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WEIGHTED COMPOSITION OPERATORS BETWEEN BERGMAN AND BLOCH SPACES

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Abstract. In this paper, we characterize the boundedness and compactness of weighted composition operators $\psi C_\varphi f = \psi(f \circ \varphi)$ acting between Bergman and Bloch spaces of holomorphic functions on the open unit disk $\mathbb{D}$.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Denote by $H(\mathbb{D})$, the space of holomorphic functions on $\mathbb{D}$. Let φ and ψ be holomorphic maps on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then we can define a linear operator $\psi C_\varphi f = \psi(f \circ \varphi)$, $f \in H(\mathbb{D})$, called a weighted composition operator. When $\psi = 1$, we just have the composition operator C_φ defined by $C_\varphi(f) = f \circ \varphi$ and when $\varphi(z) = z$ we have the multiplication operator M_ψ defined by $M_\psi(f) = \psi f$. For general back ground on composition operators, we refer [6] and references therein. Recently, several authors have studied weighted composition operators on different spaces of analytic functions. For more information on weighted composition operators, one can refer to [1], [3], [4], [5], [10], [12], [14], [15] and [16]. Weighted composition operators appear naturally in different contexts. For example, Singh and Sharma [13] related the boundedness of composition operators on Hardy space of the upper half-plane with the boundedness of weighted composition operators on the Hardy space of the open unit disk $\mathbb{D}$. Weighted composition operators also played an important role in the study of compact composition operators on Hardy spaces of upper half-plane, see [11]. Also Isometries in many Banach spaces of analytic functions are just weighted composition operators, for example see [7] and [9].

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2. Preliminaries

In this section we review the basic concepts of weighted Bergman spaces A^p_α and the α-Bloch spaces $\mathcal{B}^\alpha$. We also collect some essential facts that will be needed throughout the paper.

2.1. Weighted Bergman spaces

Let $dA(z)$ be the area measure on $\mathbb{D}$ normalized so that area of $\mathbb{D}$ is 1. For each $\alpha \in (-1, \infty)$, we set $d\nu_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z), z \in \mathbb{D}$. Then $d\nu_\alpha$ is a probability measure on $\mathbb{D}$. For $0 < p < \infty$ the weighted Bergman space A^p_α is defined as

$$A^p_\alpha = \Big\{ f \in H(\mathbb{D}) : ||f||_{A^p_\alpha} = \Big(\int_{\mathbb{D}} |f(z)|^p d\nu_\alpha(z) \Big)^{1/p} < \infty \Big\}.$$

Note that $||f||_{A^p_\alpha}$ is a true norm only if $1 \leq p < \infty$ and in this case A^p_α is a Banach space. For $0 < p < 1$, A^p_α is a non-locally convex topological vector space and $d(f,g) = ||f-g||^p_{A^p_\alpha}$ is a complete metric for it.

The growth of functions in the weighted Bergman spaces is essential in our study. To this end, the following estimates will be useful. (see [8] and [17]).

Let $f \in A^p_\alpha$. Then for every z in $\mathbb{D}$, we have

$$|f(z)| \leq \frac{||f||_{A^p_\alpha}}{(1-|z|^2)^{(2+\alpha)/p}} \tag{2.1}$$

with equality if and only if f is a constant multiple of the function

$$k_a(z) = \Big(\frac{1-|z|^2}{(1-\bar{a}z)^2} \Big)^{(2+\alpha)/p}.$$

It can be easily shown that $||k_a||^p_{A^p_\alpha} \approx 1$. Since polynomials are dense in A^p_α, it is an immediate consequence of (2.1) that for $f \in A^p_\alpha$,

$$|f(z)| = o\Big(\frac{1}{(1-|z|^2)^{(2+\alpha)/p}} \Big) \quad \text{as} \quad |z| \to 1, \tag{2.2}$$

which means that the boundary growth is not as fast as permitted by (2.1).

