain (Ω, D, g), the metric g is called *Koszul type* if there exists a closed 1-form ω such that $g = D\omega$(*cf.* [14]). In this case, we will call (Ω, D, g) a *Koszul domain.* For a homogeneous Koszul domain $(\Omega, D, g, \mathbb{G})$, if an affine Lie group $\mathbb{G}$ acts simply transitively preserving ω on Ω, then the Hessian algebra $\mathcal{G} = (\mathfrak{g}, \cdot, \langle , \rangle)$ is, in fact, a Koszul algebra since the closed 1-form gives a Lie algebra homomorphism $s : \mathfrak{g} \to \mathbb{R}$. We will denote $\langle , \rangle_s$ the Koszul type inner product determined by the Lie algebra homomorphism s. A nondegenerate inner product $\langle , \rangle$ defines an isomorphism $\phi : \mathcal{G} \to \mathcal{G}^*$ by $x \mapsto \langle \bullet, x \rangle$, where $\mathcal{G}^*$ is the set of all linear functionals on $\mathcal{G}$. Therefore, on a Koszul algebra $\mathcal{G} = (\mathfrak{g}, \cdot, \langle , \rangle_s)$, there exists an element $e \in \mathcal{G}$ such that $s = \phi(e)$. Then $s(x) = \langle x, e \rangle s = s(xe) = s(ex)$ for any $x \in \mathcal{G}$. Moreover we have the following:

Proposition 2.6. *Let $\mathcal{G} = (\mathfrak{g}, \cdot, \langle , \rangle_s)$ be a Koszul algebra, then we have*

(a) $\langle x, ee \rangle_s = \langle x, e \rangle_s$ *for all $x \in \mathfrak{g}$, thus e is an idempotent.*

(b) $\lambda_e + \lambda'_e = 1 + \rho_e$, *where λ'_e is the adjoint of λ_e with respect to $\langle\ ,\ \rangle_s$.*

Proof. (a) For any $x \in \mathcal{G}$ we have $x(ee) = e(xe) + [x, e]e$. Then, by using the fact that s is a Lie algebra homomorphism,

$$\begin{aligned} \langle x, ee \rangle_s &= s(x(ee)) = s(e(xe)) + s([x,e]e) \\ &= s(xe) + s([x,e]) = s(xe) = \langle x, e \rangle_s. \end{aligned}$$

(b) For any $x, y \in \mathcal{G}$ we have $(ey)x + y(ex) = (ye)x + e(yx)$. Then,

$$\begin{aligned}\langle \lambda_e y, x\rangle_s + \langle \lambda_e' y, x\rangle_s &= \langle \lambda_e y, x\rangle_s + \langle y, \lambda_e x\rangle_s = \langle ey, x\rangle_s + \langle y, ex\rangle_s \\ &= s((ey)x + y(ex)) = s((ye)x + e(yx)) \\ &= \langle ye, x\rangle_s + s((yx)e) = \langle \rho_e y, x\rangle_s + s(yx) \\ &= \langle \rho_e y, x\rangle_s + \langle y, x\rangle_s.\end{aligned}$$

Thus $\langle (\lambda_e + \lambda_e')(y), x\rangle_s = \langle (1 + \rho_e)(y), x\rangle_s$ for all $x, y \in \mathcal{G}$. □

The idempotent $e \in \mathfrak{g}$ is called a *principal idempotent* of $\mathcal{G}$ [15]. We note that if $\mathcal{G}$ has a right identity or left identity, then it must be the principal idempotent from the uniqueness.

Corollary 2.7. *Let $\mathcal{G} = (\mathfrak{g}, \cdot, \langle , \rangle_s)$ be an n-dimensional Koszul algebra, and let e be the principal idempotent of $\mathcal{G}$. Then*

(a) $\rho_e' = \rho_e$ *where ρ_e' is the adjoint of ρ_e with respect to $\langle\ ,\ \rangle_s$.*
(b) $2 \operatorname{tr} \lambda_e = n + \operatorname{tr} \rho_e$.

