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POISSON BRACKET DETERMINED BY A COBRACKET

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ABSTRACT. Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra. Here we give an explicit formula for the Poisson bracket on a subalgebra of $U(\mathfrak{g})^\circ$ induced by the given cobracket δ.

Let G be a connected and simply connected Poisson Lie group. Then its Lie algebra $\mathfrak{g}$ becomes a Lie bialgebra with a cobracket δ and the universal enveloping algebra $U(\mathfrak{g})$ becomes a co-Possion algebra which is deformed to a quantized universal enveloping algebra $U_q(\mathfrak{g})$. Moreover a 'good' subalgebra of the Hopf dual $U_q(\mathfrak{g})^\circ$ is considered as a quantization of the coordinate ring $\mathcal{O}(G)$ of G. (See [1], [2], [3] and [4].)

The coordinate ring $\mathcal{O}(G)$ is a Poisson algebra and almost equal to a 'good' subalgebra of the Hopf dual $U(\mathfrak{g})^\circ$ of $U(\mathfrak{g})$. Hence the Hopf dual $U(\mathfrak{g})^\circ$ becomes a Poisson algebra and there exists a Poisson bracket $\{\cdot,\cdot\}$ on $U(\mathfrak{g})^\circ$. But we do not know immediately what $\{a, b\}$ is for any $a, b \in U(\mathfrak{g})^\circ$. In Theorem, we give an explicit formula for the Poisson bracket on a subalgebra of $U(\mathfrak{g})^\circ$ induced by the given cobracket δ, which is analogous to the Sklyanin bracket in the coordinate ring $\mathcal{O}(G)$. (See [2, 2.2 A].)

Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra over a field $\mathbf{k}$, $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and Δ the comultiplication of $U(\mathfrak{g})$. Refer to [2, 1.3] for the definition of Lie bialgebra and note that $\Delta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is a homomorphism of algebra. The cobracket δ is extended uniquely to a Δ-derivation $\overline{\delta}$ from $U(\mathfrak{g})$ into $U(\mathfrak{g}) \otimes U(\mathfrak{g})$. That is,

$$\overline{\delta} : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$$

is a $\mathbf{k}$-linear map such that $\overline{\delta}|_{\mathfrak{g}} = \delta$ and $\overline{\delta}(xy) = \overline{\delta}(x)\Delta(y) + \Delta(x)\overline{\delta}(y)$ for all $x, y \in U(\mathfrak{g})$.

Let $\mathcal{C}$ be a class of finite dimensional left $U(\mathfrak{g})$-modules such that $\mathcal{C}$ is closed under finite direct sums and finite tensor products. For $M \in \mathcal{C}$, $v \in M$ and

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$f \in M^*$, a coordinate function $c^M_{f,v} \in U(\mathfrak{g})^*$ is defined by

$$c^M_{f,v}(a) = f(av), \qquad a \in U(\mathfrak{g}).$$

Note that $c^M_{f,v}$ is an element of the Hopf dual $U(\mathfrak{g})^\circ$ since the annihilator of M has a finite codimension. It is well-known that the vector space $A(\mathcal{C})$ spanned by all coordinate functions $c^M_{f,v}, M \in \mathcal{C}, v \in M, f \in M^*$ is an associative **k**-algebra with structure

$$c^M_{f,v} + c^N_{g,w} = c^{M\oplus N}_{(f,g),(v,w)}, \qquad c^M_{f,v}c^N_{g,w} = c^{M\otimes N}_{f\otimes g,v\otimes w}.$$

Note that $A(\mathcal{C})$ is a commutative **k**-algebra since $U(\mathfrak{g})$ is cocommutative. This note is to prove that $A(\mathcal{C})$ is a Poisson algebra with Poisson bracket induced by the cobracket δ. More precisely we prove that the following theorem:

Theorem. *The commutative algebra $A(\mathcal{C})$ is a Poisson algebra with Poisson bracket*

$$\{c^M_{f,v}, c^N_{g,w}\}(x) = \langle \overline{\delta}(x), c^M_{f,v} \otimes c^N_{g,w}\rangle \tag{1}$$

for all $x \in U(\mathfrak{g})$.

