Commun. Korean Math. Soc. **22** (2007), No. 3, pp. 331–351

ARITHMETIC OF INFINITE PRODUCTS AND ROGERS-RAMANUJAN CONTINUED FRACTIONS

Daeyeoul Kim*, Ja Kyung Koo**, and Yilmaz Simsek***

Abstract. Let k be an imaginary quadratic field, $\mathfrak{h}$ the complex upper half plane, and let $\tau \in \mathfrak{h} \cap k$, $q = e^{\pi i \tau}$. We find a lot of algebraic properties derived from theta functions, and by using this we explore some new algebraic numbers from Rogers-Ramanujan continued fractions.

1. Introduction

Let

$$f(a,b) = 1 + \sum_{m=1}^{\infty} (ab)^{m(m-1)/2}(a^m + b^m) = \sum_{m=-\infty}^{\infty} a^{m(m+1)/2} b^{m(m-1)/2}$$

be the theta function for $a, b \in \mathbb{C}$ with $|ab| < 1$, which was defined by Ramanujan ([5], [20]). If we set $a = qe^{2iz}$, $b = qe^{-2iz}$ for $z \in \mathbb{C}$ and $\tau \in \mathfrak{h}$ then $f(a,b) = \theta_3(z,\tau)$, where $\theta_3(z,\tau)$ is one of the classical theta series in its standard notation ([30, p.464]). Berndt proved many interesting formulas for the above theta functions in [4], [5], [6], of which we list the following properties for later use:

([5, p.35]) $$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty$$

([5, p.46]) $$f(a,b) + f(-a,-b) = 2f(a^3b, ab^3),$$

where $(a;b)_\infty = \prod_{m=0}^{\infty}(1 - ab^m)$. If $ab = cd$, then

$$f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = 2f(ac,bd)f(ad,bc) \tag{1.0}$$

$$f(a,b)f(c,d) - f(-a,-b)f(-c,-d) \tag{1.0*}$$

Received November 13, 2006.

2000 *Mathematics Subject Classification.* 11Jxx, 11R04, 11F11.

Key words and phrases. transcendental number, algebraic number, theta series, Rogers-Ramanujan continued fraction.

*Research supported in part by the Basic Research Program of the Korea Science & Engineering Foundation (R08-2003-10439-0).

**This work was supported in part by the second stage of BK 21 Project.

***He was supported by the Scientific Research Project Administration Akdeniz University.

([5, p.45]) $$= 2af(\frac{b}{c}, \frac{c}{b}abcd)f(\frac{b}{d}, \frac{d}{b}abcd).$$

Using Berndt's idea, we explore some algebraic properties on the values of those theta functions(Lemma 2.2, Theorem 2.4).

In [17], we found that $q^a \prod_{m=1}^{\infty}(1-q^{nm-t})(1-q^{nm-(n-t)})$ are algebraic numbers, where (a,n,t) are the following:

$$(-\frac{1}{12},2,1),(-\frac{1}{12},3,1),(-\frac{1}{24},4,1),(\frac{1}{60},5,1),(-\frac{11}{60},5,2),$$
$$(\frac{1}{12},6,1),(\frac{13}{84},7,1),(-\frac{11}{84},7,2),(-\frac{23}{84},7,3),$$
$$(\frac{11}{48},8,1),(-\frac{13}{48},8,3),(\frac{11}{36},9,1),(-\frac{1}{36},9,2),(-\frac{13}{36},9,4),$$
$$(\frac{23}{60},10,1),(-\frac{13}{60},10,3),(\frac{13}{24},12,1),(-\frac{11}{24},12,5),(\frac{59}{84},14,1),$$
$$(-\frac{1}{84},14,3),(-\frac{37}{84},14,5),(\frac{37}{36},18,1),(-\frac{11}{36},18,5),(-\frac{23}{36},18,7).$$

In the examples 2.6∼2.9 of Section 2, we further show that $q^a \prod_{m=1}^{\infty}(1-q^{nm-t})(1-q^{nm-(n-t)})$ are algebraic numbers for the cases (a,n,t):

$$(\frac{47}{60},15,1),(\frac{23}{60},15,2),(-\frac{13}{60},15,4),(-\frac{37}{60},15,7),$$
$$(\frac{107}{84},21,1),(\frac{71}{84},21,2),(\frac{1}{84},21,4),(-\frac{13}{84},21,5),$$
$$(-\frac{61}{84},21,8),(-\frac{73}{84},21,10),(\frac{121}{60},30,1),(-\frac{11}{60},30,7),$$
$$(-\frac{59}{60},30,11),(-\frac{71}{60},30,13),(\frac{253}{84},42,1),(\frac{109}{84},42,5),$$
$$(-\frac{47}{84},42,11),(-\frac{83}{84},42,13),(-\frac{131}{84},42,17),(-\frac{143}{84},42,19).$$

Ramanujan's lost notebook [21] has many important results on the Rogers-Ramanujan continued fraction, and some of these have been proved by several people such as Andrews [1], [2], Berndt and Chan [9], Berndt, Chan and Zhang [10], Huang [12], Kang [14], [15], and Son [28], [29]. In Section 3, we find some algebraic numbers from Rogers-Ramanujan continued fractions and we also consider the algebraic properties for certain infinite products(Theorem 3.1 and Corollary 3.2).

On the other hand, Borwein and Zhou showed that if a is an integer greater than one, r and s are positive rational numbers such that $1+a^m r - a^{2m}s \neq 0$ for all integers $m \geq 0$, then $\sum_{m=0}^{\infty} \frac{1}{1+a^m r - a^{2m}s}$ is irrational and is not a Liouville number ([11]). When $\alpha = \frac{1-\sqrt{5}}{2}$ and $\tau \in \mathfrak{h} \cap k$, if we replace $\sum$, a by $\prod$, q and set $s=1$ in the above, we can show that $\prod_{m=0}^{\infty} \frac{1}{1+q^m\alpha + q^{2m}}$ is a transcendental number, and hence $\prod_{m=0}^{\infty} \dfrac{1}{q^{\frac{1}{12}}(1+q^m\alpha+q^{2m})}$ is an algebraic number ([Theorem 3.1, (b)]).

2. Properties of theta functions

Let $\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$ and $\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}$.

According to Berndt [5, p.49] there are two identities

$$\phi(q) = \phi(q^{n^2}) + \sum_{r=1}^{n-1} q^{r^2} f(q^{n(n-2r)}, q^{n(n+2r)}) \tag{2.1}$$

and

$$\psi(q) = \frac{1}{2}\sum_{r=0}^{n-1} q^{r(r-1)/2} f(q^{n(n-2r+1)/2}, q^{n(n+2r-1)/2}), \tag{2.2}$$

where n is a positive integer.

Now, let $\alpha = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with b mod d and $|\alpha|$ be the determinant of α, and set

$$\phi_\alpha(\tau) := |\alpha|^{12}\frac{\Delta\left(\alpha\binom{\tau}{1}\right)}{\Delta\left(\binom{\tau}{1}\right)} = |\alpha|^{12} d^{-12}\frac{\Delta(\alpha\tau)}{\Delta(\tau)}$$

where $\Delta(\tau) = (2\pi)^{12} q^2 \prod_{m=1}^{\infty}(1-q^{2m})^{24}$.

Then we recall from ([19]) that

(2.3) for any $\tau \in k \cap \mathfrak{h}$ the value $\phi_\alpha(\tau)$ is an algebraic integer, which divides $|\alpha|^{12}$.

Lemma 2.0. *Let $a, n \in \mathbb{Z}^+$, $1 \le a \le n-1$ with $n \ge 2$. Then we have* $\prod_{m=0}^{\infty}(1-q^{n+a}q^{nm})(1-q^{-a}q^{nm}) = -q^{-a}\prod_{m=1}^{\infty}(1-q^{nm-a})(1-q^{nm-(n-a)})$.

Proof. It is immediate. □

Proposition 2.1 ([16]). *For $\tau \in k \cap \mathfrak{h}$,*

$$\sqrt{2}q^{\frac{1}{24}}\prod_{m=1}^{\infty}(1+q^m), \quad q^{\frac{-1}{24}}\prod_{m=1}^{\infty}(1-q^{2m-1}),$$

$$q^{\frac{-1}{24}}\prod_{m=1}^{\infty}(1+q^{2m-1}) \ \ \textit{and} \ \ \frac{1}{q^{\frac{1}{24}}\prod_{m=1}^{\infty}(1+q^m)}$$

are algebraic integers.