3. Geometry of the graph extension

Let $\mathcal{G} = (\mathfrak{g}, \cdot)$ be the graph extension of a complete unimodular Hessian algebra $(\mathfrak{a}, *, \langle , \rangle)$ where $\mathfrak{g} = \mathfrak{a} + \mathfrak{j}$ and $\mathfrak{j} = \operatorname{span}\{e\}$, which corresponds to the domain over a graph of a function $F : \mathbb{R}^n \to \mathbb{R}$. Recall that $\mathcal{G}$ is a Koszul algebra with the inner product $\langle\!\langle , \rangle\!\rangle = \operatorname{tr} \rho$ defined in (2). For $x = (a, se),\ y = (b, te) \in \mathfrak{g}$ where $a, b \in \mathfrak{a}$ and $s, t \in \mathbb{R}$,

$$\begin{aligned}&Q(x, y) \\ =\ & \langle\!\langle x, x\rangle\!\rangle \langle\!\langle y, y\rangle\!\rangle - \langle\!\langle x, y\rangle\!\rangle^2 \\ =\ & \{\langle a, a\rangle + s^2\}\{\langle b, b\rangle + t^2\} - \{\langle a, b\rangle + st\}^2 \\ =\ & s^2\langle b, b\rangle + t^2\langle a, a\rangle - 2st\langle a, b\rangle - \langle a, b\rangle^2 + \langle a, a\rangle\langle b, b\rangle.\end{aligned}$$

On the other hand, from Theorem B, we have

$$\lambda_x = \begin{pmatrix} \bar{\lambda}_a + sB & 0 \\ a^t H & s \end{pmatrix}, \quad \lambda_x' = \begin{pmatrix} \bar{\lambda}_a' + s(I - B) & a \\ 0 & s \end{pmatrix},$$

where $\bar{\lambda}_a$ is the left multiplication operator on $(\mathfrak{a}, *, \langle , \rangle)$, B denotes the restriction of λ_e on $\mathfrak{a}$, and H is the matrix representing the Hessian type inner product $\langle , \rangle$ on $\mathfrak{a}$. Then, by calculating the matrices,

$$K_x = \begin{pmatrix} \bar{K}_a + \frac{s}{2}I & \frac{1}{2}a \\ \frac{1}{2}a' H & s \end{pmatrix}, \quad K_x x = \begin{pmatrix} \bar{K}_a a + sa \\ \frac{1}{2}\langle a, a\rangle + s^2 \end{pmatrix}$$

and for $y = (b, t)$

$$K_y y = \begin{pmatrix} \bar{K}_b b + tb \\ \frac{1}{2}\langle b, b\rangle + t^2 \end{pmatrix}, \quad K_x y = \begin{pmatrix} \bar{K}_a b + \frac{s}{2}b + \frac{t}{2}a \\ \frac{1}{2}\langle a, b\rangle + st \end{pmatrix},$$

where $\bar{K}$ is the difference tensor on $\mathfrak{a}$. Therefore we have

$$\begin{aligned} &\langle\langle K_x y, K_x y\rangle\rangle - \langle\langle K_x x, K_y y\rangle\rangle \\ = &\ \langle \bar{K}_a b + \frac{s}{2}b + \frac{t}{2}a, \bar{K}_a b + \frac{s}{2}b + \frac{t}{2}a\rangle + (\frac{1}{2}\langle a,b\rangle + st)^2 \\ &-\langle \bar{K}_a a + sa, \bar{K}_b b, tb\rangle - (\frac{1}{2}\langle a,a\rangle + s^2)(\frac{1}{2}\langle b,b\rangle + t^2) \\ = &\ \langle \bar{K}_a b, \bar{K}_a b\rangle - \langle \bar{K}_a a, \bar{K}_b b\rangle \\ &-\frac{1}{4}\{s^2\langle b,b\rangle + t^2\langle a,a\rangle - 2st\langle a,b\rangle - \langle a,b\rangle^2 + \langle a,a\rangle\langle b,b\rangle\} \\ = &\ \langle -[\bar{K}_a, \bar{K}_b]b, a\rangle - \frac{1}{4}Q(x,y) \\ = &\ \langle \bar{R}(a,b)b, a\rangle - \frac{1}{4}Q(x,y), \end{aligned}$$

where $\bar{R}$ is the curvature tensor on $\mathfrak{a}$ with respect to the Hessian type inner product $\langle,\rangle$. Therefore we have the following:

