Optimal Wiener-Hopf Decoupling Controller Formula for State-space Algorithms

Kiheon Park and Jin-Geol Kim

Abstract: In this paper, an optimal Wiener-Hopf decoupling controller formula is obtained which is expressed in terms of rational matrices, thereby readily allowing the use of state-space algorithms. To this end, the characterization formula for the class of all realizable decoupling controller is formulated in terms of rational functions. The class of all stabilizing and decoupling controllers is parametrized via the free diagonal matrices and the optimal decoupling controller is determined from these free matrices.

Keywords: One-degree-of-freedom controller, optimal decoupling control, state-space formulas, Wiener-Hopf design.

1. INTRODUCTION

The design of decoupling control systems that eliminate the interactions between the various reference and manipulated signals has attracted the interest of many researchers. Desoer and Gündes [1] and Lee and Bongiorno [2] solved the two-degree-of-freedom decoupling problems. The decoupling problem for a one-degree-of-freedom control system was treated by Gómez and Goodwin [3], Youla and Bongiorno [4] and Bongiorno and Youla [5]. A notable feature of the work by Youla and Bongiorno [4] is that the solvability condition and the characterization of all decoupling controllers are explicitly expressed in a very convenient form to develop optimal controller formulas. The formula of Youla and Bongiorno [4] is derived in the frequency domain which requires the use of spectral factorization and partial fractions. The purpose of this paper is to develop state-space numerical algorithms for this formula as an alternative to the already available frequency domain numerical algorithms used for spectral factorization and partial fraction expansion. It has been shown that the Wiener-Hopf controllers described in terms of polynomial matrices can be successfully converted into the equivalent ones described in terms of rational matrices which allow the development of state-space parameter solutions [6,7]. In this paper, the optimal controller formula of Youla and Bongiorno [4] is reformulated in terms of rational matrices, thereby enabling the use of state-space algorithms. To this end, the characterization formula for the class of all realizable decoupling controllers is reformulated in terms of rational functions instead of polynomials. Youla and Bongiorno [4] used the Schur product to transform the constrained decoupling optimization problem into a solvable Wiener-Hopf problem. For the same purpose, we use the Khatri-Rao product and it turns out that this product enables us to develop the state-space computational formulas successfully. Another important contribution of this paper is to loosen the assumptions made in [4], which are needed to guarantee the existence of the optimal decoupling controller. Specifically, the assumption of the non-singularity of $\Delta_{\psi}P$ in $\mathrm{Re}\, s=0$ (Assumption 6) is removed in this paper. All of the results in this paper were developed by working exclusively in the complex s-plane and all matrix functions of $s=\sigma+j\omega$ are assumed to be real and rational . For any rational matrix $G(s)$, the notations $G'(s)$, $\det G$, and $Tr\, G$ are used for the transpose, determinant and trace of $G(s)$, respectively. The matrix $G_*(s)$ stands for $G'(-s)$. A diagonal matrix G with g_{ii} in the ith-row, ith-column, $i=1\rightarrow n$, is denoted by $diag\{g_{ii}\}$. The $n\times n$ identity matrix is denoted by I_n. When the dimension is understandable from the context, the subscript will be omitted. The Schur product of two matrices is denoted as $G\circ R$ and is the matrix whose ith-row, jth-column is given by $g_{ij}r_{ij}$. The Kronecker product of two

Manuscript received June 28, 2005; accepted May 25, 2007. Recommended by Editorial Board member Poo Gyeon Park under the direction of Editor Tae-Woong Yoon.
Kiheon Park is with the School of Information and Communication Engineering, Sungkyunkwan University, Suwon, Kyunggi-do 440-746, Korea (e-mail: khpark@ece.skku.ac.kr).
Jin-Geol Kim is with the School of Electrical Engineering, Inha University, 253, Yonghyun-dong, Nam-gu, Inchon 402-751, Korea (e-mail: john@inha.ac.kr).

matrices is denoted as $G \otimes R$. The vector $vecG = [g_1^{'} g_2^{'} \cdots g_n^{'}]$ is formed by stacking all the columns of the matrix G. The Khatri-Rao product of two matrices is denoted as $G \odot R$ and is the matrix whose ith-column is given by $g_i \otimes r_i$ where g_i and r_i are the ith-column of G and ith-column of R, respectively. For a diagonal matrix $G = diag\{g_{11}, g_{22}, \cdots, g_{nn}\}$, the vector $vecdG = \{g_{11}\ g_{22} \cdots g_{nn}\}^{'}$ is formed by stacking all the diagonal elements of the matrix G. When V is a diagonal matrix, $vec(AVD) = (D^{'} \otimes A)vecV = (D^{'} \odot A)vecdV$ [8]. Some useful formulas used in this paper are $(A \otimes B)(F \odot G) = AF \odot BG$, $I_n \odot [a_1\ a_2 \cdots a_n] = diag\{a_1, a_2, \cdots, a_n\}$ and $(F(s) \odot G(s))_*(L(s) \odot M(s)) = (F_*(s) L(s)) \circ (G_*(s)M(s))$. When D_i is diagonal, $D_1 D_2 (M \circ N) D_2 D_4 = (D_1 M D_2) \circ (D_3 N D)$ and $vecd(D_1 D_2 D_3) = D_1 D_3\ vecdD_2$. In the partial fraction expansion of $G(s)$, the contribution made by all its finite poles in $\operatorname{Re} s \le 0$ and $\operatorname{Re} s > 0$, and at $s = \infty$ are denoted by $\{G\}_+, \{G\}_-$ and $\{G\}_\infty$, respectively. A positive definite matrix Q is denoted by $Q > 0$ and $Q^{1/2}$ denotes a positive definite matrix satisfying $Q^{1/2} \cdot Q^{1/2} = Q$. A rational matrix $G(s)$ is said to be stable if it is analytic in $\operatorname{Re} s \ge 0$. A constant matrix F is sad to be stable if its eigenvalues are all in $\operatorname{Re} s < 0$. A rational function a(s) is said to be biproper if both a(s) and 1/a(s) are proper.