Lemma 2.2. *Let $\tau \in k \cap \mathfrak{h}$, $n, r \in \mathbb{Z}^+$ with $n \ge 2$.*

(a) *If n is an odd positive integer then*

$$\sum_{r=1}^{\frac{n-1}{2}}\left((-1)^{r^2} q^{\frac{r^2}{n}-\frac{n}{12}}\prod_{m=1}^{\infty}(1-q^{2nm-(n-2r)})(1-q^{2nm-(n+2r)})\right)$$

and

$$\prod_{r=1}^{\frac{n-1}{2}}\left(q^{\frac{r^2}{n}-\frac{n}{12}}\prod_{m=1}^{\infty}(1-q^{2nm-(n-2r)})(1-q^{2nm-(n+2r)})\right)$$

are algebraic numbers.

(b) *If n is an integer then*

$$\sum_{r=0}^{[\frac{n}{2}+\frac{1}{4}]} q^{\frac{r(2r-1)}{n}+\frac{1}{8n}-\frac{n}{6}}\left(\prod_{m=1}^{\infty}(1+q^{4nm-(2n-4r+1)})(1+q^{4nm-(2n+4r-1)})\right)$$
$$+\sum_{r=[\frac{n}{2}+\frac{1}{4}]+1}^{n-1}(-1)^r q^{\frac{r(2r-1)}{n}+2n-4r+1+\frac{1}{8n}-\frac{n}{6}}$$
$$\cdot\left(\prod_{m=1}^{\infty}(1+q^{4nm-(-2n+4r-1)})(1+q^{4nm-(6n-4r+1)})\right)$$

is an algebraic number where $[a]$ denote the greatest integer less than or equal to a.

(c) *If $n = 6g+1$ or $6g-1$ then*

$$\sum_{k=1}^{\frac{n-1}{2}}(-1)^{g+k}q^{\frac{k(3k-n)}{2n}}\prod_{m=1}^{\infty}\left(\frac{(1-q^{nm-2k})(1-q^{nm-(n-2k)})}{(1-q^{nm-k})(1-q^{nm-(n-k)})}\right)$$

is also an algebraic number.

Proof. (a) Since n is an odd integer, so are $n(n-2r)$ and $n(n+2r)$. If we replace q by $-q$ in (2.1), we can deduce the formula

$$\phi(-q)=\phi(-q^{n^2})+\sum_{r=1}^{n-1}(-1)^{r^2}q^{r^2}f(-q^{n(n-2r)},-q^{n(n+2r)}),$$

and replacing q^n by q in (2.1) attain the formula

$$\phi(-q^{\frac{1}{n}})=\phi(-q^n)+\sum_{r=1}^{n-1}(-1)^{r^2}q^{\frac{r^2}{n}}f(-q^{(n-2r)},-q^{(n+2r)}). \tag{2.4}$$

Thus we derive by Lemma 2.0 and (2.4) that

$$\prod_{m=1}^{\infty}(1-q^{\frac{1}{n}(2m-1)})^2(1-q^{\frac{1}{n}(2m)})-\prod_{m=1}^{\infty}(1-q^{n(2m-1)})^2(1-q^{n(2m)})$$
$$=\sum_{r=1}^{\frac{n-1}{2}}(-1)^{r^2}q^{\frac{r^2}{n}}\prod_{m=1}^{\infty}(1-q^{2nm-(n-2r)})(1-q^{2nm-(n+2r)})(1-q^{2nm})$$

$$+\sum_{r=\frac{n+1}{2}}^{n-1}(-1)^{r^2+1}q^{\frac{r^2}{2}}q^{n-2r}\prod_{m=1}^{\infty}(1-q^{2nm-(n-2r)-2n})$$
$$\cdot(1-q^{2nm-(n+2r)+2n})(1-q^{2nm}).$$

If we set $\frac{r^2}{n}=\frac{(n+r)^2}{n}+n-2r$, then we have

$$\prod_{m=1}^{\infty}(1-q^{\frac{1}{n}(2m-1)})^2(1-q^{\frac{1}{n}(2m)})-\prod_{m=1}^{\infty}(1-q^{n(2m-1)})^2(1-q^{n(2m)})$$
$$=2\sum_{r=1}^{\frac{n-1}{2}}(-1)^{r^2}q^{\frac{r^2}{n}}\prod_{m=1}^{\infty}(1-q^{2nm-(n-2r)})(1-q^{2nm-(n+2r)})(1-q^{2nm}).$$

Hence, multiplying $\dfrac{1}{q^{\frac{n}{12}}\prod_{m=1}^{\infty}(1-q^{2nm})}$ on both sides in the above, we see that

$$\left(q^{-\frac{1}{12n}}\prod_{m=1}^{\infty}(1-q^{\frac{1}{n}(2m-1)})^2\right)\left(\frac{q^{\frac{1}{12n}}(1-q^{\frac{1}{n}(2m)})}{q^{\frac{n}{12}}\prod_{m=1}^{\infty}(1-q^{2nm})}\right)$$
$$(2.5)\qquad -\left(q^{-\frac{n}{12}}\prod_{m=1}^{\infty}(1-q^{n(2m-1)})^2\right)$$
$$=2\sum_{r=1}^{\frac{n-1}{2}}(-1)^{r^2}q^{\frac{r^2}{n}-\frac{n}{12}}\prod_{m=1}^{\infty}(1-q^{2nm-(n-2r)})(1-q^{2nm-(n+2r)})$$

is an algebraic number by (2.3) and Proposition 2.1.

On the other hand, we know that $\sum_{r=1}^{\frac{n-1}{2}}(\frac{r^2}{n}-\frac{n}{12})=\frac{n-1}{24}$ and hence

$$(2.6)\qquad \prod_{r=1}^{\frac{n-1}{2}}\left[q^{\frac{r^2}{n}-\frac{n}{12}}\prod_{m=1}^{\infty}(1-q^{2nm-(n-2r)})(1-q^{2nm-(n+2r)})\right]$$
$$=q^{\frac{n-1}{24}}\prod_{m=1}^{\infty}\left(\frac{1-q^{2m-1}}{1-q^{2mn-n}}\right).$$

By Proposition 2.1, (2.5) and (2.6), we get the result.

(b) It follows from (2.2) when we replace q^n by q because the argument is quite similar to those adopted in (a).

(c) Let $n=6g+1$ be a positive integer. We see from [5, p.274] that

$$\prod_{m=1}^{\infty}\frac{(1-q^{\frac{m}{n}})}{(1-q^{nm})}=(-1)^g q^{\frac{n^2-1}{24n}}+\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k+g}$$
$$\cdot q^{(k-g)(3k-3g-1)/2n}\prod_{m=1}^{\infty}\frac{(1-q^{nm-2k})(1-q^{nm-(n-2k)})}{(1-q^{nm-k})(1-q^{nm-(n-k)})}.$$

And, multiplying both sides by $q^{-(\frac{n^2-1}{24n})}$ and using (2.3), we conclude that

$$\sum_{k=1}^{\frac{n-1}{2}}(-1)^{g+k}q^{\frac{k(3k-n)}{2n}}\prod_{m=1}^{\infty}\left(\frac{(1-q^{nm-2k})(1-q^{nm-(n-2k)})}{(1-q^{nm-k})(1-q^{nm-(n-k)})}\right)$$

is an algebraic number.

As for the case $n = 6g-1$, we can get it in like manner. □

Example 2.3. Put $n = 5$ in Lemma 2.2 (a). Then we see that

$$S := -q^{-\frac{13}{60}}\prod_{m=1}^{\infty}(1-q^{10m-3})(1-q^{10m-7}) + q^{\frac{23}{60}}\prod_{m=1}^{\infty}(1-q^{10m-1})(1-q^{10m-9})$$

and

$$T :=\left(-q^{-\frac{13}{60}}\prod_{m=1}^{\infty}(1-q^{10m-3})(1-q^{10m-7})\right)\left(q^{\frac{23}{60}}\prod_{m=1}^{\infty}(1-q^{10m-1})(1-q^{10m-9})\right)$$

are algebraic numbers. Thus we can find an algebraic equation $g(X) = X^2 - SX + T \in \overline{\mathbb{Q}}[X]$ satisfying $g(q^{\frac{23}{60}}\prod_{m=1}^{\infty}(1-q^{10m-1})(1-q^{10m-9})) = 0$. Hence,

$$q^{-\frac{13}{60}}\prod_{m=1}^{\infty}(1-q^{10m-3})(1-q^{10m-7})$$

and

$$q^{\frac{23}{60}}\prod_{m=1}^{\infty}(1-q^{10m-1})(1-q^{10m-9})$$

are algebraic numbers, too as in [17].

Theorem 2.4.