Proof. Let τ be the flip on $U(\mathfrak{g}) \otimes U(\mathfrak{g})$, that is,

$$\tau : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g}), \quad x \otimes y \mapsto y \otimes x.$$

Then $\tau \circ \overline{\delta} = -\overline{\delta}$ since $\tau \circ \Delta = \Delta$ and $\tau \circ \delta = -\delta$. Hence we have immediately that

$$\begin{aligned}\{c^M_{f,v}, c^N_{g,w}\}(x) &= \langle \overline{\delta}(x), c^M_{f,v} \otimes c^N_{g,w}\rangle = \langle \tau \circ \overline{\delta}(x), c^N_{g,w} \otimes c^M_{f,v}\rangle \\ &= -\langle \overline{\delta}(x), c^N_{g,w} \otimes c^M_{f,v}\rangle = -\{c^N_{g,w}, c^M_{f,v}\}(x)\end{aligned}$$

for all $x \in U(\mathfrak{g})$. Thus we have $\{c^M_{f,v}, c^N_{g,w}\} = -\{c^N_{g,w}, c^M_{f,v}\}$.

We will prove that (1) satisfies the Leibniz rule. Set $\tau_{12} = \tau \otimes 1$ and $\tau_{23} = 1 \otimes \tau$. Since

$$\{c^M_{f,v}c^N_{g,w}, c^L_{h,u}\}(x) = \langle (\Delta \otimes 1) \circ \overline{\delta}(x), c^M_{f,v} \otimes c^N_{g,w} \otimes c^L_{h,u}\rangle$$

and

$$\begin{aligned}&(c^M_{f,v}\{c^N_{g,w}, c^L_{h,u}\} + \{c^M_{f,v}, c^L_{h,u}\}c^N_{g,w})(x) \\ &= \langle (1 \otimes \overline{\delta}) \circ \Delta(x), c^M_{f,v} \otimes c^N_{g,w} \otimes c^L_{h,u}\rangle + \langle \tau_{23} \circ (\overline{\delta} \otimes 1) \circ \Delta(x), c^M_{f,v} \otimes c^N_{g,w} \otimes c^L_{h,u}\rangle\end{aligned}$$

for $x \in U(\mathfrak{g})$, it is enough to show that

$$(\Delta \otimes 1) \circ \overline{\delta} = (1 \otimes \overline{\delta}) \circ \Delta + \tau_{23} \circ (\overline{\delta} \otimes 1) \circ \Delta. \tag{2}$$

Set

$$\delta_1 = (\Delta \otimes 1) \circ \overline{\delta}, \quad \delta_2 = (1 \otimes \overline{\delta}) \circ \Delta, \quad \delta_3 = \tau_{23} \circ (\overline{\delta} \otimes 1) \circ \Delta.$$

Observe that $\Delta^2 = (\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$ is a homomorphism of algebra and $\delta_i, i = 1, 2, 3$, are all Δ^2-derivations. Hence it is enough to show that

$\delta_1(a) = (\delta_2 + \delta_3)(a)$ for all $a \in \mathfrak{g}$ since $U(\mathfrak{g})$ is generated by $\mathfrak{g}$. Setting $\overline{\delta}(a) = \delta(a) = \sum_i a_i \otimes b_i$, we have that

$$\begin{aligned} \delta_1(a) &= (\Delta \otimes 1) \circ \overline{\delta}(a) = \textstyle\sum_i a_i \otimes 1 \otimes b_i + 1 \otimes a_i \otimes b_i \\ \delta_2(a) &= (1 \otimes \overline{\delta}) \circ \Delta(a) = \textstyle\sum_i 1 \otimes a_i \otimes b_i \\ \delta_3(a) &= \tau_{23} \circ (\overline{\delta} \otimes 1) \circ \Delta(a) = \textstyle\sum_i a_i \otimes 1 \otimes b_i \end{aligned}$$

and thus $\delta_1(a) = (\delta_2 + \delta_3)(a)$ for all $a \in \mathfrak{g}$.

Observe that

$$\begin{aligned} \{\{c^M_{f,v}, c^N_{g,w}\}, c^L_{h,u}\}(z) &= \langle (\overline{\delta} \otimes 1) \circ \overline{\delta}(z), c^M_{f,v} \otimes c^N_{g,w} \otimes c^L_{h,u} \rangle \\ \{\{c^N_{g,w}, c^L_{h,u}\}, c^M_{f,v}\}(z) &= \langle \tau_{12} \circ \tau_{23} \circ (\overline{\delta} \otimes 1) \circ \overline{\delta}(z), c^M_{f,v} \otimes c^N_{g,w} \otimes c^L_{h,u} \rangle \\ \{\{c^L_{h,u}, c^M_{f,v}\}, c^N_{g,w}\}(z) &= \langle \tau_{23} \circ \tau_{12} \circ (\overline{\delta} \otimes 1) \circ \overline{\delta}(z), c^M_{f,v} \otimes c^N_{g,w} \otimes c^L_{h,u} \rangle \end{aligned}$$

for $z \in U(\mathfrak{g})$. Hence (1) satisfies the Jacobi identity if and only if

$$(\overline{\delta} \otimes 1) \circ \overline{\delta} + \tau_{12} \circ \tau_{23} \circ (\overline{\delta} \otimes 1) \circ \overline{\delta} + \tau_{23} \circ \tau_{12} \circ (\overline{\delta} \otimes 1) \circ \overline{\delta} = 0. \tag{3}$$