Further, if $p \geq 1$ and $f \in H(\mathbb{D})$, then $f \in A^p_\alpha$ if and only if $(1-|z|^2)f'(z)$ is in $L^p(\mathbb{D}, d\nu_\alpha)$ and

$$||f||_{A^p_\alpha} \approx |f(0)| + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^p d\nu_\alpha(z). \tag{2.3}$$

By (2.3), it follows that, whenever $f \in A^p_\alpha$, then its derivative $f' \in A^p_{p+\alpha}$ and there exists a positive constant C_p such that $||f'||_{A^p_{p+\alpha}} \leq C_p ||f||_{A^p_\alpha}$. Again by (2.1) for every z in $\mathbb{D}$, we have

$$|f'(z)| \leq \frac{||f'||_{A^p_{p+\alpha}}}{(1-|z|^2)^{(2+p+\alpha)/p}} \leq C_p \frac{||f||_{A^p_\alpha}}{(1-|z|^2)^{(2+p+\alpha)/p}}. \tag{2.4}$$

We next define weighted Bloch spaces.

2.2. α-Bloch Spaces

Let $\alpha > 0$. A function f holomorphic in $\mathbb{D}$ is said to belong to α-Bloch Space $\mathcal{B}^\alpha$ if

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| < \infty$$

and to the little α-Bloch Space $\mathcal{B}_0^\alpha$ if

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha|f'(z)| = 0.$$

For $f \in \mathcal{B}^\alpha$ define

$$||f||_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)|.$$

With this norm $\mathcal{B}^\alpha$ is a Banach space and the little α-Bloch Space is a closed subspace of the α-Bloch Space. Note that $\mathcal{B}^1 = \mathcal{B}$, the usual Bloch space.

3. Weighted composition operator from Bergman Space into the Bloch Space

Theorem 3.1. *Let $1 \le p < \infty$ and let φ and ψ be holomorphic maps on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then ψC_φ is bounded from A^p into $\mathcal{B}$ if and only if the following conditions are satisfied:*

(i) $\sup_{z\in\mathbb{D}}\big((1-|z|^2)/(1-|\varphi(z)|^2)^{2/p}\big)|\psi'(z)| < \infty$

(ii) $\sup_{z\in\mathbb{D}}\big((1-|z|^2)/(1-|\varphi(z)|^2)^{1+2/p}\big)|\psi(z)\varphi'(z)| < \infty.$

Proof. First suppose that

$$M = \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{2/p}}|\psi'(z)| < \infty$$

and

$$N = \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{1+2/p}}|\psi(z)\varphi'(z)| < \infty.$$

For $f \in A^p$, we have

$$\begin{aligned}(1-|z|^2)|(\psi C_\varphi f)'(z)| &= (1-|z|^2)|\psi'(z)f(\varphi(z)) + \psi(z)f'(\varphi(z))\varphi'(z)| \\ &\le \Big(\frac{(1-|z|^2)|\psi'(z)|}{(1-|\varphi(z)|^2)^{2/p}} + C_p\frac{(1-|z|^2)|\psi(z)|}{(1-|\varphi(z)|^2)^{1+2/p}}\Big)||f||_{A^p} \\ &\le (M + C_pN)||f||_{A^p}\end{aligned}$$

and consequently, $\psi C_\varphi f \in \mathcal{B}$. In addition to this (2.1) yields

$$|f(\varphi(0))| \le \frac{||f||_{A^p}}{(1-|\varphi(0)|^2)^{2/p}}.$$

The last two inequalities show that $||\psi C_\varphi||_{\mathcal{B}} \le M_p||f||_{A^p}$, hence ψC_φ maps A^p boundedly into $\mathcal{B}$.