(a) *Let n, u, w be positive integers with $1 \le u, w \le 2n-1$, u even and w odd. If $q^{t_u}\prod_{m=1}^{\infty}(1\pm q^{2nm-u})(1\pm q^{2nm-(2n-u)})$ and $q^{t_w}\prod_{m=1}^{\infty}(1\pm q^{2nm-w})(1\pm q^{2nm-(2n-w)})$ are algebraic numbers for $t_u, t_w \in \overline{\mathbb{Q}}$, then $q^{t_u+t_w-\frac{n}{6}}\prod_{m=1}^{\infty}(1-q^{4nm-(u+w)})(1-q^{4nm-(4n-(u+w))})(1-q^{4nm-(2n+u-w)})(1-q^{4nm-(2n-u+w)})$ is an algebraic number with double signs in the same order.*

(b) *Let n, u, w be odd positive integers with $1 \le u, w \le n-1$. If $q^{t_u}\prod_{m=1}^{\infty}(1\pm q^{nm-u})(1\pm q^{nm-(n-u)})$, $q^{t_w}\prod_{m=1}^{\infty}(1\pm q^{nm-w})(1\pm q^{nm-(n-w)})$ and $q^{t_{\frac{u+v}{2}}}\prod_{m=1}^{\infty}(1-q^{nm-(\frac{u+v}{2})})(1-q^{nm-(n-\frac{u+v}{2})})$ are algebraic numbers, then $q^{t_u+t_w-\frac{n}{12}-2t_{\frac{u+v}{2}}}\prod_{m=1}^{\infty}(1-q^{2nm-(n+u-w)})(1-q^{2nm-(n-u+w)})$ is an algebraic number with double signs in the same order.*

(c) *Let n, u be odd positive integers with $1 \le u \le n-1$. If $q^{t_u}\prod_{m=1}^{\infty}(1-q^{nm-u})(1-q^{nm-(n-u)})$ and $q^{t_{\frac{n-u}{2}}}\prod_{m=1}^{\infty}(1-q^{nm-u})(1-q^{nm-(n-\frac{n-u}{2})})$*

are algebraic numbers, then

$$q^{t_u-2(t_{\frac{n-u}{2}})}\prod_{m=1}^{\infty}(1-q^{2nm-u})(1-q^{2nm-(2n-u)})$$

is an algebraic number.

(d) *If* $q^{-\frac{1}{2}+\frac{1}{4n}+\frac{n}{6}}\prod_{m=1}^{\infty}(1-q^{2nm-1})(1-q^{2nm-(2n-1)})$ *and* $q^{-\frac{1}{2}+\frac{1}{6n}+\frac{n}{4}}\prod_{m=1}^{\infty}(1-q^{3nm-1})(1-q^{3nm-(3n-1)})$ *are algebraic numbers, then* $q^{-\frac{1}{2}+\frac{1}{12n}+\frac{n}{2}}\prod_{m=1}^{\infty}(1-q^{6nm-1})(1-q^{6nm-(6n-1)})$ *is an algebraic number.*

(e) *Let n be an even positive integer. If* $q^{\frac{2}{n}-\frac{n}{24}}\prod_{m=1}^{\infty}(1\pm q^{nm-(\frac{n-4}{2})})(1\pm q^{nm-(\frac{n+4}{2})})$, $q^{-\frac{1}{2n}-\frac{n}{24}}\prod_{m=1}^{\infty}(1\pm q^{nm-(\frac{n-2}{2})})(1\pm q^{nm-(\frac{n+2}{2})})$ *and* $q^{-\frac{1}{2}+\frac{1}{2n}+\frac{n}{12}}\prod_{m=1}^{\infty}(1-q^{nm-1})(1-q^{nm-(n-1)})$ *are algebraic numbers, then* $q^{-\frac{1}{2}+\frac{1}{4n}+\frac{n}{6}}\prod_{m=1}^{\infty}(1-q^{2nm-1})(1-q^{2nm-(2n-1)})$ *and* $q^{-\frac{3}{2}+\frac{9}{4n}+\frac{n}{6}}\prod_{m=1}^{\infty}(1-q^{2nm-3})(1-q^{2nm-(2n-3)})$ *are algebraic numbers with double signs in the same order.*

Proof. (a) By (1.0),

$$f(a,b)f(c,d)+f(-a,-b)f(-c,-d)=2f(ac,bd)f(ad,bc). \tag{2.7}$$

Let $a=q^u$, $b=q^{2n-u}$, $c=-q^w$, $d=-q^{2n-w}$,

$$A:=\prod_{m=1}^{\infty}(1+q^{2nm-u})(1+q^{2nm-(2n-u)})(1-q^{2nm-w})(1-q^{2nm-(2n-w)}),$$

$$B:=\prod_{m=1}^{\infty}(1-q^{2nm-u})(1-q^{2nm-(2n-u)})(1+q^{2nm-w})(1+q^{2nm-(2n-w)}),$$

and

$$C:=2\prod_{m=1}^{\infty}(1-q^{4nm-(u+w)})(1-q^{4nm-(4n-(u+w))})\cdot(1-q^{4nm-(2n+u-w)})(1-q^{4nm-(2n-u+w)}).$$

If we multiply through (2.7) by $\dfrac{q^{t_u+t_w-\frac{n}{6}}}{\prod_{m=1}^{\infty}(1-q^{4nm})^2}$, we get by Proposition 2.1 and the assumption that

$$\frac{q^{t_u+t_w-\frac{n}{6}}A}{\prod_{m=1}^{\infty}(1+q^{2nm})^2}+\frac{q^{t_u+t_w-\frac{n}{6}}B}{\prod_{m=1}^{\infty}(1+q^{2nm})^2}=q^{t_u+t_w-\frac{n}{6}}C$$

is an algebraic number.

(b) It goes over the same argument as in (a).

(c) It follows from the fact that

$$\prod_{m=1}^{\infty}(1-q^{2nm-u})(1-q^{2nm-(2n-u)}) = \prod_{m=1}^{\infty}\frac{(1-q^{nm-u})(1-q^{nm-(n-u)})}{(1-q^{2nm-(n-u)})(1-q^{2nm-(n+u)})}. \tag{2.8}$$

(d) For an integer $N \geq 6$ and an integer $r \not\equiv 0 \pmod N$, let $X_r(\tau)$ be the function defined by

$$X_r(\tau) = X_r(\tau, N) = e^{-2\pi i\tau(\frac{(r-1)(N-1)}{4N})}\prod_{s=0}^{N-1}\frac{K_{r,s}(\tau)}{K_{1,s}(\tau)},$$

where $K_{u,v}(\tau)$ are Klein forms of level N ([18]).

Ishida found in [13] that some polynomial $F(X,Y) \in \mathbb{Q}[X,Y]$ satisfies an affine singular model $F(X_2, X_3) = 0$ for the modular curve $X(N)$. From this and the fact that $q^{3(-\frac{1}{2}+\frac{1}{4n}+\frac{n}{6})}\prod_{m=1}^{\infty}(1-q^{6nm-3})(1-q^{26nm-(6n-3)})$ and $q^{2(-\frac{1}{2}+\frac{1}{6n}+\frac{n}{4})}\prod_{m=1}^{\infty}(1-q^{6nm-2})(1-q^{6nm-(6n-2)})$ are algebraic numbers, we can find an algebraic equation for

$$q^{-\frac{1}{2}+\frac{1}{12n}+\frac{n}{2}}\prod_{m=1}^{\infty}(1-q^{6nm-1})(1-q^{6nm-(6n-1)}).$$

Thus we get (d).

(e) Set $a = q^{\frac{n-4}{2}}$, $b = q^{\frac{n+4}{2}}$, $c = -q^{\frac{n-2}{2}}$ and $d = -q^{\frac{n+2}{2}}$ in (1.0*). Then we obtain that

$$\begin{aligned}
&\prod_{m=1}^{\infty}\left(q^{\frac{2}{n}-\frac{n}{24}}(1+q^{nm-(\frac{n-4}{2})})(1+q^{nm-(\frac{n+4}{2})})\right)\\
&\quad\times\left(q^{\frac{1}{2n}-\frac{n}{24}}(1-q^{nm-(\frac{n-2}{2})})(1-q^{nm-(\frac{n+2}{2})})\right)\left(\frac{q^{-\frac{n}{12}}}{(1+q^nm)^2}\right)\\
&\quad-\prod_{m=1}^{\infty}\left(q^{\frac{2}{n}-\frac{n}{24}}(1-q^{nm-(\frac{n-4}{2})})(1-q^{nm-(\frac{n+4}{2})})\right)\\
&\quad\times\left(q^{\frac{1}{2n}-\frac{n}{24}}(1+q^{nm-(\frac{n-2}{2})})(1+q^{nm-(\frac{n+2}{2})})\right)\left(\frac{q^{-\frac{n}{12}}}{(1+q^nm)^2}\right)\\
&= 2\prod_{m=1}^{\infty}\left(q^{-\frac{1}{2}+\frac{1}{4n}+\frac{n}{6}}(1-q^{2nm-1})(1-q^{2nm-(n-1)})\right)\\
&\quad\times\left(q^{-\frac{3}{2}+\frac{9}{4n}+\frac{n}{6}}(1-q^{2nm-3})(1-q^{2nm-(n-3)})\right) \text{ is an algebraic number.}
\end{aligned}$$

Combining this fact and the Ishida's equation for $F(X_2, X_3) = 0$, we readily get (e). □

Example 2.5. In [17], we showed that $q^{\frac{13}{84}}\prod_{m=1}^{\infty}(1 \pm q^{7m-1})(1 \pm q^{7m-6})$, $q^{\frac{-11}{84}}\prod_{m=1}^{\infty}(1 \pm q^{7m-2})(1 \pm q^{7m-5})$, $q^{\frac{-23}{84}}\prod_{m=1}^{\infty}(1 \pm q^{7m-3})(1 \pm q^{7m-4})$, $q^{\frac{26}{84}}\prod_{m=1}^{\infty}(1 \pm q^{14m-2})(1 \pm q^{14m-12})$ and $q^{-\frac{1}{84}}\prod_{m=1}^{\infty}(1 \pm q^{14m-3})(1 \pm q^{14m-11})$ are algebraic numbers with double signs in the same order. By Theorem 2.4, we further conclude that $q^{-\frac{73}{84}}\prod_{m=1}^{\infty}(1-q^{28m-5})(1-q^{28m-13})(1-q^{28m-15})(1-q^{28m-23})$ and $q^{-\frac{37}{84}}\prod_{m=1}^{\infty}(1-q^{14m-5})(1-q^{14m-9})$ are algebraic numbers.