Set

$$\begin{aligned} d_1 &= (\overline{\delta} \otimes 1) \circ \overline{\delta} \\ d_2 &= \tau_{12} \circ \tau_{23} \circ (\overline{\delta} \otimes 1) \circ \overline{\delta} \\ d_3 &= \tau_{23} \circ \tau_{12} \circ (\overline{\delta} \otimes 1) \circ \overline{\delta}. \end{aligned}$$

Hence (3) is true if and only if

$$(d_1 + d_2 + d_3)(z) = 0 \tag{4}$$

for all $z \in U(\mathfrak{g})$. Since $\overline{\delta}$ is a Δ-derivation, τ_{12} and τ_{23} are automorphisms and $U(\mathfrak{g})$ is cocommutative, we have $\Delta^2 = \tau_{12}\tau_{23}\Delta^2 = \tau_{23}\tau_{12}\Delta^2$ and

$$\begin{array}{cc} (\overline{\delta} \otimes 1)\Delta & (\Delta \otimes 1)\overline{\delta} \\ \tau_{12}\tau_{23}(\overline{\delta} \otimes 1)\Delta & \tau_{12}\tau_{23}(\Delta \otimes 1)\overline{\delta} \\ \tau_{23}\tau_{12}(\overline{\delta} \otimes 1)\Delta & \tau_{23}\tau_{12}(\Delta \otimes 1)\overline{\delta} \end{array}$$

are all Δ^2-derivations. Moreover, for all $a, b \in U(\mathfrak{g})$,

$$\begin{aligned} d_1(ab) &= \Delta^2(a) d_1(b) + ((\overline{\delta} \otimes 1)\Delta(a))((\Delta \otimes 1)\overline{\delta}(b)) \\ &\quad + ((\Delta \otimes 1)\overline{\delta}(a))((\overline{\delta} \otimes 1)\Delta(b)) + d_1(a)\Delta^2(b) \\ d_2(ab) &= (\tau_{12}\tau_{23}\Delta^2(a)) d_2(b) + (\tau_{12}\tau_{23}(\overline{\delta} \otimes 1)\Delta(a))(\tau_{12}\tau_{23}(\Delta \otimes 1)\overline{\delta}(b)) \\ &\quad + (\tau_{12}\tau_{23}(\Delta \otimes 1)\overline{\delta}(a))(\tau_{12}\tau_{23}(\overline{\delta} \otimes 1)\Delta(b)) + d_2(a)(\tau_{12}\tau_{23}\Delta^2(b)) \\ d_3(ab) &= (\tau_{23}\tau_{12}\Delta^2(a)) d_3(b) + (\tau_{23}\tau_{12}(\overline{\delta} \otimes 1)\Delta(a))(\tau_{23}\tau_{12}(\Delta \otimes 1)\overline{\delta}(b)) \\ &\quad + (\tau_{23}\tau_{12}(\Delta \otimes 1)\overline{\delta}(a))(\tau_{23}\tau_{12}(\overline{\delta} \otimes 1)\Delta(b)) + d_3(a)(\tau_{23}\tau_{12}\Delta^2(b)). \end{aligned}$$

Hence

$$\begin{aligned} (d_1 + d_2 + d_3)(ab) &= \Delta^2(a)(d_1 + d_2 + d_3)(b) + (d_1 + d_2 + d_3)(a)\Delta^2(b) \\ &\quad + x_1(a)y_1(b) + x_2(a)y_2(b) + x_3(a)y_3(b) \\ &\quad + y_1(a)x_1(b) + y_2(a)x_2(b) + y_3(a)x_3(b), \end{aligned} \tag{5}$$

where

$$\begin{aligned} x_1 &= (\overline{\delta}\otimes 1)\Delta & y_1 &= (\Delta\otimes 1)\overline{\delta} \\ x_2 &= \tau_{12}\tau_{23}(\overline{\delta}\otimes 1)\Delta & y_2 &= \tau_{12}\tau_{23}(\Delta\otimes 1)\overline{\delta} \\ x_3 &= \tau_{23}\tau_{12}(\overline{\delta}\otimes 1)\Delta & y_3 &= \tau_{23}\tau_{12}(\Delta\otimes 1)\overline{\delta}. \end{aligned}$$