Theorem 3.1. *Let $\mathcal{G} = (\mathfrak{g}, \cdot, \langle\langle,\rangle\rangle)$ be the graph extension of a complete unimodular Hessian algebra $(\mathfrak{a}, *, \langle,\rangle)$ where $\langle\langle,\rangle\rangle = \operatorname{tr}\rho$. Then the followings are equivalent:*

(a) *The curvature $\bar{R}$ of Levi-Civita connection on $\mathfrak{a}$ is flat.*
(b) *$[\bar{K}_a, \bar{K}_b] = 0$ for all $a, b \in \mathfrak{a}$, that is, the algebra $(\mathfrak{a}, \circ)$ is a commutative Frobenius algebra.*
(c) *The Hessian curvature $D\bar{K}$ on $\mathfrak{a}$ is totally symmetric.*
(d) *The sectional curvature κ on $\mathcal{G}$ is $-\frac{1}{4}$.*

Let's consider another Koszul type inner product $\langle\langle,\rangle\rangle_s$ on the graph extension $\mathcal{G} = (\mathfrak{g}, \cdot)$ where the idempotent e is the corresponding principal idempotent of $\mathcal{G}$. Assume that $s(e) = \langle\langle e, e\rangle\rangle_s = \frac{1}{\alpha}$ for $0 \neq \alpha \in \mathbb{R}$. Then $\mathfrak{a} = \ker s$ because $s(a) = \langle\langle a, e\rangle\rangle_s = s(ae) = 0$ for all $a \in \mathfrak{a}$ and the dimension argument. Thus we have the following: for $a, b \in \mathfrak{a}$,

$$\langle\langle a, b\rangle\rangle_s = s(ab) = s(a * b + \langle a,b\rangle e) = \langle a,b\rangle s(e) = \frac{1}{\alpha}\langle a,b\rangle.$$

This says that $\langle\langle,\rangle\rangle_s = \frac{1}{\alpha}\langle\langle,\rangle\rangle$, equivalently, $s = \frac{1}{\alpha}\operatorname{tr}\rho$. Let $\tilde{\lambda}'_x$ be the adjoint of λ_x with respect to $\langle\langle,\rangle\rangle_s$. Since $\langle\langle y, \tilde{\lambda}'_x z\rangle\rangle_s = \langle\langle \lambda_x y, z\rangle\rangle_s = \frac{1}{\alpha}\langle\langle \lambda_x y, z\rangle\rangle = \frac{1}{\alpha}\langle\langle y, \lambda'_x z\rangle\rangle = \langle\langle y, \lambda'_x z\rangle\rangle_s$, we have $\tilde{\lambda}'_x = \lambda'_x$ for all $x \in \mathcal{G}$. From this, we see that the Levi-Civita connection and the difference tensor are not changed. Just, $Q_s(x,y) = \frac{1}{\alpha^2}Q(x,y)$ and $\langle\langle K_x y, K_x y\rangle\rangle_s - \langle\langle K_x x, K_y y\rangle\rangle_s = \frac{1}{\alpha}\{\langle\langle K_x y, K_x y\rangle\rangle - \langle\langle K_x x, K_y y\rangle\rangle\}$. Hence the curvature $\bar{R}$ of Levi-Civita connection on $\mathfrak{a}$ is flat if and only if the sectional curvature $\kappa_s = -\frac{\alpha}{4}$. Therefore we have the following:

Theorem 3.2. *Let Ω be a homogeneous affine domain over the hypersurface Σ given as the graph of a function $F : \mathbb{R}^n \to \mathbb{R}$ such that $|\det DdF| = 1$. The Hessian metric DdF on Σ is flat if and only if Ω has a Koszul type Hessian metric whose sectional curvature is constant.*

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