Example 2.6. Let us consider some algebraic numbers from the case $n = 15$. In [17], we proved that

$$q^{\frac{1}{60}}\prod_{m=1}^{\infty}(1-q^{5m-1})(1-q^{5m-4}) \quad \text{and} \quad q^{\frac{-11}{60}}\prod_{m=1}^{\infty}(1-q^{5m-2})(1-q^{5m-3}) \tag{2.9}$$

are algebraic numbers.

Meanwhile, it can be checked by Proposition 2.1 that

$$\begin{aligned}
&\left(q^{\frac{47}{60}}\prod_{m=1}^{\infty}(1+q^{15m-1})(1+q^{15m-14})\right)\\
&\cdot\left(q^{\frac{23}{60}}\prod_{m=1}^{\infty}(1+q^{15m-2})(1+q^{15m-13})\right)\\
&\cdot\left(q^{-\frac{13}{60}}\prod_{m=1}^{\infty}(1+q^{15m-4})(1+q^{15m-11})\right)\\
&\cdot\left(q^{-\frac{37}{60}}\prod_{m=1}^{\infty}(1+q^{15m-7})(1+q^{15m-8})\right)\\
&=\prod_{m=1}^{\infty}q^{\frac{1}{3}}\left(\frac{1+q^{m}}{1+q^{15m}}\right)\left(\frac{1+q^{15m}}{1+q^{3m}}\right)\left(\frac{1+q^{15m}}{1+q^{5m}}\right)
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
&\left(q^{\frac{47}{60}}\prod_{m=1}^{\infty}(1+q^{15m-1})(1+q^{15m-14})\right)\\
&\cdot\left(q^{-\frac{13}{60}}\prod_{m=1}^{\infty}(1+q^{15m-4})(1+q^{15m-11})\right)\\
&=\prod_{m=1}^{\infty}\frac{q^{\frac{1}{60}}(1+q^{5m-1})(1+q^{5m-4})}{q^{-\frac{33}{60}}(1+q^{15m-6})(1+q^{15m-9})}
\end{aligned} \tag{2.11}$$

are algebraic numbers.

And, Rogers ([3], [22, p.332]) derived the identities

$$\prod_{m=1}^{\infty}(1+q^{15m-7})(1+q^{15m-8})(1-q^{15m}) - q\prod_{m=1}^{\infty}(1+q^{15m-2})(1+q^{15m-13})(1-q^{15m}) = \prod_{m=1}^{\infty}\frac{(1-q^{10m-4})(1-q^{10m-6})(1-q^{10m})}{1+q^m} \tag{2.12}$$

and

$$\prod_{m=1}^{\infty}(1+q^{15m-4})(1+q^{15m-11})(1-q^{15m}) - q\prod_{m=1}^{\infty}(1+q^{15m-1})(1+q^{15m-14})(1-q^{15m}) = \prod_{m=1}^{\infty}\frac{(1-q^{10m-2})(1-q^{10m-8})(1-q^{10m})}{1+q^m}. \tag{2.13}$$

It then follows from (2.3), (2.9), (2.12) and (2.13) that

$$q^{-\frac{37}{60}}\prod_{m=1}^{\infty}(1+q^{15m-7})(1+q^{15m-8}) - q^{\frac{23}{60}}\prod_{m=1}^{\infty}(1+q^{15m-2})(1+q^{15m-13}) = \prod_{m=1}^{\infty}\left(q^{-\frac{22}{60}}(1-q^{10m-4})(1-q^{10m-6})\right)\left(\frac{q^{-\frac{1}{24}}}{(1+q^m)}\right)\frac{q^{\frac{10}{24}}(1-q^{10m})}{q^{\frac{15}{24}}(1-q^{15m})} \tag{2.14}$$

and

$$q^{-\frac{13}{60}}\prod_{m=1}^{\infty}(1+q^{15m-4})(1+q^{15m-11}) - q^{\frac{47}{60}}\prod_{m=1}^{\infty}(1+q^{15m-1})(1+q^{15m-14}) = \prod_{m=1}^{\infty}\left(q^{\frac{2}{60}}(1-q^{10m-2})(1-q^{10m-8})\right)\left(\frac{q^{-\frac{1}{24}}}{(1+q^m)}\right)\frac{q^{\frac{10}{24}}(1-q^{10m})}{q^{\frac{15}{24}}(1-q^{15m})} \tag{2.15}$$

are algebraic numbers. Hence, there exists a polynomial $g(X) \in \overline{\mathbb{Q}}[X]$ satisfying $g(q^{-\frac{37}{60}}\prod_{m=1}^{\infty}(1+q^{15m-7})(1+q^{15m-8})) = 0$. Therefore, we conclude that

$$q^{\frac{47}{60}}\prod_{m=1}^{\infty}(1+q^{15m-1})(1+q^{15m-14}),$$

$$q^{\frac{23}{60}}\prod_{m=1}^{\infty}(1+q^{15m-2})(1+q^{15m-13}),$$

$$q^{-\frac{13}{60}} \prod_{m=1}^{\infty} (1+q^{15m-4})(1+q^{15m-11})$$

and

$$q^{-\frac{37}{60}} \prod_{m=1}^{\infty} (1+q^{15m-7})(1+q^{15m-8})$$

are algebraic numbers.

Example 2.7. Now, we consider the case $n = 30$. In [3] and [23, p.333] we see that

(2.16)
$$\prod_{m=1}^{\infty} \frac{(1-q^{2m})}{(1-q^{30m})(1-q^{5m-1})(1-q^{5m-4})}$$
$$= \prod_{m=1}^{\infty} (1-q^{30m-13})(1-q^{30m-17}) + q \prod_{m=1}^{\infty} (1-q^{30m-7})(1-q^{30m-23}),$$

which was described by Rogers as a remarkable identity. And we know that

$$q^{-\frac{28}{24}} \prod_{m=1}^{\infty} \left(\frac{1-q^{2m}}{1-q^{30m}}\right), \ q^{\frac{1}{60}} \prod_{m=1}^{\infty} (1-q^{5m-1})(1-q^{5m-4}),$$
$$q^{-\frac{13}{60}} \prod_{m=1}^{\infty} (1-q^{10m-3})(1-q^{10m-7})$$

and

$$q^{\frac{23}{60}} \prod_{m=1}^{\infty} (1-q^{10m-1})(1-q^{10m-9})$$

are algebraic numbers ([17]).