2. THE DECOUPLING PROBLEM AND ITS SOULUTION IN THE FREQUENCY DOMAIN

In this section the Wiener-Hopf decoupling problem for the one-degree-of freedom controller configuration shown in Fig. 1 is proposed along with its solution. In the Fig. 1 the dimension of the variables u_0, u, r, d, y, and m is $q \times 1$ and that of the variable d_0 is $p \times 1$. As is evident from this set-up, we treat the square plant case and we make the following assumption.

Assumption 1: $\det P(s) \neq 0$.

The disturbance d_0 and the noises n and m are white and process power spectral densities of Φ_d, Φ_n and Φ_m, respectively. These matrices are positive definite and Φ_d is taken to be the identity without loss of generality. The reference signal $u(s)$ is the output of the block $P_u(s)$ driven by $u_0(t) = c\delta(t)$ where $\delta(t)$ is the unit impulse and c is a column vector of random variable with zero means. Without loss of generality, the covariance matrix $< cc^* >$ is taken as the identity matrix. The block $P_u(s)$ is not a physical one, but one of the design parameters which determines the shape-deterministic part of the reference input $u(t)$ [7]. In this paper $P_u(s)$ is assumed to have poles in $\operatorname{Re} s \le 0$, which allows $j\omega$-axis poles to accommodate the tracking problems for step or ramp inputs. When the block $P_d(s)$ describes a real physical block, it should be a stable one. When it represents a mere paper model, however, this stability constraint can be relaxed. It is assumed that n, m, d_0 and $u(s)$ are statistically independent. In Fig. 1, the variables are related by

$$y(s) = T(s)(P_u u_0 + n - m) + (I - T(s))P_d d_0, \quad (1)$$

$$r(s) = P^{-1}(s)T(s)(P_u u_0 + n - m - P_d d_0), \quad (2)$$

where $T(s) = PC(I + PC)^{-1}$. A rational matrix $T(s)$ is said to be realizable for the plant $P(s)$ if the corresponding controller $C(s)$ stabilizes the feedback system in Fig. 1. By definition, a controller $C(s)$ is said to be decoupling for the plant $P(s)$ if it stabilizes the loop and produces a diagonal $T(s)$ with $\det T(s) \neq 0$. Lemma 2 describes the existence condition of a decoupling controller. Let us first define two polynomials to describe the decoupling condition. Let the i-th column of P^{-1} and i-th row of $P(s)$ be denoted by $P_{ci}(s)$ and $P_{ri}(s)$, respectively. Let $\theta_i(s)$ and $\psi_i(s)$ denote the unique monic polynomials of the minimal degrees such that $P_{ci}(s)\theta_i(s)$ and $\psi_i(s)P_{ri}(s)$ are stable, respectively. The proof of the following lemma can be seen in [4].

Lemma 1: A decoupling controller for the plant $P(s)$ exists if and only if 1) the polynomials θ_i and ψ_i are coprime, $i = 1 \to q$ and 2) the unique data construct $\sum(s) = P^{-1}\Delta_\theta \Delta_\alpha P$ is stable where $\Delta_\theta = diag\{\theta_i\}$, $\Delta_\alpha = diag\{\alpha_i\}$ and the polynomial $\alpha_i(s)$ is such that

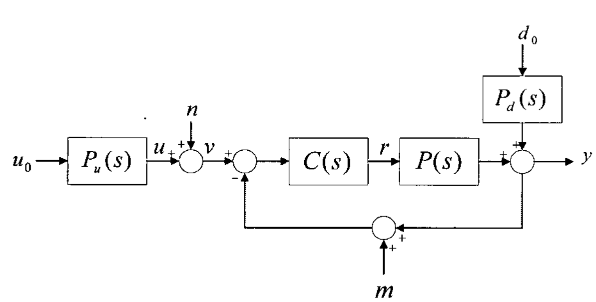

Fig. 1. A control system with a one-degree-of-freedom controller.

$$\alpha_i\theta_i + \beta_i\psi_i = 1,\ i = 1 \to q. \tag{3}$$

The existence of such α_i and β_i is always guaranteed if θ_i and ψ_i are coprime. When a decoupling controller exists, any realizable diagonal transfer matrix $T(s)$ is characterized by

$$T(s) = \Delta_\theta(\Delta_\alpha + \Delta\Delta_\psi), \tag{4}$$

where $\Delta_\psi = diag\{\psi_i\}$ and $\Delta(s)$ is an arbitrary stable diagonal matrix chosen so that $\det(I - T(s)) \not\equiv 0$. The characterization of all realizable transfer matrices in (4) is the key formula to derive the H_2 optimal controller. This formula is, however, described in terms of polynomial matrices and this causes a difficulty in developing numerical algorithms using state-space parameters. Next, we develop a rational matrix version of the formula in (4). Let $\lambda_i(s)$ denote the Wiener-Hopf spectral factor of the equation $\theta_i\theta_{i*} + \psi_i\psi_{i*} = \lambda_i\lambda_{i*}$. We define the following four rational functions:

$$\tilde{\psi}_i = \psi_i / \lambda_i,\ \tilde{\alpha}_i = \alpha_i\lambda_i + \{\lambda_i^2\gamma_i\}_\infty \psi_i / \lambda_i, \tag{5}$$

$$\tilde{\theta}_i = \theta_i / \lambda_i,\ \tilde{\beta}_i = \beta_i\lambda_i - \{\lambda_i^2\gamma_i\}_\infty \theta_i / \lambda_i, \tag{6}$$

where

$$\gamma_i = \begin{cases} \beta_i / \theta_i, & if\ \delta(\theta_i) \ge \delta(\psi_i) \\ -\alpha_i / \psi_i, & if\ \delta(\theta_i) < \delta(\psi_i). \end{cases} \tag{7}$$