Conversely suppose $\psi C_\varphi : A^p \to \mathcal{B}$ is bounded. Then taking the constant function and $f(z) = z$ respectively, in A^p, we get

$$\psi \in \mathcal{B} \quad \text{and} \quad \sup_{z\in\mathbb{D}}(1-|z|^2)|\psi(z)\varphi'(z)| < \infty. \tag{3.1}$$

Fix a point $\lambda \in \mathbb{D}$ and let

$$f(z) = \Big(\frac{1-|\varphi(\lambda)|^2}{(1-\overline{\varphi(\lambda)}z)^2}\Big)^{2/p}.$$

Then $f \in A^p$ and $||f||_{A^p} = 1$. Thus there exist a constant $C > 0$ such that

$$C \geq (1-|z|^2)|\psi'(z)f(\varphi(z)) + \psi(z)f'(\varphi(z))\varphi'(z)|.$$

That is

$$C + \frac{4(1-|z|^2)|\psi(z)\varphi'(z)||\varphi(\lambda)|(1-|\varphi(\lambda)|^2)^{2/p}}{p(1-\overline{\varphi(\lambda)}\varphi(z))^{1+4/p}} \geq \frac{(1-|z|^2)|\psi'(z)|(1-|\varphi(\lambda)|^2)^{2/p}}{((1-\overline{\varphi(\lambda)}\varphi(z))^2)^{2/p}}.$$

In particular, when $z = \lambda$, we have

$$C + \frac{4(1-|\lambda|^2)|\psi(\lambda)\varphi'(\lambda)||\varphi(\lambda)|}{p(1-|\varphi(\lambda)|^2)^{1+2/p}} \geq \frac{(1-|\lambda|^2)|\psi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{2/p}}. \tag{3.2}$$

Thus it is sufficient to prove that (ii) is true. Consider the function

$$g(z) = \frac{(1-|\varphi(\lambda)|^2)^{4/p}}{(1-\overline{\varphi(\lambda)}z)^{6/p}} - \Big(\frac{1-|\varphi(\lambda)|^2}{(1-\overline{\varphi(\lambda)}z)^2}\Big)^{2/p}.$$

Then $g \in A^p$ and $||g||_{A^p} \leq (2^{2/p}+1)^p$. Moreover, we notice that $g(\varphi(\lambda)) = 0$ and

$$|g'(\varphi(\lambda))| = \frac{2}{p}\frac{|\varphi(\lambda)|}{(1-|\varphi(\lambda)|^2)^{1+2/p}}.$$

So

$$\begin{aligned}(2^{2/p}+1)^p||\psi C_\varphi||_{\mathcal{B}} &\geq ||\psi C_\varphi g||_{\mathcal{B}} \\ &\geq \frac{2}{p}\frac{1-|\lambda|^2}{(1-|\varphi(\lambda)|^2)^{1+2/p}}|\psi(\lambda)\overline{\varphi(\lambda)}\varphi'(\lambda)|.\end{aligned}$$

Since $\lambda \in \mathbb{D}$ is arbitrary, we have

$$\sup_{\lambda\in\mathbb{D}}\Big\{\frac{1-|\lambda|^2}{(1-|\varphi(\lambda)|^2)^{1+2/p}}|\psi(\lambda)\overline{\varphi(\lambda)}\varphi'(\lambda)|\Big\} < \infty.$$

Thus for a fixed δ, $0 < \delta < 1$

$$\Big\{\frac{1-|\lambda|^2}{(1-|\varphi(\lambda)|^2)^{1+2/p}}|\psi(\lambda)\varphi'(\lambda)| : \lambda \in \mathbb{D}, |\varphi(\lambda)| > \delta\Big\} < \infty. \tag{3.3}$$

For $\lambda \in \mathbb{D}$ such that $|\varphi(\lambda)| \leq \delta$, we have

$$\frac{1-|\lambda|^2}{(1-|\varphi(\lambda)|^2)^{1+2/p}}|\psi(\lambda)\varphi'(\lambda)| \leq \frac{1}{(1-\delta^2)^{1+2/p}}(1-|\lambda|^2)|\psi(\lambda)\varphi'(\lambda)|$$

and so by (3.1)

$$\text{(3.4)} \qquad \sup\left\{\frac{1-|\lambda|^2}{(1-|\varphi(\lambda)|^{1+2/p}}|\psi(\lambda)\varphi'(\lambda)| : \lambda \in \mathbb{D}, |\varphi(\lambda)| \leq \delta\right\} < \infty.$$

