THE HYPERINVARIANT SUBSPACE PROBLEM FOR QUASI-n-HYPONORMAL OPERATORS

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Abstract. In this paper we examine the hyperinvariant subspace problem for quasi-n-hyponormal operators. The main result on this problem is as follows. If $T = N + K$ is such that N is a quasi-n-hyponormal operator whose spectrum contains an exposed arc and K belongs to the Schatten p-ideal then T has a non-trivial hyperinvariant subspace.

1. Introduction

Throughout this note, $\mathcal{H}$ denotes an infinite-dimensional separable Hilbert space. We write $\mathcal{B}(\mathcal{H})$ for the algebra of bounded linear operators on $\mathcal{H}$ and $K(\mathcal{H})$ for the ideal of compact operators on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called an n-*normal* operator if there exists a maximal abelian self-adjoint algebra $\mathcal{R}$ such that T is in the commutant of $\mathcal{R}^{(n)}$, where $\mathcal{R}^{(n)}$ denotes the direct sum of n copies of $\mathcal{R}$. From the definition we can see that $T \in \mathcal{B}(\mathcal{H})$ is n-normal if and only if it is unitarily equivalent to an $n \times n$ operator matrix (N_{ij}) acting on $\mathcal{K}^{(n)}$, where $\{N_{ij}\}$ is a collection of commuting normal operators on a separable Hilbert space $\mathcal{K}$ (cf. [5], [6, Theorem 7.17]). Moreover it was well known ([2], [1], [6, Theorem 7.2]) that each n-normal operator has an upper triangular form: i.e., if T is n-normal then T is unitarily equivalent to

$$
(1.1) \qquad \begin{pmatrix} N_{11} & N_{12} & \dots & N_{1n} \\ 0 & N_{22} & \dots & N_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & N_{nn} \end{pmatrix},
$$

where $\{N_{ij}\}_{1 \le i \le j \le n}$ consists of mutually commuting normal operators on a separable Hilbert space $\mathcal{K}$. In [4] an extended notion of n-normal operators was introduced.

Definition 1.1. Let $\mathcal{K}$ be a separable complex Hilbert space. An operator $T \in \mathcal{B}(\mathcal{H})$ is called a *quasi-n-hyponormal operator* (for $n \in \mathbb{N}$) if it is unitarily equivalent to an $n \times n$ upper triangular operator matrix (N_{ij}) acting on $\mathcal{K}^{(n)}$, where the diagonal entries N_{jj} $(j = 1, 2, \ldots, n)$ are hyponormal operators in $\mathcal{B}(\mathcal{K})$.

Clearly, the classes of hyponormal and quasi-1-hyponormal operators coincide.

2. The main result

If $T \in \mathcal{B}(\mathcal{H})$, we write $\rho(T)$ for the resolvent of T; $\sigma(T)$ for the spectrum of T; $\pi_0(T)$ for the eigenvalues of T.

The following lemma is used in proving the main result.

Lemma 2.1 ([4, Lemma 3]). *If T is quasi-n-hyponormal and $\lambda \notin \sigma(T)$ then*

$$\|(T - \lambda)^{-1}\| \leq \frac{(1 + \|T\|)^{n-1}}{\min\left\{1, \ \left[\operatorname{dist}\left(\lambda, \ \sigma(T)\right)\right]^n\right\}}.$$

The subspace $\mathcal{M}$ is said to be hyperinvariant for $T \in \mathcal{B}(\mathcal{H})$ if $\mathcal{M}$ is invariant for every operator S which commutes with T. Knowledge of the hyperinvariant subspaces of T gives information on the structure of the commutant of T. The invariant subspace problem is: *does every operator have a non-trivial invariant subspace ?* This problem remains still open. A related question is the hyperinvariant subspace problem: *does every operator which is not a multiple of the identity have a non-trivial hyperinvariant subspace ?* Also nobody knows the answer until now. Some partial solutions on those problems were given in many literature. Perhaps the most elegant results on the hyperinvariant subspace problem are the affirmative answers for non-scalar normal operators and for non-zero compact operators. Note that n-normal operators have non-trivial reducing subspaces: if $T \in \operatorname{comm} \mathcal{R}^{(n)}$ and $P \in \mathcal{R}$ then $P^{(n)}$ is a projection commuting with T. In particular one knows ([3], [6]) that if T is n-normal and is not a multiple of the identity then T has a non-trivial hyperinvariant subspace. Many authors have also examined the hyperinvariant subspace problem for some perturbations of normal operators. To see the most striking result among those attempts, we need some definitions.

