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SOME QUESTIONS ON FUZZIFICATIONS OF IDEALS IN SUBTRACTION ALGEBRAS

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ABSTRACT. In this paper, we introduce the notion of a fuzzy ideal in subtraction algebras, and give some conditions for a fuzzy set to be a fuzzy ideal in subtraction algebras. We also pose three questions on fuzzy ideals of subtraction algebras.

1. Introduction

B. M. Schein [6] considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [7] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun et al. [4] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [3], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results. In this paper, we introduce the notion of a fuzzy ideal in subtraction algebras, and give some conditions for a fuzzy set to be a fuzzy ideal in subtraction algebras. We also pose three questions on fuzzy ideals of subtraction algebras.

2. Preliminaries

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" satisfying the following conditions: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$,
(S2) $x - (x - y) = y - (y - x)$,
(S3) $(x - y) - z = (x - z) - y$.

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The last identity permits us to omit parentheses in expressions of the form $(x-y)-z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a-b=0$, where $0=a-a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X;\leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0,a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a-(a-b)$; the complement of an element $b \in [0,a]$ is $a-b$; and if $b,c \in [0,a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a-((a-b)\wedge(a-c)) \\ &= a-((a-b)-((a-b)-(a-c))). \end{aligned}$$

In a subtraction algebra, the following are true (see [4]):

(a1) $(x-y)-y=x-y$,
(a2) $x-0=x$ and $0-x=0$,
(a3) $(x-y)-x=0$,
(a4) $x-(x-y)\leq y$,
(a5) $(x-y)-(y-x)=x-y$,
(a6) $x-(x-(x-y))=x-y$,
(a7) $(x-y)-(z-y)\leq x-z$,
(a8) $x\leq y$ if and only if $x=y-w$ for some $w\in X$,
(a9) $x\leq y$ implies $x-z\leq y-z$ and $z-y\leq z-x$ for all $z\in X$,
(a10) $x,y\leq z$ implies $x-y=x\wedge(z-y)$,
(a11) $(x\wedge y)-(x\wedge z)\leq x\wedge(y-z)$.

Definition 2.1. [4] A nonempty subset A of a subtraction algebra X is called an *ideal* of X, denoted by $A \lhd X$, if it satisfies:

(b1) $a-x\in A$ for all $a\in A$ and $x\in X$,
(b2) for all $a,b\in A$, whenever $a\vee b$ exists in X then $a\vee b\in A$.

Proposition 2.2. [4] *A nonempty subset A of a subtraction algebra X is an ideal of X if and only if it satisfies:*

(b3) $0\in A$,
(b4) $(\forall x\in X)(\forall y\in A)(x-y\in A \Rightarrow x\in A)$.

Proposition 2.3. [4] *Let X be a subtraction algebra and let $x,y\in X$. If $w\in X$ is an upper bound for x and y, then the element*

$$x\vee y := w-((w-y)-x)$$

is a least upper bound for x and y.

3. Fuzzy ideals

In what follows let X be a subtraction algebra unless otherwise specified.

Definition 3.1. A fuzzy set $\mathscr{A}$ in X is called a *fuzzy ideal* of X if it satisfies:

(c1) $(\forall x,y\in X)\ (\mathscr{A}(x-y)\geq\mathscr{A}(x))$,
(c2) $(\forall x,y\in X)\ (\exists\, x\vee y \Rightarrow \mathscr{A}(x\vee y)\geq\min\{\mathscr{A}(x),\mathscr{A}(y)\})$.

Example 3.2. Consider a subtraction algebra $X = \{0, 1, 2\}$ with the following Cayley table:

$-$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Let $\mathscr{A}$ be a fuzzy set in X defined by $\mathscr{A}(0) = 0.7$, $\mathscr{A}(1) = 0.2$, and $\mathscr{A}(2) = 0.5$. Then it is easy to verify that $\mathscr{A}$ is a fuzzy ideal of X.

We give some conditions for a fuzzy set to be a fuzzy ideal in subtraction algebras.

Proposition 3.3. *If a fuzzy set $\mathscr{A}$ in X satisfies*

(c3) $(\forall\, x, a, b \in X)\ (\mathscr{A}(x - ((x - a) - b)) \geq \min\{\mathscr{A}(a), \mathscr{A}(b)\})$,

then $\mathscr{A}$ is a fuzzy ideal of X.

Proof. Let $\mathscr{A}$ be a fuzzy set in X satisfying (c3). Then

$$\begin{aligned} \mathscr{A}(x - y) &= \mathscr{A}((x - y) - (((x - y) - x) - x)) \\ &\geq \min\{\mathscr{A}(x), \mathscr{A}(x)\} = \mathscr{A}(x) \end{aligned}$$

by applying (a2), (a3) and (c3). Therefore (c1) is valid. Now, suppose $x \vee y$ exists for $x, y \in X$. Putting $w := x \vee y$, we get $x \vee y = w - ((w - x) - y)$ by Proposition 2.3. It follows from (c3) that

$$\mathscr{A}(x \vee y) = \mathscr{A}(w - ((w - x) - y)) \geq \min\{\mathscr{A}(x), \mathscr{A}(y)\},$$

and so (c2) is valid. Hence $\mathscr{A}$ is a fuzzy ideal of X. □

Question 1. Does any fuzzy ideal of a subtraction algebra satisfy the condition (c3)?