Park *et al.* [9] showed that $\tilde{\theta}_i, \tilde{\psi}_i, \tilde{\alpha}_i,$ and $\tilde{\beta}_i$ are stable proper rational functions when θ_i and ψ_i are coprime and they satisfy the equality

$$\tilde{\alpha}_i\tilde{\theta}_i + \tilde{\beta}_i\tilde{\psi}_i = 1,\ i = 1 \to q. \tag{8}$$

Let $\tilde{\Delta}_\theta = diag\{\tilde{\theta}_i\}$, $\tilde{\Delta}_\psi = diag\{\tilde{\psi}_i\}$, $\tilde{\Delta}_\alpha = diag\{\tilde{\alpha}_i\}$, and $\tilde{\Delta}_\beta = diag\{\tilde{\beta}_i\}$. Then it follows from (5)~(7) that

$$\tilde{\Delta}_\theta = \Delta_\theta\Delta_\lambda^{-1},\ \tilde{\Delta}_\psi = \Delta_\psi\Delta_\lambda^{-1},\ \Delta_\lambda = diag\{\lambda_i\}, \tag{9}$$

$$\tilde{\Delta}_\alpha = \Delta_\alpha\Delta_\lambda + \{\Delta_\lambda^2\Delta_\gamma\}_\infty\Delta_\psi\Delta_\lambda^{-1},\ \Delta_\gamma = diag\{\gamma_i\}, \tag{10}$$

$$\begin{aligned} \tilde{\Delta}_\beta &= \Delta_\beta\Delta_\lambda - \Delta_\lambda^{-1}\Delta_\theta\{\Delta_\lambda^2\Delta_\gamma\}_\infty \\ &and\ \tilde{\Delta}_\alpha\tilde{\Delta}_\theta + \tilde{\Delta}_\beta\tilde{\Delta}_\psi = I. \end{aligned} \tag{11}$$

Note that $\tilde{\Delta}_\theta, \tilde{\Delta}_\psi, \tilde{\Delta}_\alpha,$ and $\tilde{\Delta}_\beta$ are stable proper rational matrices and their state-space parameters are easily computed by using the algorithms in [9]. Now, we have the following Lemma which can be proved from (4) by using the relations (9) ~ (11).

Lemma 2: The class of all realizable $T(s)$ in (4) is also characterized by the formula

$$T(s) = \tilde{\Delta}_\theta(\tilde{\Delta}_\alpha + \tilde{\Delta}\tilde{\Delta}_\psi), \tag{12}$$

where $\tilde{\Delta}(s)$ is an arbitrary diagonal stable rational matrix chosen so that $\det(I - T(s)) \not\equiv 0$. Though the formula in (12) is a stable rational matrix version of the one in (4), we still need to modify it to develop numerical algorithms using state-space parameters. Let $a_i(s)$ and $b_i(s)$ be arbitrary monic strict-Hurwitz polynomials such that θ_i / a_i and ψ_i / b_i are biproper. Let us define $\Delta_a = diag\{a_i\}$ and $\Delta_b = diag\{b_i\}$. Then, the realizable $T(s)$ in (12) can be modified as

$$T(s) = \tilde{\Delta}_\theta\tilde{\Delta}_\alpha + \hat{\Delta}_\theta\hat{\Delta}\hat{\Delta}_\psi, \tag{13}$$

where $\hat{\Delta}_\theta := \Delta_\theta\Delta_\alpha^{-1}$ and $\hat{\Delta}_\psi := \Delta_l^{-1}\Delta_\psi$ are biproper and $\hat{\Delta} := \Delta_a\Delta_\lambda\tilde{\Delta}\Delta_\lambda\Delta_b$ is an arbitrary stable diagonal matrix. The modified formula in (13) will be used to find the optimal H_2 solution. Next, we present the H_2 optimal decoupling control problem. A meaningful cost function is given by

$$E = \|Q_e e(s)\|_2^2 + \hat{k}^2 \|Q_r(s)r(s)\|_2^2, \tag{14}$$

where

$$\begin{aligned} e(s) &= u(s) - y(s) \\ &= (I - T)(P_u u_0 - P_d d_0) - T(n - m). \end{aligned} \tag{15}$$

The optimal realizable decoupling $T(s)$ is the one that minimizes the cost function E. The optimal problem is solvable under the following assumptions.

Assumption 2: The constant weighting matrix Q_e is nonsingular. The stable weighting matrix $Q_r(s)$ is chosen so that $\hat{k}Q_r(s)P^{-1}(s)$ is proper (such a choice is always possible).

Assumption 3: Consider the fraction $P(s) = A^{-1}(s)B(s)$ where (A, B) is a left coprime poly-nomial matrix pair. The matrices $A(s)P_u(s)$ and $A(s)P_d(s)$ are stable.