If T is a compact operator, arrange the eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$, repeated according to multiplicity, in decreasing order, and let λ_n denote the n-th eigenvalues of $|T|$. For each $p \geq 1$, define

$$|T|_p := \left[\sum_{n=1}^{\infty} \lambda_n^p\right]^{\frac{1}{p}}$$

and

$$\mathcal{C}_p := \{T : \ |T|_p < \infty\}.$$

Then $\mathcal{C}_p$ forms an ideal in $\mathcal{B}(\mathcal{H})$, called the Schatten p-ideal. It is well-known (cf. [6, Theorem 6.12]) that if $T = N + K \in \mathcal{B}(\mathcal{H})$ is such that

   (i) N is a normal operator and $K \in \mathcal{C}_p$ for some $p \geq 1$;
   (ii) $\sigma(N)$ contains an exposed arc (i.e., a subset whose intersection with some open disk in $\mathbf{C}$ is a smooth Jordan arc),

then T has a non-trivial hyperinvariant subspace.

We will show that this theorem remains true if "normal" is replaced by "quasi-n-hyponormal".

If $T \in \mathcal{C}_k$, with k an integer greater than 1, and if $\{\lambda_1, \lambda_2, \ldots\}$ is an enumeration of the non-zero eigenvalues of T, repeated according to multiplicity, define

$$\delta_k(T) := \prod_{j=1}^{\infty} \left[(1 + \lambda_j)\exp\left(-\lambda_j + \frac{\lambda_j^2}{2} + \cdots + \frac{(-1)^{k-1}\lambda_j^{k-1}}{k-1} \right) \right].$$

(If $\pi_0(T) \subset \{0\}$, define $\delta_k(T) = 1$.) It was well-known that (cf. [6, Lemmas 6.6 and 6.7])

   (2.1) $\delta_k(T)$ is an absolutely convergent infinite product;
   (2.2) there exists a constant $c_k > 0$ such that $|\delta_k(T)| \leq \exp\left(c_k |T|_k^k \right)$;
   (2.3) there exists a constant $M_k > 0$ such that
   $$\|\delta_k(T)(1 + T)^{-1}\| \leq \exp\left(M_k |T|_k^k \right)$$
   whenever $-1 \notin \sigma(T)$;
   (2.4) if $B \in \mathcal{C}_k$ and $A \in \mathcal{B}(\mathcal{H})$ is arbitrary then $\delta_k\left(B(z - A)^{-1} \right)$ is an analytic function of z on $\rho(A)$.

The following lemma is an extension of [6, Lemma 6.11]; the proof is similar to that of [6, Lemma 6.11].

Lemma 2.2. *Let $T = N + K$, where N is quasi-n-hyponormal and $K \in \mathcal{C}_k$ for some $k \geq 1$. Suppose $\pi_0(T) = \emptyset$ and $\sigma(N)$ contains an exposed arc J. Let $z_0 \in J$ and let L be any closed bounded line segment with z_0 as an endpoint which is not tangent to J and is such that $L \cap \sigma(N) = \{z_0\}$. Then there exists a constant $c > 0$ such that*

$$\|(z - T)^{-1}\| \leq \exp\left(\frac{c}{|z - z_0|^{nk+1}} \right) \quad \text{for all } z \in L \setminus \{z_0\}.$$