Thus we claim by (2.16) that

$$\begin{aligned} D &:= q^{-\frac{28}{24}} \prod_{m=1}^{\infty} \left(\frac{1-q^{2m}}{1-q^{30m}}\right) \left(\frac{q^{\frac{1}{60}}}{(1-q^{5m-1})(1-q^{5m-4})}\right) \\ &= q^{-\frac{71}{60}} \prod_{m=1}^{\infty} (1-q^{30m-13})(1-q^{30m-17}) \\ &\quad + q^{-\frac{11}{60}} \prod_{m=1}^{\infty} (1-q^{30m-7})(1-q^{30m-23}) \end{aligned} \tag{2.17}$$

is an algebraic number. And,

$$E := \prod_{m=1}^{\infty} \frac{q^{-\frac{13}{60}}(1-q^{10m-3})(1-q^{10m-7})}{q^{\frac{69}{60}}(1-q^{30m-3})(1-q^{30m-27})}$$

$$(2.18)\qquad = \prod_{m=1}^{\infty}\left(q^{-\frac{71}{60}}(1-q^{30m-13})(1-q^{30m-17})\right)\cdot\left(q^{-\frac{11}{60}}(1-q^{30m-7})(1-q^{30m-23})\right)$$

is an algebraic number. Hence (2.17) and (2.18) enable us to get a polynomial $g(X) = X^2 - DX + E \in \overline{\mathbb{Q}}[X]$ satisfying $g(q^{-\frac{71}{60}}\prod_{m=1}^{\infty}(1-q^{30m-13})(1-q^{30m-17})) = 0$. Thus, we find that

$$q^{-\frac{71}{60}}\prod_{m=1}^{\infty}(1-q^{30m-13})(1-q^{30m-17}) \text{ and } q^{-\frac{11}{60}}\prod_{m=1}^{\infty}(1-q^{30m-7})(1-q^{30m-23})$$

are algebraic numbers. And by Lemma 2.2 (a) we derive that

$$q^{-\frac{59}{60}}\prod_{m=1}^{\infty}(1-q^{30m-11})(1-q^{30m-19}) - q^{\frac{121}{60}}\prod_{m=1}^{\infty}(1-q^{30m-1})(1-q^{30m-29})$$

is an algebraic number. Moreover, it is not difficult to see that

$$\left(q^{-\frac{59}{60}}\prod_{m=1}^{\infty}(1-q^{30m-11})(1-q^{30m-19})\right)\cdot\left(-q^{\frac{121}{60}}\prod_{m=1}^{\infty}(1-q^{30m-1})(1-q^{30m-29})\right)$$
$$= \frac{q^{\frac{23}{60}}\prod_{m=1}^{\infty}(1-q^{10m-1})(1-q^{10m-9})}{q^{-\frac{13}{60}}\prod_{m=1}^{\infty}(1-q^{30m-9})(1-q^{30m-21})} \text{ is an algebraic number.}$$

It then follows that

$$(2.19)\quad q^{-\frac{59}{60}}\prod_{m=1}^{\infty}(1-q^{30m-11})(1-q^{30m-19}),\ q^{\frac{121}{60}}\prod_{m=1}^{\infty}(1-q^{30m-1})(1-q^{30m-29})$$

are algebraic numbers.

If we use Son's identity ([28])

$$f^3(-q^{11},-q^{19}) + q^9 f^3(-q,-q^{29}) = \left(\sum_{m=0}^{\infty}\frac{q^{3m(m+1)}}{(q^3;q^3)_m}\right)(q^6;q^6)_\infty\frac{(q^{10};q^{10})_\infty^3}{(q^{30};q^{30})_\infty},$$

we have by (2.19) that

$$q^{\frac{11}{20}}\sum_{m=0}^{\infty}\frac{q^{3m(m+1)}}{(q^3;q^3)_m}$$
$$= \prod_{m=1}^{\infty}(q^{-\frac{59}{60}}(1-q^{30m-11})(1-q^{30m-19}))^3\left(q^{\frac{10}{24}}\frac{(1-q^{30m})}{(1-q^{10m})}\right)^3\left(q\frac{(1-q^{30m})}{(1-q^{6m})}\right)$$
$$+ \prod_{m=1}^{\infty}(q^{\frac{121}{60}}(1-q^{30m-1})(1-q^{30m-29}))^3\left(q^{\frac{10}{24}}\frac{(1-q^{30m})}{(1-q^{10m})}\right)^3\left(q\frac{(1-q^{30m})}{(1-q^{6m})}\right)$$

is an algebraic number. Here we note that combining Slater identity A.14 ([17], [24], [27]) and Son's identity we can find a relation between sums and infinite products, i.e.,

$$\begin{aligned} & q^{\frac{11}{20}} \sum_{m=0}^{\infty} \frac{q^{3m(m+1)}}{(q^3;q^3)_m} \\ = & \prod_{m=1}^{\infty} (q^{-\frac{59}{60}}(1-q^{30m-11})(1-q^{30m-19}))^3 \left(q^{\frac{10}{24}} \frac{(1-q^{30m})}{(1-q^{10m})}\right)^3 \left(q \frac{(1-q^{30m})}{(1-q^{6m})}\right) \\ & + \prod_{m=1}^{\infty} (q^{\frac{121}{60}}(1-q^{30m-1})(1-q^{30m-29}))^3 \left(q^{\frac{10}{24}} \frac{(1-q^{30m})}{(1-q^{10m})}\right)^3 \left(q \frac{(1-q^{30m})}{(1-q^{6m})}\right) \\ = & \prod_{m=1}^{\infty} (q^{\frac{1}{20}}(1-q^{15m-3})(1-q^{15m-12})) \left(q^{\frac{1}{2}} \frac{(1-q^{15m})}{(1-q^{3m})}\right). \end{aligned}$$

Example 2.8. Slater found in [26](or [3]) that

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{q^{m^2}}{(q)_{2m}} = & \prod_{m=1}^{\infty} \frac{(1+q^{30m-14})(1+q^{30m-16})(1-q^{30m})}{(1-q^m)} \\ & - \prod_{m=1}^{\infty} \frac{q^2(1+q^{30m-4})(1+q^{30m-26})(1-q^{30m})}{(1-q^m)}, \end{aligned}$$

$$\sum_{m=1}^{\infty} \frac{q^{3m^2}}{(q^4)_{4,m}(q)_{2,m}} = \prod_{m=1}^{\infty} \frac{(1+q^{2m-1})}{(1+(-1)^m q^{5m-1})(1-(-1)^m q^{5m-4})}$$

and

$$\sum_{m=1}^{\infty} \frac{q^{\frac{1}{2}(m^2+m)}}{(q)_m (q)_{2,m}} = \prod_{m=1}^{\infty} \frac{(1+q^m)(1-q^{7m-2})(1-q^{7m-5})(1-q^{7m})}{(1-q^m)(1+q^{7m-1})(1+q^{7m-6})}$$

where $(a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ and $(a)_{r,n} = (1-a)(1-aq^r)\cdots(1-aq^{r(n-1)})$. By (2.3) and Example 2.7 we see that

$$\begin{aligned} & q^{-\frac{1}{40}} \sum_{m=1}^{\infty} \frac{q^{m^2}}{(q)_{2m}} \\ = & \prod_{m=1}^{\infty} \left(q^{-\frac{74}{60}}(1+q^{30m-14})(1+q^{30m-16})\right) \left(q^{\frac{29}{24}} \frac{(1-q^{30m})}{(1-q^m)}\right) \\ & - \prod_{m=1}^{\infty} \left(q^{\frac{46}{60}}(1+q^{30m-4})(1+q^{30m-26})\right) \left(q^{\frac{29}{24}} \frac{(1-q^{30m})}{(1-q^m)}\right) \end{aligned}$$

is an algebraic number and hence $\sum_{m=1}^{\infty} \frac{q^{m^2}}{(q)_{2m}}$ is transcendental.

We showed in [17] that

$$q^{\frac{23}{60}} \prod_{m=1} (1+q^{10m-1})(1+q^{10m-9}) \quad \text{and} \quad q^{-\frac{11}{60}} \prod_{m=1} (1-q^{5m-2})(1-q^{5m-3})$$

are algebraic numbers. Thus,

$$\begin{aligned} &q^{-\frac{7}{120}} \sum_{m=1}^{\infty} \frac{q^{3m^2}}{(q^4)_{4,m}(q)_{2,m}} \\ &= \prod_{m=1}^{\infty} \frac{q^{-\frac{1}{24}}(1+q^{2m-1})}{\left(q^{-\frac{22}{60}}(1-q^{10m-4})(1-q^{10m-6})\right)\left(q^{\frac{23}{60}}(1+q^{10m-1})(1+q^{10m-9})\right)} \end{aligned}$$

is an algebraic number, and so $\sum_{m=1}^{\infty} \frac{q^{3m^2}}{(q^4)_{4,m}(q)_{2,m}}$ is a transcendental number. And, by (2.3), Proposition 2.1 and Example 2.5 we derive that

(2.20)
$$\begin{aligned} q^{\frac{1}{168}} \sum_{m=1}^{\infty} \frac{q^{\frac{1}{2}(m^2+m)}}{(q)_m(q)_{2,m}} = \prod_{m=1}^{\infty} &\left(q^{\frac{1}{24}}(1+q^m)\right)\left(\frac{q^{-\frac{11}{84}}(1-q^{7m-2})(1-q^{7m-5})}{q^{\frac{13}{84}}(1+q^{7m-1})(1-q^{7m-6})}\right) \\ &\cdot \left(q^{\frac{6}{24}}\frac{(1-q^m)}{(1-q^{7m})}\right) \end{aligned}$$

is an algebraic number. On the other hand, we know from [3] and [22] that

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{q^{\frac{1}{2}(m^2+m)}}{(q)_m(q)_{2,m}} = &\prod_{m=1}^{\infty} \frac{(1+q^m)(1+q^{21m-10})(1+q^{21m-11})(1-q^{21m})}{(1-q^m)} \\ &- q\prod_{m=1}^{\infty} \frac{(1+q^m)(1+q^{21m-4})(1+q^{21m-17})(1-q^{21m})}{(1-q^m)}. \end{aligned}$$