Let us define

$$M_1 = \begin{bmatrix} Q_e \\ \hat{k}Q_r P^{-1} \end{bmatrix} \text{ and } M_2 = \begin{bmatrix} P_u\ \Omega_n\ \Omega_m\ P_d \end{bmatrix}, \tag{16}$$

where $\Omega_n = \Phi_n^{1/2}$ and $\Omega_m = \Phi_m^{1/2}$. Let $\mathrm{Re}\, s = 0$ denote the finite part of the purely imaginary $s = j\omega$ axis.

Assumption 4: The matrix $G_\theta := \Delta_{\theta*}M_{1*}M_1\Delta_\theta$ is nonsingular in $\mathrm{Re}\, s = 0$.

Assumption 5: The matrix $G_\psi := \Delta_\psi M_2 M_{2*}\Delta_{\psi*}$ is nonsingular in $\operatorname{Re} s = 0$.

Assumption 6: The matrices $P_u(s)$ and $P_d(s)$ are strictly proper.

For later use, we define $\Omega(s)$ as the Wiener-Hopf spectral factor of the equation

$$\hat{\Delta}_{\psi *}\hat{\Delta}_{\theta *}(M_2' \odot M_1)_*(M_2' \odot M_1)\hat{\Delta}_\theta\hat{\Delta}_\psi = \hat{G}_\psi' \circ \hat{G}_\theta = \Omega_*\Omega \tag{17}$$

and define U, M_0, and M_p as

$$U = (M_2' \odot M_1)\hat{\Delta}_\psi\hat{\Delta}_\theta\Omega^{-1}, \tag{18}$$

$$M_0 = M_p - M_1\tilde{\Delta}_\theta\tilde{\Delta}_\alpha M_2, \quad M_p = \begin{bmatrix} Q_eP_u & 0 & 0 & Q_eP_d \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{19}$$

As the consequences of Assumptions 3 ~ 5, we have the following lemma whisch is useful to prove Theorem 1.

Lemma 3: 1) $\tilde{\Delta}_\psi P_u, \tilde{\Delta}_\psi P_d$, and $\tilde{\Delta}_\psi M_2$ are stable. 2) If the plant $P(s)$ admits a decoupling controller, then $P^{-1}\tilde{\Delta}_\theta\tilde{\Delta}_\alpha P_u$ and $P^{-1}\tilde{\Delta}_\theta\tilde{\Delta}_\alpha P_d$ are stable. 3) Ω is stable. 4) Ω^{-1} is stable and hence U is inner. 5) M_0 is stable. 6) Ω^{-1} is proper.

Proof: 1) Since $\tilde{\Delta}_\psi P = \tilde{\Delta}_\psi A^{-1}B$ is stable, $\tilde{\Delta}_\psi A^{-1}$ is stable and, therefore, $\tilde{\Delta}_\psi A^{-1}AP_u$, $\tilde{\Delta}_\psi A^{-1}AP_d$ and, hence, $\tilde{\Delta}_\psi M_2$ are stable. 2) $P^{-1}\tilde{\Delta}_\theta\tilde{\Delta}_\alpha P = P^{-1}\Delta_\theta \Delta_\alpha P + P^{-1}\Delta_\theta\Delta_\lambda^{-1}\{\Delta_\lambda^2\Delta_\gamma\}_\infty\Delta_\lambda^{-1}\Delta_\psi P$ is stable since $P^{-1}\Delta_\theta\Delta_\alpha P$ is stable by Lemma 1. Hence, $P^{-1}\tilde{\Delta}_\theta\tilde{\Delta}_\alpha A^{-1}$ is stable so that $P^{-1}\tilde{\Delta}_\theta\tilde{\Delta}_\alpha A^{-1}AP_u$ and $P^{-1}\tilde{\Delta}_\theta\tilde{\Delta}_\alpha A^{-1}AP_d$ are stable. 3) Since $M_1\tilde{\Delta}_\theta$ and $\tilde{\Delta}_\psi M_2$ are stable, $\hat{G}_\theta$ and $\hat{G}_\psi$ are analytic in $\operatorname{Re} s = 0$ and hence Ω is stable. 4) By Assumptions 4 and 5, $\hat{G}_\psi' \circ \hat{G}_\theta$ is nonsingular in $\operatorname{Re} s = 0$ and hence Ω^{-1} is analytic in $\operatorname{Re} s = 0$ and is stable. That U is inner is obvious from (17). 5) Using the equality in (11), we obtain

$$M_0 = M_p - M_1\tilde{\Delta}_\theta\tilde{\Delta}_\alpha M_2 = \begin{bmatrix} Q_e\tilde{\Delta}_\beta\tilde{\Delta}_\psi P_u & -Q_e\tilde{\Delta}_\theta\tilde{\Delta}_\alpha\Omega_m & -Q_e\tilde{\Delta}_\theta\tilde{\Delta}_\alpha\Omega_m & Q_e\tilde{\Delta}_\beta\tilde{\Delta}_\psi P_d \\ -\hat{k}Q_rP^{-1}\tilde{\Delta}_\theta\tilde{\Delta}_\alpha P_u & -\hat{k}Q_rP^{-1}\tilde{\Delta}_\theta\tilde{\Delta}_\alpha\Omega_n & -\hat{k}Q_rP^{-1}\tilde{\Delta}_\theta\tilde{\Delta}_\alpha\Omega_m & \hat{k}Q_rP^{-1}\tilde{\Delta}_\theta\tilde{\Delta}_\alpha P_d \end{bmatrix}.$$

