Commun. Korean Math. Soc. **22** (2007), No. 3, pp. 369–371

ANOTHER METHOD FOR A KUMMER-TYPE TRANSFORMATION FOR A $_2F_2$ HYPERGEOMETRIC FUNCTION

JUNESANG CHOI AND ARJUN K. RATHIE

ABSTRACT. Very recently, by employing an addition theorem for the confluent hypergeometric function, Paris has obtained a Kummer-type transformation for a $_2F_2(x)$ hypergeometric function with general parameters in the form of a sum of $_2F_2(-x)$ functions. The aim of this note is to derive his result without using the addition theorem.

1. Introduction and results required

We start with a Kummer-type transformation for a $_2F_2(x)$ hypergeometric function with general parameters in the form of a sum of $_2F_2(-x)$ functions due to Paris [1, Eq.(3)]:

$$(1.1)\qquad {}_2F_2\left(\begin{matrix} a, & d \\ b, & c \end{matrix}\,\middle|\, x\right) = e^x \sum_{n=0}^{\infty} \frac{(c-d)_n}{(c)_n\, n!} (-x)^n\, {}_2F_2\left(\begin{matrix} b-a, & d \\ b, & c+n \end{matrix}\,\middle|\, -x\right),$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ $(n = 0, 1, 2, \ldots)$ is the Pochhammer symbol. Paris [1] also considered several interesting special cases of (1.1). This result (1.1) was established with the help of the integral representation for $_2F_2$ [3, Eq.(4.8.3.11)]:

$$(1.2)\qquad \begin{aligned} &{}_2F_2\left(\begin{matrix} a, & d \\ b, & c \end{matrix}\,\middle|\, x\right) \\ &= \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)} \int_0^1 t^{a-1}(1-t)^{b-a-1}\, {}_1F_1(d;\, c;\, xt)\, dt \end{aligned}$$

Received May 4, 2005.
2000 *Mathematics Subject Classification.* Primary 33C65; Secondary 33C60, 33C70, 33C05.
Key words and phrases. generalized hypergeometric function, Kummer's first theorem for $_1F_1$, Kummer-type transformation, addition theorem for $_1F_1$.

and

$$(1.3)\quad {}_2F_2\left(\begin{matrix} a, & d \\ b, & c \end{matrix}\,\middle|\, x\right) = \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)} \int_0^1 t^{b-a-1}(1-t)^{a-1}\,{}_1F_1\left(d\,;\,c\,;\,x-xt\right)dt$$

provided $\Re(b) > 0$ and $\Re(a) > 0$, and the addition theorem for the confluent hypergeometric function in the form [2, Eq.(2.3.5)]:

$$(1.4)\quad {}_1F_1\left(d\,;\,c\,;\,x-xt\right) = e^x \sum_{n=0}^{\infty} \frac{(c-d)_n}{(c)_n\, n!}\,(-x)^n\,{}_1F_1\left(d\,;\,c+n\,;\,-xt\right).$$

Paris [1] remarked that the special case of (1.1) when $c = d$ reduces to the well-known Kummer's first theorem [3]:

$$(1.5)\quad {}_1F_1\left(a\,;\,b\,;\,x\right) = e^x\,{}_1F_1\left(b-a\,;\,b\,;\,-x\right).$$

Here we aim at showing that (1.1) can be derived by using (1.5) instead of (1.4).

2. Derivation of (1.1)

Start with the left-hand side of (1.1) and use (1.2), it becomes

$${}_2F_2\left(\begin{matrix} d, & a \\ c, & b \end{matrix}\,\middle|\, x\right) = \frac{\Gamma(c)}{\Gamma(d)\,\Gamma(c-d)} \int_0^1 t^{d-1}(1-t)^{c-d-1}\,{}_1F_1\left(a\,;\,b\,;\,xt\right)dt,$$

which can be written as

$$(2.1)\quad {}_2F_2\left(\begin{matrix} d, & a \\ c, & b \end{matrix}\,\middle|\, x\right) = \frac{\Gamma(c)}{\Gamma(d)\,\Gamma(c-d)}\, e^x \int_0^1 t^{d-1}(1-t)^{c-d-1}\,e^{-x}\,{}_1F_1\left(a\,;\,b\,;\,xt\right)dt.$$

Using (1.5) in the integrand of the integral in (2.1), we have

$$(2.2)\quad {}_2F_2\left(\begin{matrix} d, & a \\ c, & b \end{matrix}\,\middle|\, x\right) = \frac{\Gamma(c)}{\Gamma(d)\,\Gamma(c-d)}\, e^x \int_0^1 t^{d-1}(1-t)^{c-d-1}\,e^{-x(1-t)}\,{}_1F_1\left(b-a\,;\,b\,;\,-xt\right)dt.$$

Now expand $e^{-x(1-t)}$ in (2.2) as the Maclaurin series, after a little simplification, we obtain

$$
\begin{aligned}
&{}_2F_2\left(\begin{matrix} d, & a \\ c, & b \end{matrix} \,\middle|\, x\right) \\
(2.3) \qquad &= \frac{\Gamma(c)}{\Gamma(d)\,\Gamma(c-d)}\, e^x \\
&\quad \cdot \sum_{r=0}^{\infty} \frac{(-x)^r}{r!} \int_0^1 t^{d-1}(1-t)^{c-d+r-1}\, {}_1F_1\,(b-a\,;\, b\,;\, -xt)\, dt.
\end{aligned}
$$

Substituting $1 - t = u$ in (2.3) and simplifying, we get

$$
\begin{aligned}
&{}_2F_2\left(\begin{matrix} d, & a \\ c, & b \end{matrix} \,\middle|\, x\right) \\
&= \frac{\Gamma(c)}{\Gamma(d)\,\Gamma(c-d)}\, e^x \\
&\quad \cdot \sum_{r=0}^{\infty} \frac{(-x)^r}{r!} \int_0^1 u^{c-d+r-1}(1-u)^{d-1}\, {}_1F_1\,(b-a\,;\, b\,;\, -x(1-u))\, du.
\end{aligned}
$$

Finally, applying (1.3) to the integral part in the last identity, we have

$$
{}_2F_2\left(\begin{matrix} d, & a \\ c, & b \end{matrix} \,\middle|\, x\right) = e^x \sum_{r=0}^{\infty} \frac{(-x)^r}{r!} \frac{(c-d)_r}{(c)_r}\, {}_2F_2\left(\begin{matrix} b-a, & d \\ b, & c+r \end{matrix} \,\middle|\, -x\right).
$$

This completes the proof of (1.1).

References

[1] R. B. Paris, *A Kummer-type transformation for a $_2F_2$ hypergeometric function*, J. Comput. Appl. Math. **173** (2005), 379–382.

[2] L. J. Slater, *Confluent Hypergeometric Functions*, Cambridge University Press, Cambridge, 1960.

[3] ———, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1966.

Junesang Choi
Department of Mathematics
Dongguk University
Kyongju 780-714, Korea
E-mail address: junesang@mail.dongguk.ac.kr

Arjun K. Rathie
Department of Mathematics
Govt. P. G. College
Sujangarh, Distt. Churu, Rajasthan State, India
E-mail address: akrathie@rediffmail.com