Hence, we claim by (2.20) and Example 2.5, that

$$\begin{aligned} H &:= q^{-\frac{73}{84}} \prod_{m=1}^{\infty} (1+q^{21m-10})(1+q^{21m-11}) - q^{\frac{1}{84}} \prod_{m=1}^{\infty} (1+q^{21m-4})(1+q^{21m-17}) \\ &= \left(q^{\frac{1}{168}} \sum_{m=1}^{\infty} \frac{q^{\frac{1}{2}(m^2+m)}}{(q)_m(q)_{2,m}}\right)\left(\frac{1}{q^{\frac{1}{24}}\prod_{m=1}^{\infty}(1+q^m)}\right)\left(q^{-\frac{20}{24}}\frac{(1-q^m)}{(1-q^{21m})}\right) \end{aligned}$$

and

$$\begin{aligned} I &:= \left(q^{-\frac{73}{84}} \prod_{m=1}^{\infty} (1+q^{21m-10})(1+q^{21m-11})\right)\left(-q^{\frac{1}{84}}(1+q^{21m-4})(1+q^{21m-17})\right) \\ &= -\prod_{m=1}^{\infty} \frac{q^{-\frac{23}{84}}(1+q^{7m-3})(1+q^{7m-4})}{q^{\frac{39}{84}}(1+q^{21m-3})(1+q^{21m-18})} \end{aligned}$$

are algebraic numbers. Therefore we obtain an equation $g(X) = X^2 - HX + I$ satisfying $g(q^{-\frac{73}{84}} \prod_{m=1}^{\infty}(1+q^{21m-10})(1+q^{21m-11})) = 0$, from which we derive that $q^{-\frac{73}{84}}\prod_{m=1}^{\infty}(1+q^{21m-10})(1+q^{21m-11})$ and $q^{\frac{1}{84}}\prod_{m=1}^{\infty}(1+q^{21m-4})(1+q^{21m-17})$ are algebraic numbers.

Next, it follows from Slater identities A.80, 82 and routine calculations that

$$\left(q^{\frac{25}{168}}\sum_{m=0}\frac{(-q;q)_m q^{\frac{1}{2}m(m+1)}}{(q;q)_{2m+1}}\right)\left(\prod_{m=1}^{\infty}\frac{q^{\frac{1}{24}}(1-q^m)}{q^{\frac{21}{24}}(1-q^{21m})}\right)\left(\frac{1}{q^{\frac{1}{24}}(1+q^m)}\right)$$
$$=\left(q^{-\frac{61}{84}}\prod_{m=1}^{\infty}(1+q^{21m-8})(1+q^{21m-13})\right)-\left(q^{\frac{107}{84}}\prod_{m=1}^{\infty}(1+q^{21m-1})(1+q^{21m-20})\right),$$

$$\left(q^{\frac{121}{168}}\sum_{m=0}\frac{(-q;q)_m q^{\frac{1}{2}m(m+3)}}{(q;q)_{2m+1}}\right)\left(\prod_{m=1}^{\infty}\frac{q^{\frac{1}{24}}(1-q^m)}{q^{\frac{21}{24}}(1-q^{21m})}\right)\left(\frac{1}{q^{\frac{1}{24}}(1+q^m)}\right)$$
$$=\left(q^{-\frac{13}{84}}\prod_{m=1}^{\infty}(1+q^{21m-5})(1+q^{21m-16})\right)-\left(q^{\frac{71}{84}}\prod_{m=1}^{\infty}(1+q^{21m-2})(1+q^{21m-19})\right),$$

$$-\left(q^{-\frac{61}{84}}\prod_{m=1}^{\infty}(1+q^{21m-8})(1+q^{21m-13})\right)\left(q^{\frac{107}{84}}\prod_{m=1}^{\infty}(1+q^{21m-1})(1+q^{21m-20})\right)$$
$$=-\left(\prod_{m=1}^{\infty}\frac{q^{\frac{13}{84}}(1+q^{7m-1})(1+q^{7m-6})}{q^{-\frac{33}{84}}(1+q^{21m-6})(1+q^{21m-15})}\right)$$

and

$$-\left(q^{-\frac{13}{84}}\prod_{m=1}^{\infty}(1+q^{21m-5})(1+q^{21m-16})\right)\left(q^{\frac{71}{84}}\prod_{m=1}^{\infty}(1+q^{21m-2})(1+q^{21m-19})\right)$$
$$=-\left(\prod_{m=1}^{\infty}\frac{q^{-\frac{11}{84}}(1+q^{7m-2})(1+q^{7m-5})}{q^{-\frac{23}{84}}(1+q^{21m-9})(1+q^{21m-12})}\right)$$

are algebraic numbers. Therefore, we conclude that $q^{-\frac{13}{84}}\prod_{m=1}^{\infty}(1+q^{21m-5})(1+q^{21m-16})$, $q^{\frac{71}{84}}\prod_{m=1}^{\infty}(1+q^{21m-2})(1+q^{21m-19})$, $q^{-\frac{61}{84}}\prod_{m=1}^{\infty}(1+q^{21m-8})(1+q^{21m-13})$ and $q^{\frac{107}{84}}\prod_{m=1}^{\infty}(1+q^{21m-1})(1+q^{21m-20})$ are algebraic numbers.

Example 2.9. According to Slater ([27]), we have the following identity

$$\begin{aligned}&\prod_{m=1}^{\infty}(1+q^{42m-19})(1+q^{42m-23})(1-q^{42m})\\ &-q^3\prod_{m=1}^{\infty}(1+q^{42m-5})(1+q^{42m-37})(1-q^{42m})\\ &=\prod_{m=1}^{\infty}\frac{(1-q^{2m})}{(1+q^{2m-1})}\sum_{n=0}\frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_{2n}}.\end{aligned}\tag{2.21}$$

From the infinite products derived from theta functions we know that

$$q^{-\frac{13}{168}}\sum_{n=0}\frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_{2n}}$$

is an algebraic number ([17]). Then by (2.21) we get that

(2.22)

$$\begin{aligned}&\left(q^{-\frac{13}{168}}\sum_{n=0}\frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_{2n}}\right)\left(\prod_{m=1}^{\infty}\frac{q^{\frac{2}{24}}(1-q^{2m})}{q^{\frac{42}{24}}(1-q^{42m})}\right)\left(\prod_{m=1}^{\infty}\frac{q^{\frac{1}{24}}}{(1+q^{2m-1})}\right)\\&=\left(q^{-\frac{143}{84}}\prod_{m=1}^{\infty}(1+q^{42m-19})(1+q^{42m-23})\right)\\&\quad-\left(q^{\frac{109}{84}}\prod_{m=1}^{\infty}(1+q^{42m-5})(1+q^{42m-37})\right)\end{aligned}$$

is an algebraic number and we can also check by Example 2.8 that

(2.23)

$$\begin{aligned}&-\prod_{m=1}^{\infty}\left(q^{-\frac{143}{84}}(1+q^{42m-19})(1+q^{42m-23})\right)\left(q^{\frac{109}{84}}(1+q^{42m-5})(1+q^{42m-37})\right)\\&=-\frac{\left(q^{\frac{71}{84}}\prod_{m=1}^{\infty}(1+q^{21m-2})(1+q^{21m-19})\right)\left(q^{-\frac{13}{84}}\prod_{m=1}^{\infty}(1+q^{21m-5})(1+q^{21m-16})\right)}{\left(q^{\frac{107}{42}}\prod_{m=1}^{\infty}(1+q^{42m-2})(1+q^{42m-40})\right)\left(q^{-\frac{61}{42}}\prod_{m=1}^{\infty}(1+q^{42m-16})(1+q^{42m-26})\right)}\end{aligned}$$

is an algebraic number. Therefore, by (2.22) and (2.23) $\prod_{m=1}^{\infty}q^{-\frac{143}{84}}(1+q^{42m-19})(1+q^{42m-23})$ and $\prod_{m=1}^{\infty}q^{\frac{109}{84}}(1+q^{42m-5})(1+q^{42m-37})$ are algebraic numbers. In a similar way, due to Slater identities (A. 118), (A. 119) ([24], [27]) and [17] concerning algebraic numbers for infinite sums, we obtain that

(2.24)

$$\begin{aligned}&\left(q^{\frac{11}{168}}\sum_{n=0}\frac{(-q;q^2)_n q^{n(n+2)}}{(q^2;q^2)_{2n}}\right)\left(\prod_{m=1}^{\infty}\frac{q^{\frac{2}{24}}(1-q^{2m})}{q^{\frac{42}{24}}(1-q^{42m})}\right)\left(\prod_{m=1}^{\infty}\frac{q^{\frac{1}{24}}}{(1+q^{2m-1})}\right)\\&=\left(q^{-\frac{131}{84}}\prod_{m=1}^{\infty}(1+q^{42m-17})(1+q^{42m-25})\right)\\&\quad-\left(q^{-\frac{47}{84}}\prod_{m=1}^{\infty}(1+q^{42m-11})(1+q^{42m-31})\right),\end{aligned}$$