This matrix is stable according to 1) and 2). 6) Since $\Omega(s)$ is proper and $\Omega(\infty)$ is invertible, Ω^{-1} is proper. In fact, $\hat{G}_\psi$ and $\hat{G}_\theta$ are proper so that $\hat{G}_\psi' \circ \hat{G}_\theta = \Omega_*\Omega$, and hence $\Omega(s)$, are proper. Also, $\Omega_*(\infty)\Omega(\infty) = \hat{G}_\psi'(\infty) \circ \hat{G}_\theta(\infty)$ is positive definite since $G_\psi'(\infty)$ and $G_\theta(\infty)$ are positive definite [10]. □

Next, substituting (1) and (2) into (14) and (15), we have $E = \|M_p - M_1TM_2\|_2^2$. Invoking the realizable diagonal transfer matrix formula for $T(s)$ in (13), we obtain $E = \|M_0 - M_1\hat{\Delta}_\theta\hat{\Delta}\hat{\Delta}_\psi M_2\|_2^2$. Since the vectorization does not affect the H_2 norm, $E = \|vecM_0 - (M_2' \odot M_1)vecd(\hat{\Delta}_\theta\hat{\Delta}\hat{\Delta}_\psi)\|_2^2 = \|vecM_0 - (M_2' \odot M_1)\hat{\Delta}_\theta\hat{\Delta}_\psi\, vecd\hat{\Delta}\|_2^2 = \|vecM_0 - U\Omega vecd\hat{\Delta}\|_2^2$, which is the standard form of the H_2 problem form. Our H_2 problem is to find the free stable parameter $\hat{\Delta}$ that minimizes E. The procedure used to obtain the optimal $\hat{\Delta}$, and hence the optimal T, is straight-forward [6] and hence only its solution formula is presented here.

Theorem 1: Let the plant $P(s)$ admit a decoupling controller (Lemma 1). Under Assumptions 2~6, the optimal decoupling problem is solvable and the optimal closed transfer matrix is obtained as

$$vecdT_{opt}(s) = \hat{\Delta}_\psi\hat{\Delta}_\theta\Omega^{-1}(\{b\}_+ + \{c\}_-), \tag{20}$$

where

$$b = U_*vecM_P, \tag{21}$$

$$c = \Omega\hat{\Delta}_\theta^{-1}\tilde{\Delta}_\theta\tilde{\Delta}_\alpha\hat{\Delta}_\psi^{-1}e_q,\ e_q = [1\,1 \cdots 1]'. \tag{22}$$

The optimal closed transfer matrix $T_{opt}(s)$ is strictly proper.

The solution formulas obtained in this section are based on frequency domain algorithms such as spectral factorizations and partial fractions. It is desirable to develop state-space numerical algorithms as alternatives to the frequency domain numerical algorithms. We seek to develop state-space domain formula in the next section.

3. NUMERICAL COMPUTATION IN THE STATE-SPACE DOMAIN

In this section, the state–space representations of the formulas in Section 2 are sought to facilitate their computation. The notation $M = ARE(F, R, Q)$ implies that M is the so-called stabilizing solution of the algebraic Riccati equation $F'M + MF - MRM + Q = 0$. That is, M is a solution to the ARE and $F - RM$ is stable. This solution exists and is unique provided that (F, R) is stabilizing and the associated Hamiltonian matrix has no $j\omega - axis$ eigenvalues [11]. To simplify notation, the convention $H(sI - F)^{-1}G + J =: \left[\begin{array}{c|c} F & G \\ \hline H & J \end{array}\right]$ or $H(sI - F)^{-1}G + J =: [F, G, H, J]$ will be used. When $P_i(s) = [F_i, G_i, H_i, J_i]$, $i = 1,2$, we define $prod(P_1(s), P_2(s))$ as the state-space parameters of the product $P_1(s)P_2(s)$. That is,

$$prod(P_1, P_2) = \left[\begin{array}{cc|c} F_1 & G_1H_2 & G_1J_2 \\ 0 & F_2 & G_2 \\ \hline H_1 & J_1H_2 & J_1J_2 \end{array}\right] \tag{23}$$

and this computation can be easily programmed as a MATLAB function. The operation $prod(P_1, prod(P_2, P_3))$ is conveniently written as $prod(P_1, P_2, P_3)$. In this section, numerical algorithms are developed to find the state-space parameters of the realizable $T(s)$ in (13) and the optimal solution $vecd\ T_{opt}(s)$ in (20). Our main concern is to compute $vecd\ T_{opt}(s)$ and it is assumed that the simple quantities such as $\{P_d\}_+, \{P_d\}_-, \theta_i$ and ψ_i are calculated by hand. We assume that the plant $P(s)$ and the disturbance transfer matrix $P_d(s)$ are characterized by the structure, $\dot{x}(t) = Fx(t) + Gr(t) + G_d d_0(t)$, $y(t) = Hx(t) + Jr(t)$. Then it follows that $P(s) = [F, G, H, J]$ and $P_d(s) = [F, G_d, H, 0]$. Let the state-space parameters of $P_u(s)$ be $[F_u, G_u, H_u, 0]$. As explained in Section 2, F_u is assumed to have its eigenvalues in $\mathrm{Re}\, s \le 0$. Also, let $\{P_d(s)\}_+ = [F_{d+}, G_{d+}, H_{d+}, 0]$ and $\{P_d(s)\}_- = [F_{d-}, G_{d-}, H_{d-}, 0]$.