Proof. Observe that $\sigma(T) \subseteq \sigma(N) \cup \pi_0(T)$. So if $\pi_0(T) = \emptyset$, then $\sigma(T) \subset \sigma(N)$. Let D be an open disk such that $D \cap \sigma(N) = J$. Given a line segment L as in the statement of the theorem, let D' be a disk, with radius $\leq \frac{1}{2}$, tangent to J at z_0 and contained in D which meets L and whose intersection with J is $\{z_0\}$. Define the function δ on $\rho(N)$ by

$$\delta(z) := \delta_k\left(-K(z - N)^{-1} \right).$$

Then by (2.4), δ is analytic on $\rho(N)$. Observe that for $z \in \rho(N)$,

$$(z - T)^{-1} = (z - N)^{-1} \left(1 - K(z - N)^{-1} \right)^{-1}.$$

Since $-1 \notin \sigma(-K(z-N)^{-1})$, it follows that $\delta(z) \neq 0$ because δ_k converges absolutely. So, for $z \in \rho(N)$,

$$(z-T)^{-1} = \delta(z)^{-1}(z-N)^{-1}\delta(z)\left(1 - K(z-N)^{-1}\right)^{-1}.$$

By (2.2) and Lemma 2.1,

$$|\delta(z)| \leq \exp\left(c_1|K(z-N)^{-1}|_k^k\right) \quad \text{(some constant } c_1\text{)}$$

$$\leq \exp\left(c_1|K|_k^k\,\|(z-N)^{-1}\|^k\right)$$

$$\leq \exp\left(\frac{c_2}{\min\left\{1,\ \left[\text{dist}\,(z,\ \sigma(N))\right]^{nk}\right\}}\right) \quad (c_2 := c_1|K|_k^k(1+\|N\|)^{(n-1)k}).$$

Since for $z \in D'$, $\text{dist}\,(z, \sigma(N)) \geq \text{dist}\,(z, \partial D')$, we have

$$|\delta(z)| \leq \exp\left(\frac{c_2}{\min\left\{1,\ \left[\text{dist}\,(z,\ \partial D')\right]^{nk}\right\}}\right) \leq \exp\left(\frac{c_2}{\left[\text{dist}\,(z,\ \partial D')\right]^{nk}}\right).$$

Since $\delta(z) \neq 0$ throughout the disk D', it has an analytic logarithm $\alpha(z)$, i.e., $\exp(\alpha(z)) = \delta(z)$. Since $|\delta(z)| = \exp(\text{Re}\,\alpha(z))$, we have

$$\text{Re}\,\alpha(z) \leq \frac{c_2}{\left[\text{dist}\,(z,\ \partial D')\right]^{nk}}.$$

Using an argument involved with the Borel-Carathéodory inequality (see [6, Lemma 6.10]), we can get

$$|\alpha(z)| \leq \frac{c_3}{|z-z_0|^{nk+1}} \quad \text{for some constant } c_3 \text{ and } z \in D' \cap L.$$

Hence, $\text{Re}\,\alpha(z) \geq \frac{-c_3}{|z-z_0|^{nk+1}}$, so that

$$|\delta(z)^{-1}| = \exp(-\text{Re}\,\alpha(z)) \leq \exp\left(\frac{c_3}{|z-z_0|^{nk+1}}\right).$$

Therefore, now, for $z \in D' \cap L$,

$$\|(z-T)^{-1}\|$$

$$= \|\delta(z)^{-1}(z-N)^{-1}\delta(z)\left(1 - K(z-N)^{-1}\right)^{-1}\|$$

$$\leq \exp\left(\frac{c_3}{|z-z_0|^{nk+1}}\right)\|(z-N)^{-1}\|\,\|\delta(z)\left(1 - K(z-N)^{-1}\right)^{-1}\|$$

$$\leq \exp\left(\frac{c_3}{|z-z_0|^{nk+1}}\right)\|(z-N)^{-1}\|\exp\left(c_4\,|K|_k^k\,\|(z-N)^{-1}\|^k\right)$$

for some constant c_4 (by (2.3)).