(2.25)

$$\begin{aligned}&-\prod_{m=1}^{\infty}\left(q^{-\frac{131}{84}}(1+q^{42m-17})(1+q^{42m-25})\right)\left(q^{-\frac{47}{84}}(1+q^{42m-11})(1+q^{42m-31})\right)\\&=-\frac{\left(q^{\frac{11}{84}}\prod_{m=1}^{\infty}(1+q^{21m-4})(1+q^{21m-17})\right)\left(q^{-\frac{73}{84}}\prod_{m=1}^{\infty}(1+q^{21m-10})(1+q^{21m-11})\right)}{\left(q^{\frac{71}{42}}\prod_{m=1}^{\infty}(1+q^{42m-4})(1+q^{42m-38})\right)\left(q^{-\frac{13}{42}}\prod_{m=1}^{\infty}(1+q^{42m-10})(1+q^{42m-32})\right)},\end{aligned}$$

$$\left(q^{\frac{107}{168}}\sum_{n=0}\frac{(-q;q^2)_n q^{n(n+2)}}{(q^2;q^2)_{2n+1}}\right)\left(\prod_{m=1}^{\infty}\frac{q^{\frac{2}{24}}(1-q^{2m})}{q^{\frac{42}{24}}(1-q^{42m})}\right)$$

(2.26)
$$\cdot\left(\prod_{m=1}^{\infty}\frac{q^{\frac{1}{24}}}{(1+q^{2m-1})}\right)$$
$$=\left(q^{-\frac{83}{84}}\prod_{m=1}^{\infty}(1+q^{42m-13})(1+q^{42m-29})\right)$$
$$-\left(q^{\frac{253}{84}}\prod_{m=1}^{\infty}(1+q^{42m-1})(1+q^{42m-41})\right),$$

and
(2.27)
$$-\prod_{m=1}^{\infty}\left(q^{-\frac{83}{84}}(1+q^{42m-13})(1+q^{42m-29})\right)\left(q^{\frac{253}{84}}(1+q^{42m-1})(1+q^{42m-41})\right)$$
$$=-\frac{\left(q^{\frac{107}{84}}\prod_{m=1}^{\infty}(1+q^{21m-1})(1+q^{21m-20})\right)\left(q^{-\frac{61}{84}}\prod_{m=1}^{\infty}(1+q^{21m-8})(1+q^{21m-13})\right)}{\left(q^{\frac{11}{42}}\prod_{m=1}^{\infty}(1+q^{42m-8})(1+q^{42m-34})\right)\left(q^{-\frac{73}{42}}\prod_{m=1}^{\infty}(1+q^{42m-20})(1+q^{42m-22})\right)}.$$

Thus we claim by (2.24)∼(2.27) that

$$q^{-\frac{131}{84}}\prod_{m=1}^{\infty}(1+q^{42m-17})(1+q^{42m-25}),\quad q^{-\frac{47}{84}}\prod_{m=1}^{\infty}(1+q^{42m-11})(1+q^{42m-31}),$$
$$q^{-\frac{83}{84}}\prod_{m=1}^{\infty}(1+q^{42m-13})(1+q^{42m-29}),\quad q^{\frac{253}{84}}(1+q^{42m-1})(1+q^{42m-41})$$

are algebraic numbers.

3. Rogers-Ramanujan identities

The Rogers-Ramanujan continue fraction, defined by

$$R(q):=\frac{q^{\frac{1}{5}}}{1}\,\frac{q}{+1+}\,\frac{q^2}{1}\,\frac{q^3}{+\;1\;+\cdots},$$

first appeared in a paper of Rogers ([23]). Using such identities, Rogers proved that

$$R(q)=q^{\frac{1}{5}}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}.$$

By (2.9),

(3.1)
$$R(q)=\frac{q^{\frac{1}{60}}\prod_{m=1}^{\infty}(1-q^{5m-1})(1-q^{5m-4})}{q^{-\frac{11}{60}}\prod_{m=1}^{\infty}(1-q^{5m-2})(1-q^{5m-3})}$$

is an algebraic number. Let

$$T(q) := \frac{q^{\frac{1}{5}}}{1}\, \frac{q}{-1+}\, \frac{q^2}{1-}\, \frac{q^3}{1+\cdots}.$$

Berndt ([7, p.16]) wrote that

$$\begin{aligned} &T(q)R(q)(T(q)-R(q))^4 - T(q)^2R(q)^2(T(q)-R(q))^2 \\ &+ 2T(q)^3R(q)^3 = (T(q)-R(q))(1+T(q)^5R(q)^5). \end{aligned} \tag{3.2}$$

From (3.1) and (3.2), it follows that $T(q)$ is an algebraic number.

Set

$$f(-q) := f(-q;-q^2) = \sum_{m=-\infty}^{\infty} (-1)^m q^{m(3m-1)/2} = (q;q)_\infty.$$

Note that the latter equality is known as Pentagonal number theorem.

Theorem 3.1. *Let $\alpha = \frac{1-\sqrt{5}}{2}$, $\beta = \frac{1+\sqrt{5}}{2}$ and $\zeta = e^{\frac{2\pi i}{5}}$.*

(a) $q^{\frac{9}{40}} \sum_{m=-\infty}^{\infty} (-1)^m(10m+3)q^{(5m+3)m/2}$,

$q^{\frac{1}{40}} \sum_{m=-\infty}^{\infty} (-1)^m(10m+1)q^{(5m+1)m/2}$ *are transcendental numbers.*

(b) $\prod_{m=0}^{\infty} \frac{1}{q^{\frac{1}{12}}(1+\alpha q^m + q^{2m})}$ *and* $\prod_{m=0}^{\infty} \frac{1}{q^{\frac{1}{12}}(1+\beta q^m + q^{2m})}$ *are algebraic numbers.*

(c) $q^{\frac{1}{12}} \prod_{m=1}^{\infty}(1-\zeta^2 q^m)(1-\zeta^3 q^m)$ *and* $q^{\frac{1}{12}} \prod_{m=1}^{\infty}(1-\zeta q^m)(1-\zeta^4 q^m)$ *are algebraic numbers.*

Proof. (a) We know from [8] that

$$\sum_{m=-\infty}^{\infty} (-1)^m(10m+3)q^{(5m+3)m/2} = \left(\frac{3}{R^2(q)} + R^3(q)\right) q^{\frac{2}{5}} f^3(-q^5)$$

and

$$\sum_{m=-\infty}^{\infty} (-1)^m(10m+1)q^{(5m+1)m/2} = \left(\frac{3}{R^2(q)} - 3R^3(q)\right) q^{\frac{3}{5}} f^3(-q^5).$$

Since $\left(q^{\frac{5}{24}} f^3(-q^5) = \prod_{m=1}^{\infty}(1-q^{5m})\right)^3$ is transcendental ([17]) and R(q) is algebraic, we conclude that

$$q^{\frac{9}{40}} \sum_{m=-\infty}^{\infty} (-1)^m(10m+3)q^{(5m+3)m/2} \text{ and } q^{\frac{1}{40}} \sum_{m=-\infty}^{\infty} (-1)^m(10m+1)q^{(5m+1)m/2}$$

are transcendental numbers.

(b) It follows from [8] that

$$\frac{1}{\sqrt{R(q)}} - \alpha\sqrt{R(q)} = \frac{1}{q^{\frac{1}{10}}}\sqrt{\frac{f(-q)}{f(-q^5)}}\prod_{m=1}^{\infty}\frac{1}{1+\alpha q^{\frac{m}{5}}+q^{\frac{2m}{5}}}$$

and

$$\frac{1}{\sqrt{R(q)}} - \beta\sqrt{R(q)} = \frac{1}{q^{\frac{1}{10}}}\sqrt{\frac{f(-q)}{f(-q^5)}}\prod_{m=1}^{\infty}\frac{1}{1+\beta q^{\frac{m}{5}}+q^{\frac{2m}{5}}}.$$

Meanwhile, we know that $q^{-\frac{4}{24}}\prod_{m=1}^{\infty}\left(\frac{1-q^m}{1-q^{5m}}\right)$ and $R(q)$ are algebraic numbers. Since the values of the Dedekind $\eta(\tau)$-function are nonzero ([25]),

$$\prod_{m=1}^{\infty}\frac{1}{q^{\frac{1}{60}}(1+\alpha q^{\frac{m}{5}}+q^{\frac{2m}{5}})} \quad\text{and}\quad \prod_{m=1}^{\infty}\frac{1}{q^{\frac{1}{60}}(1+\beta q^{\frac{m}{5}}+q^{\frac{2m}{5}})}$$

are algebraic numbers. Then the fact that τ, $5\tau(\in \mathfrak{h}\cap k)$ and $2+\alpha$, $2+\beta$ are algebraic numbers implies the result.