Next, the state-space parameters of $\tilde{\Delta}_\theta$, $\tilde{\Delta}_\psi$, $\tilde{\Delta}_\alpha$ and $\tilde{\Delta}_\beta$ are found. Using the formulas of Park *et al.* [9], we obtain the state-space parameters of $\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\theta}_i$ and $\tilde{\psi}_i$ as follows:

1) When $\delta(\theta_i) \ge \delta(\psi_i)$, find a minimal realization of $\psi_i(s)/\theta_i(s)$ as $[F_{di}, G_{di}, H_{di}, J_{di}]$. Then,

$$\begin{bmatrix} \tilde{\alpha}_i & \tilde{\beta}_i \\ -\tilde{\psi}_i & \tilde{\theta}_i \end{bmatrix} = \left[\begin{array}{c|cc} F_{di} - K_{2di}H_{di} & G_{di} - K_{2di}J_{di} & K_{2di} \\ \hline Q_{di}^{1/2}K_{1di} & Q_{di}^{1/2} & 0 \\ -G_{mdi}^{-1/2}H_{di} & -G_{mdi}^{-1/2}J_{di} & G_{mdi}^{-1/2} \end{array}\right]$$

$$=: \left[\begin{array}{c|cc} F_{ai} & G_{ai} & G_{bi} \\ \hline H_{1i} & J_{11i} & J_{12i} \\ H_{2i} & J_{21i} & J_{22i} \end{array}\right], \tag{24}$$

where

$$Q_{di} = I + J'_{di}J_{di}, G_{mdi} = I + J_{di}J'_{di}, \tag{25}$$

$$\begin{cases} K_{1di} = Q_{di}^{-1}(J'_{di}H_{di} + G'_{di}M_{1di}) \\ K_{2di} = (G_{di}J'_{di} + M_{2di}H'_{di})G_{mdi}^{-1}, \end{cases} \tag{26}$$

$$M_{1di} = ARE(F_{di} - G_{di}Q_{di}^{-1}J'_{di}H_{di}, G_{di}Q_{di}^{-1}G'_{di}, H'_{di}G_{mdi}^{-1}H_{di}), \tag{27}$$

$$M_{2di} = ARE((F_{di} - G_{di}J'_{di}G_{mdi}^{-1}H_{di})', H'_{di}G_{mdi}^{-1}H_{di}, G_{di}Q_{di}^{-1}G'_{di}). \tag{28}$$

2) When $\delta(\theta_i) < \delta(\psi_i)$, let a minimal realization of $\theta_i(s)/\psi_i(s)$ be $[F_{di}, G_{di}, H_{di}, 0]$. Then

$$\begin{bmatrix} \tilde{\beta}_i & \tilde{\alpha}_i \\ -\tilde{\theta}_i & \tilde{\psi}_i \end{bmatrix} = \left[\begin{array}{c|cc} F_{di} - K_{2di}H & G_{di} & K_{2di} \\ \hline K_{1di} & I & 0 \\ -H_{di} & 0 & I \end{array}\right] =: \left[\begin{array}{c|cc} F_{ai} & G_{bi} & G_{ai} \\ \hline H_{1i} & J_{12i} & J_{11i} \\ -H_{2i} & -J_{22i} & -J_{21i} \end{array}\right], \tag{29}$$

where $K_{1di} = G'_{di}M_{1di}, K_{2di} = M_{2di}H'_{di}$, $M_{1di} = ARE(F_{di}, G_{di}G'_{di}, H'_{di}H_{di})$, and $M_{2di} = ARE(F'_{di}, H'_{di}H_{di}, G_{di}G'_{di})$.

3) When $\theta_i = \psi_i = 1$, we set $\tilde{\alpha}_i = 0$, $\tilde{\beta}_i = 1$, $\tilde{\theta}_i = 1$, and $\tilde{\psi}_i = 1$. Hence,

$$\begin{bmatrix} \tilde{\alpha}_i & \tilde{\beta}_i \\ -\tilde{\psi}_i & \tilde{\theta}_i \end{bmatrix} = \left[\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & -1 & 1 \end{array}\right] =: \left[\begin{array}{c|cc} F_{ai} & G_{ai} & G_{bi} \\ \hline H_{1i} & J_{11i} & J_{12i} \\ H_{2i} & J_{21i} & J_{22i} \end{array}\right]. \tag{30}$$

Now, it follows that

$$\tilde{\Delta}_\theta = [F_\theta, G_\theta, H_\theta, J_\theta], \tilde{\Delta}_\psi = [F_\theta, G_\psi - H_\theta, J_\psi], \tag{31}$$

$$\tilde{\Delta}_\alpha = [F_\theta, G_\psi, H_\alpha, J_\alpha], \tilde{\Delta}_\beta = [F_\theta, G_\theta, H_\alpha, J_\beta], \tag{32}$$

where

$F_\theta = diag\{F_{ai}\}, G_\theta = diag\{G_{bi}\}, H_\theta = diag\{H_{2i}\}$, $J_\theta = diag\{J_{22i}\}, G_\psi = diag\{G_{ai}\}, J_\psi = diag\{-J_{21i}\}$, $H_\alpha = diag\{H_{1i}\}, J_\alpha = diag\{J_{11i}\}$, and $J_\beta = diag\{J_{12i}\}$.