Consequently by (3.3) and (3.4), we have

$$\sup_{\lambda\in\mathbb{D}} \frac{1-|\lambda|^2}{(1-|\varphi(\lambda)|^2)^{1+2/p}}|\psi(\lambda)\varphi'(\lambda)| < \infty$$

and so by (3.2)

$$\sup_{\lambda\in\mathbb{D}} \frac{(1-|\lambda|^2)}{(1-|\varphi(\lambda)|^2)^{2/p}}|\psi'(\lambda)| < \infty.$$

□

Theorem 3.2. *Let φ and ψ be holomorphic maps on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Suppose that ψC_φ maps A^p boundedly into $\mathcal{B}$. Then ψC_φ maps A^p compactly into $\mathcal{B}$. if and only if the following conditons are satisfied.*

(i) $\lim_{|\phi(z)|\to 1} \left((1-|z|^2)/(1-|\varphi(z)|^2)^{2/p}\right)|\psi'(z)| = 0.$

(ii) $\lim_{|\phi(z)|\to 1} \left((1-|z|^2)/(1-|\varphi(z)|^2)^{1+2/p}\right)|\psi(z)\varphi'(z)| = 0.$

Proof. Let $\{f_n\}$ be a bounded sequence in A^p that converges to 0 uniformly on compact subset of $\mathbb{D}$. Let M $= \sup_n ||f_n||_{A^p} < \infty$. Since $\epsilon > 0$, there exists an r such that if $|\varphi(z)| > r$, then

$$\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{2/p}}|\psi'(z)| < \epsilon \quad \text{and} \quad \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{1+2/p}}|\psi(z)\varphi'(z)| < \epsilon.$$

By (2.3) and (2.4), we have

$$|f_n(z)| \leq \frac{||f_n||_{A^p}}{(1-|z|^2)^{2/p}} \quad \text{and} \quad |f_n'(z)| \leq C_p\frac{||f_n||_{A^p}}{(1-|z|^2)^{1+2/p}}.$$

Thus for $z \in \mathbb{D}$ such that $|\varphi(z)| > r$, we have

$$\begin{aligned}
&(1-|z|^2)|(\psi C_\varphi f_n)'(z)| \\
&\qquad = (1-|z|^2)|\psi'(z)f_n(\varphi(z)) + \psi(z)f_n'(\varphi(z))\varphi'(z)| \\
&\qquad \leq \frac{(1-|z|^2)|\psi'(z)|}{(1-|\varphi(z)|^2)^{2/p}}||f_n||_{A^p} + C_p\frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2)^{1+2/p}}||f_n||_{A^p} \\
&\qquad \leq \epsilon M + C_p\varepsilon M \text{ for all } n.
\end{aligned}$$

On the other hand, since f_n and f'_n converges to zero uniformly on $\{w : |w| \leq r\}$, there exists an n_0 such that if $|\varphi(z)| \leq r$ and $n \geq n_0$, then $|f_n(\varphi(z))| < \epsilon$ and $|f'_n(\varphi(z))| < \epsilon$. Also condition (i) and (ii) of Theorem 3.1 implies that

$$A = \sup_{z\in\mathbb{D}}(1-|z|^2)|\psi'(z)| < \infty \quad \text{and} \quad B = \sup_{z\in\mathbb{D}}(1-|z|^2)|\psi(z)\varphi'(z)| < \infty.$$

Thus we deduce that

$$\begin{aligned}(1-|z|^2)&|(\psi C_\varphi f_n)'(z)|\\ &\leq (1-|z|^2)|\psi'(z)||f_n(\varphi(z))| + (1-|z|^2)|\psi(z)\varphi'(z)||f'_n(\varphi(z))|\\ &\leq (A+B)\varepsilon.\end{aligned}$$

The above argument, together with the fact that

$$\psi C_\varphi f_n(0) = \psi(0)f_n(\varphi(0)) \to 0, \text{ as } n \to \infty$$

yield

$$||\psi C_\varphi f_n||_{\mathcal{B}} \to 0 \text{ as } n \to \infty.$$

Conversely, suppose ψC_φ maps A^p compactly into $\mathcal{B}$. Let $\{z_n\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Let

$$f_n(z) = \Big(\frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)}\Big)^{2/p}, z \in \mathbb{D}.$$