Proposition 3.4. *Every fuzzy ideal $\mathscr{A}$ of X satisfies the following inequality:*

$$(\forall\, x \in X)(\mathscr{A}(0) \geq \mathscr{A}(x)).$$

Proof. If we take $y := x$ in (c1), then $\mathscr{A}(0) = \mathscr{A}(x - x) \geq \mathscr{A}(x)$ for all $x \in X$. □

Proposition 3.5. *Let $\mathscr{A}$ be a fuzzy set in X such that*

(k1) $(\forall\, x \in X)(\mathscr{A}(0) \geq \mathscr{A}(x))$,

(k2) $(\forall\, x, y, z \in X)(\mathscr{A}(x - z) \geq \min\{\mathscr{A}((x - y) - z), \mathscr{A}(y)\})$.

Then we have the following fact that

$$(\forall\, a, x \in X)(x \leq a \Rightarrow \mathscr{A}(x) \geq \mathscr{A}(a)).$$

Proof. Let $a, x \in X$ be such that $x \leq a$. Then

$$\begin{aligned} \mathscr{A}(x) &= \mathscr{A}(x - 0) \geq \min\{\mathscr{A}((x - a) - 0), \mathscr{A}(a)\} \\ &= \min\{\mathscr{A}(0), \mathscr{A}(a)\} = \mathscr{A}(a) \end{aligned}$$

by (a2), (k1) and (k2), proving the proposition. □

Theorem 3.6. *If a fuzzy set* $\mathscr{A}$ *in* X *satisfies* (k1) *and* (k2), *then* $\mathscr{A}$ *is a fuzzy ideal of* X.

Proof. Let $\mathscr{A}$ be a fuzzy set in X satisfying (k1) and (k2), and let $x, y \in X$. Then $x - y \leq x$ by (a3). It follows from Proposition 3.5 that $\mathscr{A}(x-y) \geq \mathscr{A}(x)$, i.e., (c1) is valid. Also, we have $\mathscr{A}(x \vee y) \geq \mathscr{A}(x)$ whenever $x \vee y$ exists in X by using Proposition 3.5, and so $\mathscr{A}(x \vee y) \geq \min\{\mathscr{A}(x), \mathscr{A}(y)\}$. Thus (c2) is valid. Therefore $\mathscr{A}$ is a fuzzy ideal of X. □

Question 2. Does any fuzzy ideal of a subtraction algebra satisfy the condition (k2)?

Theorem 3.7. *If* $\mathscr{A}$ *is a fuzzy ideal of* X, *then*

$$(\forall \alpha \in [0,1])(U(\mathscr{A};\alpha) \neq \emptyset \Rightarrow U(\mathscr{A};\alpha) \lhd X).$$

Proof. Suppose that $\mathscr{A}$ is a fuzzy ideal of X and let $\alpha \in [0,1]$ be such that $U(\mathscr{A};\alpha) \neq \emptyset$. For $x \in X$ and $a \in U(\mathscr{A};\alpha)$, we have $\mathscr{A}(a) \geq \alpha$ and so $\mathscr{A}(a - x) \geq \mathscr{A}(a) \geq \alpha$ by (c1). Hence $a - x \in U(\mathscr{A};\alpha)$. Assume that $a \vee b$ exists in X for all $a, b \in U(\mathscr{A};\alpha)$. Using (c2), we have

$$\mathscr{A}(a \vee b) \geq \min\{\mathscr{A}(a), \mathscr{A}(b)\} \geq \alpha,$$

and thus $a \vee b \in U(\mathscr{A};\alpha)$. Therefore $U(\mathscr{A};\alpha) \lhd X$. □

Question 3. Does the converse of Theorem 3.7 hold?

Theorem 3.8. *For a nonzero element* w *of* X, *let* $\mathscr{A}$ *be a fuzzy set in* X *defined by*

$$\mathscr{A}(x) = \begin{cases} \alpha & \text{if } x \in (w], \\ \beta & \text{otherwise,} \end{cases}$$

where $(w] := \{x \in X \mid x \leq w\}$ *and* $\alpha, \beta \in [0,1]$ *with* $\alpha > \beta$. *Then* $\mathscr{A}$ *is a fuzzy ideal of* X.

Proof. Let $x, y \in X$. If $x \notin (w]$, then $\mathscr{A}(x) = \beta \leq \mathscr{A}(x - y)$. Assume that $x \in (w]$. Then $x - y \leq x \leq w$, and so $x - y \in (w]$. Thus $\mathscr{A}(x-y) = \alpha = \mathscr{A}(x)$. Therefore (c1) is valid. Now if $x \notin (w]$ or $y \notin (w]$, then

$$\min\{\mathscr{A}(x), \mathscr{A}(y)\} = \beta \leq \mathscr{A}(x \vee y)$$

whenever $x \vee y$ exists in X. Suppose that $x, y \in (w]$. Then $x \leq w$ and $y \leq w$, and so $x \vee y$ exists by Proposition 2.3. Since $x \vee y = w - ((w - y) - x)$, it follows from (a3) that $x \vee y \leq w$, i.e., $x \vee y \in (w]$, and hence $\mathscr{A}(x \vee y) = \alpha = \min\{\mathscr{A}(x), \mathscr{A}(y)\}$. Consequently, $\mathscr{A}$ is a fuzzy ideal of X. □

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