Next, we find the state-space parameters of $\Omega(s)$ and $U(s)$. Let a minimal realization of $Q_r(s)P^{-1}(s)$ be found as $Q_rP^{-1}=[F_q,G_q,H_q,J_q]$. It is easy to obtain the state-space parameter of $M_1(s)$ and $M_2(s)$ as

$$M_1=\left[\begin{array}{c|c} F_g & G_g \\ \hline 0 & Q_e \\ \hat{k}H_q & \hat{k}J_q \end{array}\right] \text{ and } M_2=\left[\begin{array}{cc|cccc} F_u & 0 & G_u & 0 & 0 & 0 \\ 0 & F & 0 & 0 & 0 & G_d \\ \hline H_u & H & 0 & \Omega_n & \Omega_n & 0 \end{array}\right]. \tag{33}$$

Next, we define $[F_1,G_1,H_1,J_1]:=prod(M_1,\tilde{\Delta}_\theta)$ and $[F_2,G_2,H_2,J_2]:=prod(\tilde{\Delta}_\psi,M_2)$. Let the state-space parameters of $\tilde{\Delta}_\theta(s)$ and $\tilde{\Delta}_\psi(s)$ be found as $\tilde{\Delta}_\theta(s)=[\hat{F}_\theta,\hat{G}_\theta,\hat{H}_\theta,I]$ and $\tilde{\Delta}_\psi(s)=[\hat{F}_\psi,\hat{G}_\psi,\hat{H}_\psi,I]$. We define $[\hat{F}_1,\hat{G}_1,\hat{H}_1,\hat{J}_1]:=prod(M_1,\hat{\Delta}_\theta)$ and $[\hat{F}_2,\hat{G}_2,\hat{H}_2,\hat{J}_2]:=prod(\hat{\Delta}_\psi,M_2)$. In the computation of $M_1\tilde{\Delta}_\theta$ and $\tilde{\Delta}_\psi M_2$, $M_1\hat{\Delta}_\theta$ and $\hat{\Delta}_\psi M_2$ it is important to perform a minimal realization operation [12] for the state-space parameters to force the matrices $F_1,F_2,\hat{F}_1$, and $\hat{F}_2$ to be stable. The minimal realization algorithms in actual use are not perfectly accurate. This does not, however, cause a significant problem as long as the resulting F matrices remain stable.

Next, using the state-space formula for the Khatri-Rao product [6,Lemma 6], we obtain

$$M_2'\hat{\Delta}_\psi' \odot M_1\hat{\Delta}_\theta=\left[\begin{array}{cc|c} \hat{F}_2'\otimes I & \hat{J}_2'\otimes\hat{H}_1 & \hat{H}'\odot\hat{J}_1 \\ 0 & I\otimes\hat{F}_1 & I\odot\hat{G}_1 \\ \hline \hat{G}_2'\otimes I & \hat{J}_2'\otimes\hat{H}_1 & \hat{J}_2'\odot\hat{J}_1 \end{array}\right] =:\left[\begin{array}{c|c} F_3 & G_3 \\ \hline H_3 & J_3 \end{array}\right]. \tag{34}$$

The introduction of the Khatri-Rao product brings inflates of the size of the state-space parameters, but they have many zero terms, so that the minimal realization operation works fairly well. Since $\Omega_*\Omega=(M_2'\hat{\Delta}_\psi'\odot M_1\hat{\Delta}_\theta)_*(M_2'\hat{\Delta}_\psi'\odot M_1\hat{\Delta}_\theta)$, we obtain the following results from Lemma 7 of [6];

$$\Omega(s)=[F_3,G_3,H_\omega,J_\omega], \tag{35}$$

where

$$\begin{cases} H_\omega=R_3^{-1/2}(J_3'H_3+G_3'M_3) \\ J_\omega=R_3^{1/2},R_3=J_3'J_3, \end{cases} \tag{36}$$

$$M_3=ARE(F_3-G_3R_3^{-1}J_3'H_3,\ G_3R_3^{-1}G_3',H_3'(I-J_3R_3^{-1}J_3')H_3), \tag{37}$$

and

$$U(s)=\left[\begin{array}{c|c} F_3-G_3K_3 & G_3R_3^{-1/2} \\ \hline H_3-J_3K_3 & J_3R_3^{-1/2} \end{array}\right]=:\left[\begin{array}{c|c} F_4 & G_4 \\ \hline H_4 & J_4 \end{array}\right] \tag{38}$$

with $K_3=J_\omega^{-1}H_\omega$. It follows from (35) that $\Omega^{-1}(s)=\left[F_4,G_4,-K_3,J_\omega^{-1}\right]$. The next step is to calculate $\{b\}_+$ and $\{c\}_-$. Since $b=U_*vecM_p=U_*(vec\{M_p\}_+ +vec\{M_p\}_-)$, $\{b\}_+=\{U_*vec\{M_p\}_+\}_+$ and the state-space parameters of $U_*,\{M_p\}_+$ and $vec\{M_p\}_+$ are given by

$$U_*(s)=\left[-F_4',-H_4',G_4',J_4'\right], \tag{39}$$

$$\{M_p\}_+=\left[\begin{array}{cc|cccc} F_u & 0 & G_u & 0 & 0 & 0 \\ 0 & F_{d+} & 0 & 0 & 0 & G_{d+} \\ \hline Q_eH_u & Q_eH_{d+} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] =:\left[\begin{array}{c|c} F_p & G_p \\ \hline H_p & 0 \end{array}\right] \tag{40}$$

and $vec\{M_p\}_+=\left[I\otimes F_p,vecG_p,I\otimes H_p,0\right]$. The computation of $\{b\}_+=\{U_*vec\{M_p\}_+\}_+$ can be done by adopting the partial fraction technique of [6] and the result is

$$\begin{aligned}\{b\}_+&=[I\otimes F_p,vecG_p,J_4'(I\otimes H_p)+G_4'M_4,0] \\ &=:[F_b,G_b,H_b,0],\end{aligned} \tag{41}$$