Then $f_n \in A^p$. Since ψC_φ maps A^p compactly into $\mathcal{B}$, the functions f_n have unit norm and $f_n \to 0$ uniformly on compact subset of $\mathbb{D}$, it follows that $||\psi C_\varphi f_n||_{\mathcal{B}} \to 0$ as $n \to \infty$. Thus

$$||\psi C_\varphi f_n||_{\mathcal{B}} \geq \Big|\frac{(1-|z_n|^2)|\psi'(z_n)|}{(1-|\varphi(z_n)|^2)^{2/p}} - \frac{2(1-|z_n|^2)|\psi(z_n)||\overline{\varphi(z_n)}\varphi'(z_n)|}{p(1-|\varphi(z_n|^2)^{1+2/p}}\Big|$$

and so, we have

$$\text{(3.5)} \qquad \frac{2(1-|z_n|^2)|\psi(z_n)||\varphi(z_n)\varphi'(z_n)|}{p(1-|\varphi(z_n|^2)^{1+2/p}} + ||\psi C_\varphi f_n|| \geq \frac{(1-|z_n|^2)|\psi'(z_n)|}{(1-|\varphi(z_n)|^2)^{2/p}}.$$

Thus it is sufficient to prove that (ii) is true. Consider the function

$$g_n(z) = \frac{(1-|\varphi(z_n)|^2)^{4/p}}{(1-\overline{\varphi(z_n)}z)^{6/p}} - \Big(\frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^2}\Big)^{2/p}$$

for a sequence z_n in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Then $\{g_n\}$ is a bounded sequence in A^p and $g_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Moreover, we notice that $g_n(\varphi(z_n)) = 0$ and

$$|g'(\varphi(z_n))| = \frac{2}{p}\frac{|\varphi(z_n)|}{(1-|\varphi(z_n)|^2)^{1+2/p}}.$$

So

$$\text{(3.6)} \qquad (2^{2/p}+1)^p||\psi C_\varphi||_{\mathcal{B}} \geq \frac{2}{p}\frac{1-|\lambda|^2}{(1-|\varphi(\lambda)|^2)^{1+2/p}}|\psi(\lambda)\overline{\varphi(\lambda)}\varphi'(\lambda)|.$$

Consequently by (3.5) and (3.6), we have

$$\lim_{|\phi(z)|\to 1}\left((1-|z|^2)/(1-|\varphi(z)|^2)^{2/p}\right)|\psi'(z)| = 0$$

and

$$\lim_{|\phi(z)|\to 1}\left((1-|z|^2)/(1-|\varphi(z)|^2)^{1+2/p}\right)|\psi(z)\varphi'(z)| = 0.$$

□

4. Weighted composition operators from a Bergman space into the little Bloch space

Theorem 4.1. *Let $1 \le p < \infty$ and let φ and ψ be holomorphic maps on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then ψC_φ is bounded from A^p into $\mathcal{B}_0$ if and only if the following conditions are satisfied.*

(i) $\sup_{z\in\mathbb{D}} \dfrac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{2/p}}|\psi'(z)| < \infty$

(ii) $\sup_{z\in\mathbb{D}} \dfrac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{1+2/p}}|\psi(z)\varphi'(z)| < \infty$

(iii) $\psi \in \mathcal{B}_0$.

(iv) $\lim_{|z|\to 1}(1-|z|^2)|\psi(z)\varphi'(z)| = 0.$

Proof. First suppose that ψC_φ maps A^p boundedly into $\mathcal{B}_0$. Then (i) and (ii) can be proved exactly in the same way as in the proof of the Theorem 3.1. By taking $f(z) = c$, we get $\psi \in \mathcal{B}_0$, which is (iii). Again by taking $f(z) = z$, we get