Thus by Lemma 2.1 we have that for some constants c_5, c_6,

$$\|(z - T)^{-1}\|$$

$$\leq \exp\left(\frac{c_3}{|z - z_0|^{nk+1}}\right) \frac{c_5}{\min\left\{1, \left[\text{dist}(z, \sigma(N))\right]^n\right\}}$$

$$\times \exp\left(\frac{c_6}{\min\left\{1, \left[\text{dist}(z, \sigma(N))\right]^{nk}\right\}}\right)$$

$$\leq \exp\left(\frac{c_3}{|z - z_0|^{nk+1}}\right) \frac{c_5}{\left[\text{dist}(z, \sigma(N))\right]^n} \exp\left(\frac{c_6}{\left[\text{dist}(z, \sigma(N))\right]^{nk}}\right)$$

since $\text{dist}(z, \sigma(N)) \leq \text{dist}(z, \{z_0\}) \leq 1$ for $z \in L \cap D'$. Since L is not tangent to J, $\frac{|z - z_0|}{\text{dist}(z, \sigma(N))}$ is bounded for $z \in L \cap D'$. It thus follows that for $z \in L \cap D'$,

$$\|(z - T)^{-1}\|$$

$$\leq \exp\left(\frac{c_3}{|z - z_0|^{nk+1}}\right) \exp\left(\frac{c_7}{|z - z_0|^n}\right) \exp\left(\frac{c_8}{|z - z_0|^{nk}}\right)$$

$$\leq \exp\left(\frac{c}{|z - z_0|^{nk+1}}\right) \quad \text{for some constant } c > 0 \text{ (since } |z - z_0| \leq 1\text{)}.$$

This completes the proof. $\square$

The following is a well-known theorem.

Lemma 2.3 ([6, Theorem 6.3]). *Let $T \in \mathcal{B}(\mathcal{H})$ be such that $\sigma(T)$ contains an exposed arc J. Suppose that for each point $z_0 \in J$ and each closed line segment L which meets $\sigma(T)$ only in $\{z_0\}$ and which is not tangent to J, there exists a constant $c > 0$ such that*

$$(2.5) \qquad \|(z - T)^{-1}\| \leq \exp\left(\frac{c}{|z - z_0|^k}\right)$$

for some k and all $z \in L \setminus \{z_0\}$. Then T has a non-trivial hyperinvariant subspace.

We are ready for:

Theorem 2.4. *Let $T = N + K \in \mathcal{B}(\mathcal{H})$ be such that*

    (i) *N is a quasi-n-hyponormal operator and $K \in \mathcal{C}_p$ for some $p \geq 1$;*

    (ii) *$\sigma(N)$ contains an exposed arc J.*

Then T has a non-trivial hyperinvariant subspace.

Proof. We can write

$$
N \equiv \begin{pmatrix} N_1 & * & \ldots & * \\ 0 & N_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \ldots & 0 & N_n \end{pmatrix} \qquad (N_i \text{ hyponormal}).
$$

Since $\pi_0(N) \subset \bigcup_{j=1}^{n} \pi_0(N_j)$ and each N_j is a hyponormal operator acting on a separable space, it follows that $\pi_0(N)$ is countable. But since $\sigma(N) \subset \sigma(T) \bigcup \pi_0(N)$, we have that $J \subset \sigma(T)$. If $\pi_0(T) \neq \emptyset$ then evidently T has a nontrivial hyperinvariant subspace. If instead $\pi_0(T) = \emptyset$ then since $\sigma(T) \subset \sigma(N)$, it follows that J is an exposed arc of $\sigma(T)$. By Lemma 2.2, the resolvent of T satisfies the condition (2.5) with k the least integer greater than $np + 1$. So the result follows at once from Lemma 2.3. ☐

We conclude with two corollaries.

Corollary 2.5. *Let* $T = N + K \in \mathcal{B}(\mathcal{H})$ *be such that*

(i) N *is a hyponormal operator and* $K \in \mathcal{C}_p$ *for some* $p \geq 1$;
(ii) $\sigma(N)$ *contains an exposed arc.*

Then T *has a non-trivial hyperinvariant subspace.*

Proof. Immediate from Theorem 2.4. ☐

Corollary 2.6. *Let* $T = N + K \in \mathcal{B}(\mathcal{H})$ *be such that*

(i) N *is an n-normal operator and* $K \in \mathcal{C}_p$ *for some* $p \geq 1$;
(ii) $\sigma(N)$ *is contained in a smooth Jordan curve.*

Then T *has a non-trivial invariant subspace.*

Proof. Immediate from the same argument as [6, Corollary 6.14] together with Theorem 2.4. ☐

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