Note that every element of $U(\mathfrak{g})$ can be written by a $\mathbf{k}$-linear combination of products $z = a_1\cdots a_n$ of elements $a_i\in\mathfrak{g}$. Set $n=\ell(z)$. We will use induction on $\ell(z)$ to prove (4). If $\ell(z)=1$ then (4) is true since $\mathfrak{g}^*$ is a Lie algebra and $\overline{\delta}(z)=\delta(z)$. Suppose that (4) is true for all elements with length less than n and let $\ell(z)=n$. Then $z=ab$ for some a,b such that $\ell(a)=n-1$ and $\ell(b)=1$. Thus $(d_1+d_2+d_3)(a)=0$ and $(d_1+d_2+d_3)(b)=0$ by the induction hypothesis and it is enough to show that

$$x_1(a)y_1(b)+x_2(a)y_2(b)+x_3(a)y_3(b)=0 \tag{6}$$

and

$$y_1(a)x_1(b)+y_2(a)x_2(b)+y_3(a)x_3(b)=0 \tag{7}$$

by (5).

Suppose $\ell(a)=1$, $\ell(b)=1$ and let

$$\overline{\delta}(a)=\delta(a)=\sum a_1\otimes a_2,\quad \overline{\delta}(b)=\delta(b)=\sum b_1\otimes b_2.$$

Then we have

$$x_1(a)\Delta^2(d)y_1(b)+x_2(a)\Delta^2(d)y_2(b)+x_3(a)\Delta^2(d)y_3(b)=0 \tag{8}$$

for all $d\in U(\mathfrak{g})$ since

$$\begin{aligned} &x_1(a)\Delta^2(d)y_1(b)+x_2(a)\Delta^2(d)y_2(b)+x_3(a)\Delta^2(d)y_3(b)\\ =\ &(\sum a_1\otimes a_2\otimes 1)\Delta^2(d)(\sum b_1\otimes 1\otimes b_2+\sum 1\otimes b_1\otimes b_2)\\ &+(\sum a_1\otimes 1\otimes a_2+\sum 1\otimes a_1\otimes a_2)\Delta^2(d)(\sum b_1\otimes b_2\otimes 1)\\ &+(\sum 1\otimes a_1\otimes a_2)\Delta^2(d)(\sum b_2\otimes b_1\otimes 1+\sum b_2\otimes 1\otimes b_1)\\ &+(\sum a_2\otimes a_1\otimes 1+\sum a_2\otimes 1\otimes a_1)\Delta^2(d)(\sum 1\otimes b_1\otimes b_2)\\ &+(\sum a_2\otimes 1\otimes a_1)\Delta^2(d)(\sum 1\otimes b_2\otimes b_1+\sum b_1\otimes b_2\otimes 1)\\ &+(\sum a_2\otimes 1\otimes a_1+\sum a_1\otimes a_2\otimes 1)\Delta^2(d)(\sum b_2\otimes 1\otimes a_1)\\ =\ &0 \end{aligned}$$

by the skew symmetry, $\tau\delta=-\delta$. Hence (6) is true for the case $\ell(a)=1$ and $\ell(b)=1$ and for the case $\ell(a)=0$ and $\ell(b)=1$ since $\Delta^2(1)=1\otimes 1\otimes 1$ and $\overline{\delta}(1)=0$.

Suppose that $\ell(a) > 1$, $\ell(b) = 1$ and that $a = cd$ for some c, d with $\ell(c) = 1$. Then

$$\begin{aligned}
& x_1(a)y_1(b) + x_2(a)y_2(b) + x_3(a)y_3(b) \\
= & [\Delta^2(c)x_1(d) + x_1(c)\Delta^2(d)]y_1(b) \\
& \qquad + [\Delta^2(c)x_2(d) + x_2(c)\Delta^2(d)]y_2(b) \\
& + [\Delta^2(c)x_3(d) + x_3(c)\Delta^2(d)]y_3(b) \\
= & \Delta^2(c)[x_1(d)y_1(b) + x_2(d)y_2(b) + x_3(d)y_3(b)] \\
& + [x_1(c)\Delta^2(d)y_1(b) + x_2(c)\Delta^2(d)y_2(b) + x_3(c)\Delta^2(d)y_3(b)] \\
= & 0
\end{aligned}$$

by (8) and the induction hypothesis. Therefore (6) is true for all a and b with arbitrary $\ell(a)$ and $\ell(b) = 1$, as claimed. The equation (7) is proved as in (6). Therefore (1) satisfies the Jacobi identity. It completes the proof of the theorem. □

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