where M_4 is the unique solution of the equation $F_4'M_4+M_4(I\otimes F_p)=-H_4'(I\otimes H_p)$. Next, notice that $\{c\}_-=\{\Omega\hat{\Delta}_\theta^{-1}\tilde{\Delta}_\theta\tilde{\Delta}_\alpha\hat{\Delta}_\psi^{-1}\}_-e_q$ and $\Omega\hat{\Delta}_\theta^{-1}\tilde{\Delta}_\theta\tilde{\Delta}_\alpha$ is stable. Let us define $[F_5,G_5,H_5,J_5]=:prod(\Omega,\hat{\Delta}_\theta^{-1},\tilde{\Delta}_\theta,\tilde{\Delta}_\alpha)$ where $\hat{\Delta}_\theta^{-1}=[\hat{F}_\theta-\hat{G}_\theta\hat{H}_\theta,\hat{G}_\theta,-\hat{H}_\theta,I]$. In computing $prod(\Omega,\hat{\Delta}_\theta^{-1},\tilde{\Delta}_\theta,\tilde{\Delta}_\alpha)$, minimal realization should be performed to force the matrix to be stable. Since $\hat{\Delta}_\psi^{-1}=[\hat{F}_\psi-\hat{G}_\psi\hat{H}_\psi,\hat{G}_\psi,-\hat{H}_\psi,I]$ is unstable, the computation of $\{c\}_-$ can be done similarly to that of $\{b\}_+$ and the result is

$$\begin{aligned}\{c\}_-&=[\hat{F}_\psi-\hat{G}_\psi\hat{H}_\psi,\hat{G}_\psi e_q,H_5M_5-J_5\hat{H}_\psi,0] \\ &=:[F_6,G_6,H_6,0],\end{aligned} \tag{42}$$

where M_5 is the solution of $F_5M_5 - M_5(\hat{F}_\psi - \hat{G}_\psi\hat{H}_\psi) = G_5\hat{H}_\psi$. Let us define $[F_7, G_7, H_7, J_7] =: prod(\hat{\Delta}_\psi, \hat{\Delta}_\theta, \Omega^{-1})$. In this computation, we always observe that the auxiliary variables $\Delta_a(s)$ and $\Delta_b(s)$ are cancelled out when minimal realization is performed for $[F_7, G_7, H_7, J_7]$. Since

$$\{b\}_+ + \{c\}_- = \left[\begin{array}{cc|c} F_b & 0 & G_b \\ 0 & F_6 & G_6 \\ \hline H_b & H_6 & 0 \end{array}\right] =: \left[\begin{array}{c|c} F_8 & G_8 \\ \hline H_8 & 0 \end{array}\right], \quad (43)$$

it follows that $vecdT_{opt}(s) = \hat{\Delta}_\psi\hat{\Delta}_\theta\Omega^{-1}(\{b\}_+ + \{c\}_-) = prod([F_7, G_7, H_7, J_7], [F_8, G_8, H_8, 0]) =: [F_9, G_9, H_9, 0]$. In this computation, it should be confirmed that F_9 is a stable matrix after the minimal realization operation.

4. CONCLUSION

State-space algorithms to compute the optimal one-degree-of-freedom decoupling controller of Youla and Bongiorno [4] were developed. To develop the state-space algorithms, the characterization formula for the class of all realizable decoupling controllers is reformulated in terms of rational functions instead of the polynomials that are favored by Youla and Bongiorno [4]. The class of all stabilizing and decoupling controllers is characterized in terms of $T(s)$ via the free diagonal parameter matrix $\hat{\Delta}(s)$ defined in (13). The optimal decoupling controller is determined from these free parameters. The inner-outer factorization and the Kharti-Rao product expression for the application of the vectorization operation to a diagonal matrix are the key steps to obtain the optimal solution. A compact set of assumptions is given to assure the existence of the optimal solution is given.

A possible future research work is to adapt the work in this paper to the standard plant model [11,13] that can accommodate a wider range of control problems including the nonunity feedback configuration.

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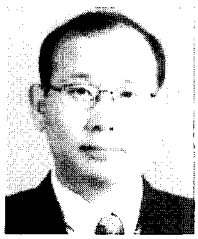
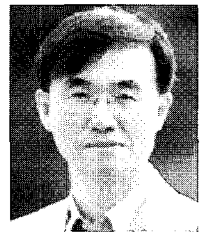


Kiheon Park received the B.S. and M.S. degrees in Electrical Engineering from Seoul National University in 1978 and 1980, respectively, and the Ph.D. degree in Systems Engineering from Polytechnic University, Farmingdale, NY, in 1987. During the years 1988-1990, he was involved in a factory automation project at the Electronic and Telecom-munications Research Institute, Dae-Jeon, Korea. He is currently a Professor of the School of Information and Communication Engineering at Sungkyunkwan University, Suwon, Korea. His research interests include the Wiener-Hopf design of the linear multivariable control systems, decoupling control systems, and industrial control networks. Dr. Park was the recipient of Korea Electronic Association Scholarship from 1983 to 1986.



Jin-Geol Kim received the B.S. degree in Electrical Engineering from Seoul National University, Seoul, Korea in 1978. He received the M.S. and Ph.D. degrees in Electrical and Computer Engineering from the University of Iowa, Iowa City, IA, U.S.A in 1984 and 1988, respectively. He received the M.S. degree in Mathematics from the University of Iowa, Iowa City, IA, U.S.A in 1985. Since 1988 he has been with the School of Electrical Engineering, Inha University, Inchon, Korea. His main research interests include nonlinear control algorithms, mobile/biped robots, and intelligent control systems.