$$\lim_{|z|\to 1}(1-|z|^2)|\psi(z)\varphi'(z) + \psi'(z)\varphi(z)| = 0.$$

Since $\psi \in \mathcal{B}_0$ and $|\varphi(z)| < 1$, we get

$$\lim_{|z|\to 1}(1-|z|^2)|\psi(z)\varphi'(z)| = 0.$$

Next, suppose that $(i) - (iv)$ are satisfied. Take any $\varepsilon > 0$. Let $f \in A^p$. Then by (2.2) there is $\delta_1 \in (0,1)$ such that for any $z \in \mathbb{D}$, $|z| > \delta_1$, we have $|f(z)| < \varepsilon/(1-|z|^2)^{2/p}$ Thus for $|\varphi(z)| > \delta_1$, by (i), we can find a constant $M > 0$ such that

$$(1-|z|^2)|\psi'(z)f(\varphi(z))| < \varepsilon|\psi'(z)|\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{2/p}} \le \varepsilon M. \tag{4.1}$$

On the other hand, since by (iii) $\psi \in \mathcal{B}_0$, so for above ε, there is $\delta_2 \in (0,1)$ such that $|z| > \delta_2$ implies that $(1-|z|^2)|\psi'(z)| < \varepsilon$. Thus for $|\varphi(z)| \le \delta_1$, if $|z| > \delta_2$, we have a constant $N > 0$ such that

$$(1-|z|^2)|\psi'(z)f(\varphi(z))| < ||f||_{A^p}|\psi'(z)|\frac{(1-|z|^2)}{(1-\delta_1^2)^{2/p}} \le \varepsilon N. \tag{4.2}$$

By combining (4.1) and (4.2), we see that whenever $|z| > \delta_2$, we have

$$(1-|z|^2)|\psi'(z)f(\varphi(z))| \leq \max(M,N)\varepsilon.$$

Since $f \in A^p$, there is $\delta_3 \in (0,1)$ such that for any $z \in \mathbb{D}$, $|z| > \delta_3$, we have $|f'(z)| < \varepsilon/(1-|z|^2)^{1+2/p}$. Thus for $|\varphi(z)| > \delta_3$, by (ii), there is a constant $M' > 0$ such that

$$(4.3) \qquad \begin{aligned}(1-|z|^2)|\psi'(z)f(\varphi(z))| &< \varepsilon|\psi(z)\varphi'(z)|\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{1+2/p}} \\ &\leq \varepsilon M'.\end{aligned}$$

On the other hand by (iv), there is $\delta_4 \in (0,1)$, such that $|z| > \delta_4$ implies that $(1-|z|^2)|\psi(z)\varphi'(z)| < \varepsilon$. Thus for $|\varphi(z)| \leq \delta_3$ and $|z| > \delta_4$, we have a constant $N' > 0$ such that

$$(4.4) \qquad \begin{aligned}(1-|z|^2)|\psi(z)f'\varphi(z)\varphi'(z)| &\leq ||f||_{A^p}|\psi(z)\varphi'(z)|\frac{(1-|z|^2)}{(1-{\delta_3}^2)^{1+2/p}} \\ &\leq \varepsilon N'.\end{aligned}$$

By combining (4.3) and (4.4), we see that for $\delta = \max(\delta_2, \delta_4)$, if $|z| > \delta$, then there is a constant $C > 0$ such that

$$(1-|z|^2)|\psi'(z)f(\varphi(z)) + \psi(z)f'(\varphi(z))\varphi'(z)| < C\varepsilon,$$

which means

$$\lim_{|z|\to 1}(1-|z|^2)(\psi C_\varphi f)'(z) = 0.$$

Thus $\psi C_\varphi f \in B_0$. Hence by Closed Graph Theorem ψC_φ maps A^p boundedly into $\mathcal{B}_0$. □

Theorem 4.2. *Let $1 \leq p < \infty$. Let φ and ψ be holomorphic maps on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Suppose that ψC_φ maps A^p boundedly into $\mathcal{B}_0$. Then the weighted composition operator ψC_φ maps A^p compactly into $\mathcal{B}_0$ if and only if*

(i) $\displaystyle\lim_{|z|\to 1}\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{2/p}}|\psi'(z)| = 0$

(ii) $\displaystyle\lim_{|z|\to 1}\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{1+2/p}}|\psi(z)\varphi'(z)| = 0.$