(c) We find in [8] that

$$f(-q^2,-q^3) - \alpha q^{15} f(-q,-q^4) = f(-\zeta^2,-\zeta^3 q^{\frac{1}{5}})/(1-\zeta^2)$$

and

$$f(-q^2,-q^3) - \beta q^{15} f(-q,-q^4) = f(-\zeta,-\zeta^4 q^{\frac{1}{5}})/(1-\zeta).$$

If we multiply both sides by $q^{\frac{1}{60}}\prod_{m=1}^{\infty}\frac{1}{1-q^m}$, then

$$\begin{aligned}&\left(q^{-\frac{11}{60}}\prod_{m=1}^{\infty}(1-q^{5m-2})(1-q^{5m-3})\right)\left(\frac{q^{\frac{5}{24}}(1-q^{5m})}{q^{\frac{1}{24\cdot 5}}(1-q^{\frac{1}{5}m})}\right)\\ &+\alpha\left(q^{\frac{1}{60}}\prod_{m=1}^{\infty}(1-q^{5m-1})(1-q^{5m-4})\right)\left(\frac{q^{\frac{5}{24}}(1-q^{5m})}{q^{\frac{1}{24\cdot 5}}(1-q^{\frac{1}{5}m})}\right)\\ &= q^{\frac{1}{60}}\prod_{m=1}^{\infty}(1-\zeta^2 q^{\frac{1}{5}m})(1-\zeta^3 q^{\frac{1}{5}m})\end{aligned}$$

is an algebraic number. Replacing τ by 5τ, we conclude that $q^{\frac{1}{12}}\prod_{m=1}^{\infty}(1-\zeta^2 q^m)(1-\zeta^3 q^m)$ is an algebraic number. The other case can be done in a similar way. □

Corollary 3.2. *If* $\zeta = e^{\frac{2\pi i}{5}}$, *then* $\frac{\prod_{m=1}^{\infty}(1-\zeta^2 q^m)(1-\zeta^3 q^m)}{\prod_{m=1}^{\infty}(1-\zeta q^m)(1-\zeta^4 q^m)}$ *is an algebraic number and*
$q^a\frac{\prod_{m=1}^{\infty}(1-\zeta^2 q^m)(1-\zeta^3 q^m)}{\prod_{m=1}^{\infty}(1-\zeta q^m)(1-\zeta^4 q^m)}$ *is a transcendental number with* $a \in \mathbb{Q}-\{0\}$.

Proof. It is immediate by Theorem 3.1 (c). □

Remark. Let m be a positive integer not divisible by 5, $\zeta = e^{\frac{2\pi i}{5}}$ and $\alpha = \frac{1-\sqrt{5}}{2}$. Then we know from [8] the following identity

$$\prod_{j=0}^{4}(1+\alpha\zeta^{mj}q^{\frac{n}{5}}+\zeta^{2mj}q^{\frac{2n}{5}}) = (1-q^n)^2.$$

Hence, we get that

$$\prod_{\substack{m=1 \\ m\not\equiv 0 \pmod 5}}^{\infty}\left(\prod_{j=0}^{4}(1+\alpha\zeta^{mj}q^{\frac{m}{5}}+\zeta^{2mj}q^{\frac{2m}{5}})\right) = \prod_{m=1}\left(\frac{1-q^m}{1-q^{5m}}\right)^2$$

and

$$\prod_{m=1}^{\infty}\left(\prod_{j=0}^{4}(1+\alpha\zeta^{mj}q^{\frac{m}{5}}+\zeta^{2mj}q^{\frac{2m}{5}})\right) = \prod_{m=1}\left(\frac{1-q^m}{1-q^{5m}}\right)^2\left(1+\alpha q^m+q^{2m}\right)^5.$$

By Theorem 3.1 and (2.3) we conclude that

$$q^{\frac{1}{12}}\prod_{m=1}^{\infty}\left(\prod_{j=0}^{4}(1+\alpha\zeta^{mj}q^{\frac{m}{5}}+\zeta^{2mj}q^{\frac{2m}{5}})\right)$$

is an algebraic number.

References

[1] G. E. Andrews, *An introduction to Ramanujan's lost notebook*, Amer. Math. Monthly **86** (1979), 89–108.

[2] ———, *Ramanujan's lost notebook. III. The Rogers-Ramanujan continued fraction*, Adv. Math. **41** (1981), 186–208.

[3] W. N. Bailey, *On the simplification of some identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. (3) (1951), 217–221.

[4] B. C. Berndt, *Ramanujan's Notebooks II*, Springer-Verlag, 1989.

[5] ———, *Ramanujan's Notebooks III*, Springer-Verlag, 1991.

[6] ———, *Ramanujan's Notebooks IV*, Springer-Verlag, 1993.

[7] ———, *Ramanujan's Notebooks V*, Springer-Verlag, 1997.

[8] B. C. Berndt, S.-S. Huang, J. Sohn, and S. H. Son, *Some theorems on the Rogers - Ramanujan continued fraction in Ramanujan's lost notebook*, Trans. Amer. Math. Soc. **352** (2000), 2157–2177.

[9] B. C. Berndt and H. H. Chan, *Some values for the Rogers-Ramanujan continued fraction*, Canad. J. Math. **47** (1995), 897–914.

[10] B. C. Berndt, H. H. Chan, and L.-C. Zhang, *Explicit evaluations of the Rogers-Ramanujan continued fraction*, J. Reigne Angew. Math. **480** (1996), 141–159.

[11] P. B. Borwein and P. Zhou, *On the irrationality of a certain q series*, Proc. Amer. Math. Soc. **127** (1999), 1605–1613.

[12] S.-S. Huang, *Ramanujan's evaluations of the Rogers-Ramanujan type continued fractions at primitive roots of unity*, Acta Arith. **80** (1997), 49–60.

[13] N. Ishida, *Generators and equations for modular function fields of principal congruence subgroups*, Acta Arith. **85** (1998), 197–207.

[14] S.-Y. Kang, *Some theorems on the Rogers-Ramanujan continued fraction and associated theta function identities in Ramanujan's lost notebook*, The Ramanujan Journal **3** (1999), 91–111.
[15] ———, *Ramanujan's formulas for the explicit evaluation of the Rogers-Ramanujan continued fraction and theta functions*, Acta Arith. **90** (1999), 49–68.
[16] D. Kim and J. K. Koo, *Algebraic integer as values of elliptic functions*, Acta Arith. **100** (2001), 105–116.
[17] ———, *On the infinite products derived from theta series I*, J. Korean Math. Soc. **44** (2007), 55–107.
[18] D. Kubert and S. Lang, *Units in the modular function fields*, Math. Ann. **218** (1975), 175–189.
[19] S. Lang, *Elliptic Functions*, Addison-Wesley, 1973.
[20] S. Ramanujan, *Notebooks (2 volumes)*, Tata Institute of Fundamental Research, Bombay, 1957.
[21] ———, *The Lost Notebook and Other Unpublished Papers*, Narosa, 1988.
[22] L. J. Rogers, *On two theorems of combinatory analysis and some allied identities*, Proc. London Math. Soc. (1) **16** (1917), 315–336.
[23] ———, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. (1) **25** (1894), 318–343.
[24] A. V. Sills, *Finite Rogers-Ramanujan type identities*, Electron. J. Combin. **10** (2003), 1–122.
[25] J. Silverman, *The Arithmetic of Elliptic Curves*, Springer -Verlag, New York, 1986.
[26] L. J. Slater, *A new proof of Roger's transformations of series*, Proc. London Math. Soc. sereis 2 **53** (1951), 461–475.
[27] ———, *Further identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. series 2 **54** (1952), 147–167.
[28] S. Son, *Cubic identities of theta functions*, The Ramanujan Journal **2** (1998), 303–316.
[29] ———, *Some theta function identities related to Rogers-Ramanujan continued fraction*, Proc. Amer. Math. Soc. **126** (1998), 2895–2902.
[30] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, Cambridge Press, 1978.

Daeyeoul Kim
National Institute for Mathematical Sciences 385-16
Tajeon 305-340, Korea
and
Department of Math. and Institute of Pure and Applied Mathematics
Chonbuk National University
Chonju 561-756, Korea
E-mail address: daeyeoul@chonbuk.ac.kr

Ja Kyung Koo
Korea Advanced Institute of Science and Technology
Department of Mathematics
Tajeon 305-701, Korea
E-mail address: jkkoo@math.kaist.ac.kr

Yilmaz Simsek
University of Akdeniz, Faculty of Science
Department of Mathematics
07058 Antalya, Turkey
E-mail address: ysimsek@akdeniz.edu.tr