Proof. By lemma 5.2 in [14], a closed set K in $\mathcal{B}_0$ is compact if and only if it is bounded and satisfies

$$\lim_{|z|\to 1}\sup_{f\in K}(1-|z|^2)^\alpha|f(z)| = 0.$$

Thus set $\{\psi C_\varphi f : f \in A^p, ||f||_{A^p} \leq 1\}$ has compact closure in $\mathcal{B}_0$ if and only if

$$\lim_{|z|\to 1}\sup\{(1-|z|^2)|(\psi C_\varphi f)'(z)| : f \in A^p, ||f||_{A^p} \leq M\} = 0,$$

for some $M > 0$. Suppose that $f \in \mathcal{B}_0$ is such that $||f||_{A^p} \leq 1$, and ψ and φ satisfies (i) and (ii). Then

$$\begin{aligned}(1-|z|^2)&|(\psi C_\varphi f)'(z)| \\ &= (1-|z|^2)|\psi'(z)f(\varphi(z)) + \psi(z)f'(\varphi(z))\varphi'(z)| \\ &\leq \Big(\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{2/p}}|\psi'(z)| + \frac{(1-|z|^2)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{1+2/p}}\Big)||f||_{A^p}.\end{aligned}$$

Thus $\sup\{(1-|z|^2)|(\psi C_\varphi f)'(z)| : f \in A^p, ||f||_{A^p} \leq 1\}$

$$\leq \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{2/p}}|\psi'(z)| + \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{1+2/p}}|\psi(z)\varphi'(z)|$$

and it follows that

$$\lim_{|z|\to 1} \sup\{(1-|z|^2)|(\psi C_\varphi f)'(z)| : f \in A^p, ||f||_{A^p} \leq 1\} = 0.$$

Hence ψC_φ maps A^p compactly into $\mathcal{B}_0$. Conversely, suppose that ψC_φ maps A^p compactly into $\mathcal{B}_0$. Using the same test function as in the proof of Theorem 3.2, we see that

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{2/p}}|\psi'(z)| = 0. \tag{4.5}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{1+2/p}}|\psi(z)\varphi'(z)| = 0. \tag{4.6}$$

Since ψC_φ maps A^p boundedly into $\mathcal{B}_0$, Theorem 4.1 implies that $\psi \in \mathcal{B}_0$ and

$$\lim_{|z|\to 1} (1-|z|^2)|\psi(z)\varphi'(z)| = 0. \tag{4.7}$$

It is easy to show that $\psi \in \mathcal{B}_0$ and (4.3) is equivalent to (i). and (4.6) and (4.7) is equivalent to (ii). □

Proposition 4.3. *Let φ and ψ be holomorphic maps on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then ψC_φ maps $\mathcal{B}$ into A^p and is compact if the following conditions are satisfied:*

(i) $\sup_{z\in\mathbb{D}} (1-|z|^2)|\psi'(z)|log(2/(1-|\varphi(z)|^2)) < \infty$;

(ii) $\sup_{z\in\mathbb{D}} ((1-|z|^2)/(1-|\varphi(z)|^2))|\psi(z)\varphi'(z)| < \infty$.

Proof. Let $f \in \mathcal{B}$, then by Theorem 1 of [16] ψC_φ maps $\mathcal{B}$ into itself and thus also into a large space A^p. Since convergence in either space implies uniform convergence on compact sets, it follows from the closed graph theorem that ψC_φ is a bounded operator from $\mathcal{B}$ into A^p. In order to see that ψC_φ is a compact operator from $\mathcal{B}$ in to A_p, choose q such that $q > p$ and factorize ψC_φ through the intermediate space A^p:

$$\mathcal{B} \xrightarrow{\widetilde{\psi C_\varphi}} A^q \xrightarrow{I} A^p.$$

Hence $\widetilde{\psi C_\varphi}$ is the composition operator from $\mathcal{B}$ to A^q and I is the injection map. We have just seen that former is a bounded composition operator while latter is compact by Lemma 3 from [2] applied to the open unit disk. Since $\psi C_\varphi = I o\ \widetilde{\psi C_\varphi}$, it is compact. □

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